Expectiles-type risk measures: an alternative to value-at-risk and expected shortfall

Author: **Charlotte Jamotton**
Advisor: **Prof. Pierre Devolder**

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Abstract

Embraced by the international banking standards known as the Basel Accords, value-at-risk (VaR) has long been promoted as the benchmark risk measurement tool in insurance and financial market sectors. Consistently criticised for its lack of subadditivity and its failure to account for the severity of losses in the far tail of the distribution, value-at-risk lost some ground to the tail-sensitive expected shortfall (tail VaR). Its sensitivity to the size of potential losses beyond VaR should be the key incentive for tail VaR to receive the favour of the regulator and prudential risk managers. In that respect, the Basel IV standards, whose implementation is due in January 2023, amended the global bank capital requirements by shifting the quantitative risk metrics system from 99% value-at-risk to 97.5% expected shortfall. Similarly, while capital requirements are still determined on the basis of a 99.5% value-at-risk over one year under the EU insurance regulatory regime Solvency II, the Swiss Solvency Test requires the use of a 99% tail VaR. At the same time, the (erroneous) belief that expected shortfall could not be backtested because of its non-elicitability fostered an interest in emerging alternatives, namely expectiles as the sole elicitable law invariant coherent risk measures.

In light of this, one may legitimately wonder what the best risk measure might be. Subject to the important caveat that no risk measure can achieve superiority (there is no panacea), the aim of this thesis is to provide a compendium comparison between expectiles and traditional measures (value-at-risk, expected shortfall, and variance) through a number of applications in risk management. In that regard, we will review the commonly accepted desirable properties of risk measures such as coherence, convexity, law-invariance, comonotonic additivity, robustness, and elicitability.

While expectiles may address some of the flaws in value-at-risk and expected shortfall, they might fail to detect risk concentrations and difficulties in their interpretation gave rise to criticisms. We also find that the backtesting-related issues and lack of robustness of expected shortfall are indeed not problematic. While non-robustness to outliers might actually enable to foresee future large losses, non-elicitability does not actually preclude backtesting. There is thus no sufficient evidence to justify an all-inclusive replacement of value-at-risk and expected shortfall by expectiles.

The main contribution of this work is a paper by J.M. Chen (2018), entitled *On exactitude in financial regulation : value at risk, expected shortfall and expectiles.*

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Contents

Introduction

In his paper titled *World Risk Society* published in 1992, the sociologist Beck labeled our society as the *risk society*. His seminal work speaks to many of the issues before us today. Indeed, for Beck (1992, p.2), the consequences of modernity “are a set of risks (. . .), the likes of which we have never previously faced (. . .) their spatial consequences are (. . .) not amenable to limitation - as they cross national boundaries”. This quote resonates very much with the Covid-19 crisis. He then goes on to say: “These dangers can no longer be limited in time - as future generations are affected”, which is clearly a concern of climate risk. He further states that, “unlike in an earlier modernity, no one can be held accountable for the hazards of the risk society” which is largely in line with today’s questions about the liable party for self-driving cars. As early as 1992, Beck had thus managed to capture the current concerns about risk management. The increasing need for risk management and risk measures arises in many different contexts. For instance, in order to determine the extra regulatory capital a financial institution has to hold as a buffer against unforeseen events, a proper measure needs to be established to quantify the institution’s risk exposure. Measuring risks, that is choosing the right risk measure, is precisely at the core of our thesis.

The paper is organized as follows:

- Chapter 1 introduces the nature of risk itself and the notion of risk measure.
- Chapter 2 delves into the risk measure properties deemed as desirable. Among others, the axioms of the so-called coherent risk measures will be formally defined.
- Chapter 3 reviews and compares the traditional risk measures with respect to their properties.
- Chapter 4 focuses on the expectiles, as an alternative to value-at-risk and expected shortfall.
- Chapter 5 models the capital requirement of a life insurer under the Solvency II norm and assesses whether the expectile may supersede the value-at-risk.

The paper ends with a discussion of the (dis)advantages of the different risk measures and a best practise recommendation for choosing a risk measure to that effect.

Chapter 1

Introduction to risk, risk management, and risk measures

Each and every financial institution (e.g. a bank or an insurance company) faces uncertain future events, commonly known as risks, that may negatively impact the company's wealth and hence its solvency (Devolder, 2021). The notion of **risk** was first defined by Knight (1921) as a "quantity susceptible of measurement". By further describing risk as measurable uncertainty, and uncertainty as non-measurable risk, Knight shaped the distinction between risk and uncertainty. Since a risk is not certain, the likelihood of occurrence of its potential outcomes can (only) be estimated (Devolder, 2021). Knowledge of the risk's probability distribution allows one to measure (estimate) its risk exposure and to set risk tolerance boundaries. Dealing with the management of risk and uncertainty is precisely the purpose of **risk management**.

Risk management first consists in identifying all relevant and emerging risks that may adversely affect a company. While the impact of climate change is becoming patently clear, some more unforeseeable events may still take an industry by surprise. The global impact of Covid-19 is a case in point. While the pandemic risk was completely identified by professionals and, with that in mind, Swiss Re had bought securitization based on a mortality index to cover extreme mortality losses due to a global pandemic, the lockdown consequences were, conversely, not anticipated. The main challenge clearly lies in the business' ability to timely identify the scope of each risk to which they are or could be exposed to.

Once identified, risks will be assessed and measured. This second step is crucial since it is primarily used to determine capital requirements to prevent insolvency in the event of unexpected losses. Likewise, risk measures are also often used as a management tool to set risk exposure tolerances within a firm.

Lastly, the fast-moving risk landscape is the reason why risk management is the bedrock of any insurance company. While climate risk and the scale of natural catastrophes are on the rise, the development of disruptive technologies, such as Artificial Intelligence and autonomous vehicles, also led to the emergence of new risks. To ensure the continued relevancy of each risk impact, close monitoring of and reporting on the risks management is of utmost importance.

The consequences of poor or inadequate risk management systems were notably showcased by the collapse of the British merchant bank Barings in 1995, Orange County's bankruptcy in 1994, the blow-up of LTCM in 1998, and, more recently, the liquidation of Intégrale in 2022. These are stark and prime examples of how the response of a business depends largely on the quality of its risk management.

Although most of these striking events occurred in the 1990s, the demand for risk management had already been heightened by the 1970s oil crises and the abolition of the Bretton–Woods system of fixed exchange rates. The latter put an end to the reduced risk when investing and trading with other countries by rendering the dollar a fiat currency. Together with the excessive energy prices volatility, Bretton Woods' collapse fueled a widespread need for hedging products. However, "when hedges do not exist, for instance due to the structural incompleteness of the corresponding market" - i.e. no replicating securities - "a careful **measure of risk** needs to be established" (Denuit et al., 2006, p.59).

While risk measures may be used for a variety of purposes, their interpretation given by Artzner et al. (1999) serves as a common nexus. Whether it is for determining insurance premiums, margin requirements for investors, capital adequacy, economic or risk capital, a risk measure is a management tool that gives the least amount needed for a financial loss to be considered acceptable. (Embrechts et al., 2005)

1.1 Definition of risk and risk measures

Overall, the need for measures of risk is largely attributable to risk analysts' main concern, namely big losses (Osband, 2011). Formally, the concept of risk measures can be deduced from the following definition of risk.

Definition 1 (Risk)**.** A risk is modeled by a random variable, denoted by any capital letter, on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ where Ω represents all possible states of nature. The random variable L describes the loss, modeled with a positive sign $(L > 0)$, on a fixed time horizon.

In the present paper, when we mention a non-negative random variable we mean a risk, and vice versa. Since risk managers are mainly concerned with large losses, we will also focus on the right-tail of loss distributions with the convention that losses are positive numbers.

The definition of a risk as a loss random variable may be broadened to the set $\mathscr{L} \in L(\Omega, \mathscr{F}, \mathbb{P})$ of risks (market risk, credit risk, model risk . . .) we want to study, where $L(\Omega, \mathscr{F}, \mathbb{P})$ represents the set of all $\mathbb{P}-$almost surely finite random variables on $(\Omega, \mathscr{F})$. The interpretation of this set of random variables $\mathscr{L}$ stems from the aforementioned definition of a risk L; while L describes the loss over some time horizon Δ, $\mathscr{L}$ describes all the portfolio losses over Δ (Embrechts et al., 2005; Wüthrich & Merz, 2013).

Any risk $L \in \mathscr{L}$ is completely described by its distribution function $F_L(x) = P(L \leq x)$. Knowledge of the risks distribution function may enable, under stochastic dominance rules and expected utility theory, to compare two risks L_1 and L_2. For instance, the *loss* random variable L_1 has first-order stochastic dominance over L_2 when $F_{L_1}(x) \geq F_{L_2}(x)$. Formally,

$$L_1 \preceq_{st} L_2 \text{ if and only if } P(L_1 > x) \leq P(L_2 > x) \text{ for all } x \in \mathbb{R} \tag{1.1}$$

Loosely speaking, L_2 is said to be smaller than L_1 in the distribution order because the probability that the loss L_2 exceeds some threshold x is greater than the corresponding probability for L_1. Since we are working with losses, the smaller the loss, the "better" the risk. Henceforth, L_1 is preferred over L_2 (Chen, 2018).

However, as pointed out by Wüthrich and Merz (2013, p.261), "the distribution functions are not always adequate to compare two risks". Although stochastic dominance makes risk ranking possible, it does not give any information about the risk severity and frequency. Nor does it give any indication of the required amount of buffer capital. Thereupon, an alternative consists of quantifying the risk exposure by one number only, which leads us to the concept of risk measures.

Definition 2 (Risk measure)**.** A risk measure ϱ is a risk functional mapping loss distributions (from the set of risks $\mathscr{L}$) to the non-negative real numbers $\mathbb{R}^+$ (Devolder, 2021; Wüthrich & Merz, 2013).

$$\varrho : L \in \mathscr{L} \longrightarrow \varrho(L) \in \mathbb{R}^+ \tag{1.2}$$

Simply put, applying a risk measure ϱ to a loss distribution is a means for summarising "the probability distribution attached to a risk" into a *single* non-negative number (Devolder, 2021). This shortcut, although convenient in many cases (allowing for calculations of regulatory requirements, provisions, insurance risk premiums, etc.), results unavoidably in the loss of information. Indeed, risk measures are unable to grasp in a single number the whole scope of a risk, and only focus on a particular aspect of it. (Denuit et al., 2006)

While numerous risk measures have been developed and extensively studied over the years, the debate surrounding the appropriate measure to use is still ongoing. Value-at-risk, which was widely accepted as the industry standard, following the famous *Weatherstone 4.15* report at JPMorgan, remains the most popular. The axiomatic system established by Artzner et al. (1999) is also considered as a landmark in the development of the theory of risk measures. This approach drove the emergence of another compelling risk measure, known as expected shortfall. In recent years, expectiles have also attracted the attention of the actuarial literature, following the discussion around the non-elicitability of expected shortfall.

Chapter 2

Desirable properties of risk measures

The following chapter offers a complete overview of the so-called desirable properties any "good" risk measure should have. In the first Section, we present the four axioms of coherence developed by Artzner et al. (1999) in their breakthrough paper titled *Coherent Measures of Risk*. In Sections 2 and 3, we extend this axiomatic approach by introducing the larger classes of convex and law invariant risk measures. Section 4 introduces the concept of comonotonicity as an alternative means to relax the (contentious) subadditivity axiom. In Sections 5 and 6, we discuss robustness and elicitability upon which expected shortfall was deemed to be flawed.

Before defining the four properties of coherence, any reasonable risk measures should fulfill two axioms relating to loading:

Axiom 1. *Non-excessive loading or no-ripoff*

The risk measure applied to the risk's distribution shall not exceed the maximal potential loss.

$$\varrho(L) \leq \max(L) \quad \forall L \tag{2.1}$$

In other words, the amount of risk capital $\varrho(L)$ that should be added as a buffer, against unexpected losses induced by one's risk exposure, shall not exceed the maximal potential loss. Keeping more capital would be pointless and cost-ineffective (lock up capital is pricey).

Axiom 2. *Non-negative loading*

The risk measure applied to the risk's distribution shall be greater than the expected loss.

$$\varrho(L) \geq \mathbb{E}(L) \quad \forall L \tag{2.2}$$

If the minimal capital tied up $\varrho(L)$ is smaller than the expected loss $\mathbb{E}(L)$, the ruin will become certain over the long term.

2.1 Coherent risk measures

A risk measure ϱ is called coherent when it satisfies the four axioms introduced by Artzner et al. (1999): translation invariance, monotonicity, subadditivity, and positive homogeneity.

These four axioms capture essential properties of financial markets described below. Nevertheless, the lack of coherence does not preclude a risk measure from being used as evidenced by value-at-risk. Indeed, VaR is considered a standard in bank and insurance industries, regardless of its violation of the subadditivity axiom pointed out by Artzner et al. (1999).

Axiom 3. *Translation invariance (TI)*

A risk measure ϱ is translation invariant (also known as consistent) if, for all random loss variables $L \in \mathscr{L}$ and every non-random $c \in \mathbb{R}$, we have:

$$\varrho(L + c) = \varrho(L) + c \tag{2.3}$$

Any deterministic increase c (for which there is no uncertainty) in the risk liability will alter the risk measure accordingly by the same amount. Similarly, the addition of a deterministic cash amount c to the initial risky position should trigger the same decrease c in the actual risk (Denuit et al., 2006; Devolder, 2021; Wüthrich & Merz, 2013).

The concept of economic capital can be inferred from this translativity axiom. As already mentioned, the economic capital is the amount of capital a company needs in order to survive the riskiness of its operations, and hence stay solvent. Assuming we bring the risk measure of L as capital such that $c = -\varrho(L)$, the adjusted loss position is then $\tilde{L} = L - \varrho(L)$. Given that $\varrho(c) = c$ if $c \in \mathbb{R}$, we have:

$$\varrho(\tilde{L}) = \varrho\left(L - \varrho(L)\right) = \varrho(L) - \varrho(L) = 0 \tag{2.4}$$

This results in a neutral position where the new position $\tilde{L}$ is "acceptable" to an external or internal risk controller "without further injection of capital" (Embrechts et al., 2005, p.239).

Remark 1. Whilst $\varrho(L)$ indicates the risk capital to be held *as of now*, the amounts c and X are to be paid in the *future*. Yet there is no discounting of c and X. The shortness of the time horizon used in insurance calculations (annual contracts with tacit renewal) obviates the need to discount c and X. This amounts to assuming the interest rate is equal to 0%. We will be retaining this assumption of no discounting throughout our applications.

Axiom 4. *Monotonicity (M)*

Axiom 4 states that, given the risk L_2 always exceeds L_1, the risk measures should be similarly ordered. Mathematically, the risk measure ϱ is monotonic if, for all random variables L_1 and $L_2 \in \mathscr{L}$ and $P(L_1 \leq L_2) = 1$, it holds that

$$\varrho(L_1) \leq \varrho(L_2) \tag{2.5}$$

Using the utility theory, this is tantamount to saying position L_1 has first-order stochastic dominance over position L_2 in that L_2 accounts for higher losses than L_1 as defined in (1.1).

Monotonicity is easily intelligible from a portfolio perspective as it entails that a greater amount of risk capital is required to cushion the position that leads to higher losses. (Embrechts et al., 2005)

Axiom 5. *Subadditivity (S)*

The subadditivity axiom reflects the principle of diversification, that is "a merger does not create extra risk" as argued by Artzner et al. (1999). Therefore, merged risks should be less risky, and hence require less buffer capital, than the sum of the individual risks pre-merger.

Formally, the risk measure ϱ is said to be subadditive if, for any loss random variables L_1, $L_2 \in \mathscr{L}$, we have:

$$\varrho(L_1 + L_2) \leq \varrho(L_1) + \varrho(L_2) \tag{2.6}$$

Specifically, the sum of the individual risks $\varrho(L_1) + \varrho(L_2)$ should be regarded as the upper bound of the risk of the portfolio $\varrho(L_1 + L_2)$ made up of those individual risks. The diversification benefit is then measured as the difference $\varrho(L_1) + \varrho(L_2) - \varrho(L_1 + L_2) \geq 0$

Remark 2. When the equality holds in equation (2.6), we speak about *additivity*. See additivity for comonotonic risks in section 2.4 for further comments.

The subadditivity hypothesis is quite sensible in finance. In the worst case scenario, we lose the entire initial investment. The maximal loss is therefore bounded. However, in theory, such a bound does not always exist in insurance. For instance, third-party motor liability for corporal injuries is known to be unlimited by law in Belgium. If we were to merge two insurance portfolios X and Y, where X is a third-party motor liability portfolio, losses generated by X may be so serious they drag along the merged portfolio towards an insolvent situation. In this instance, it is preferable to keep X and Y apart.

Along the same line of reasoning, an important part of the debate surrounding the relevance of the subadditivity axiom revolves around the risk measure value-at-risk. Although widely used, VaR fails to be subadditive in most practical cases (see part 3.2). In theory, non-subadditivity, and hence the lack of coherence, does not preclude the use of a risk measure. Be that as it may, the following arguments seek to explain why subadditivity may still be regarded as a reasonable and valuable condition for risk measures.

As a whole, lack of subadditivity implies the risk measure might not capture some essential properties of financial markets. As mentioned above, Axiom 5 reflects the established principle of modern portfolio theory whereby risk can be reduced by diversification.

In terms of internal risk management, the equation (2.6) means the overall risk of an enterprise is smaller or equal to the sum of the individual risks of each of its departments. This effectively means risk management systems can be decentralised. However, when subadditivity is not met, the existence of an upper bound for the overall risk of the firm, obtained by aggregating the risk of the different business units, is no longer clear. In this instance, a decentralised risk management system may fail (Denuit et al., 2006; Embrechts et al., 2005).

The use of a non-subadditive risk measure may also spur a company to legally split-up into separatly-run entities in the interest of reducing its solvency capital requirements (SCR). Without subadditivity, the summation of SCRs of distinct legal entities or different lines of business may underestimate the overall SCR of the insurance company.

Those are a few of the most prominent reasons why opting for a subadditive risk measure should be considered as a legitimate choice.

However, as explained by Denuit et al. (2006, p.65), the use of subadditive risk measures may also provide a false sense of security. By aggregating risks that are positively dependent, the business actually increases the riskiness of its position, which “should induce higher capital requirements”. This argument, along with the one illustrated by the third-party motor liability, conflicts with the general idea behind the subadditivity axiom.

Axiom 6. *Positive homogeneity (PH)*

The risk measure ϱ is homogeneous if multiplying any risk $L \in \mathscr{L}$ by any positive constant $\lambda \geq 0$ results in a "linear increase in risk" of the same amount λ (Chen, 2018, p.3).

$$\varrho(\lambda L) = \lambda \varrho(L) \tag{2.7}$$

The homogeneity principle thus states that the risk measure is always directly proportional to the risk exposure. If a business doubles its exposure to a risky position L, it will, in turn, need twice as much risk capital.

This axiom also typically implies independence with respect to the monetary unit - which does not mean insensitivity to the currency (unless the exchange rates are set in advance) since λ must be deterministic, and exchange rates are dynamic (L is a *future* cash-flow).

It can be easily showed that the justification of axiom 6 ensues from the subadditivity axiom,

$$\varrho(\lambda L) = \varrho\,\underbrace{(L + L + \cdots + L)}_{\lambda\ terms} \leq \lambda \varrho(L) \text{ for } \lambda \in \mathbb{N} \tag{2.8}$$

The equality in the above equation (2.8) should hold, and results in positive homogeneity since "there is no netting or diversification between the losses in this portfolio" (Embrechts et al., 2005).

That being said, the positive homogeneity axiom has often been criticised for not taking into consideration the liquidity problems triggered by large concentrations of risks. In plain terms, the bigger the position size, the harder the liquidation of the position. Concerns on the liquidity risk omission in coherent risk measures were raised by many authors, including Acerbi and Scandolo (2008) and McNeil et al. (2015). The main line of reasoning is that the risk of the position should at least not be decreasing with its size i.e. $\varrho(\lambda X) > \lambda \varrho(X)$ for large values of the multiplier λ - which is inconsistent with the use of the subadditivity axiom to prove positive homogeneity in equation (2.8). This issue motivated the weaker property of convexity covered in the next section.

2.2 Convex risk measures

Since the positive homogeneity axiom is not always reckoned as appropriate, the larger class of convex risk measures has often been used as an alternative, including by Deprez and Gerber (1985). A risk measure ϱ is said to be convex if it satisfies axioms 3 and 4 and the weaker property of convexity defined below as axiom 7. The compliance to the positive homogeneity axioms is no longer required because axiom 7 acts as a less stringent substitute.

Axiom 7. *Convexity (C)*

A risk measure ϱ is convex if, for all random variables L_1 and $L_2 \in \mathscr{L}$, and any constant $\lambda \in [0, 1]$, the following inequality holds,

$$\varrho\left(\lambda L_1 + (1-\lambda) L_2\right) \leq \lambda \varrho(L_1) + (1-\lambda)\varrho(L_2) \tag{2.9}$$

Axiom (C) can be justified using axioms 5 (S) and 6 (PH). By first applying axiom 5 (S), we get $\varrho\left(\lambda L_1 + (1-\lambda) L_2\right) \leq \varrho\left(\lambda L_1\right) + \varrho\left((1-\lambda) L_2\right)$. Given axiom 6 (PH), we know that

$\varrho(\lambda L_1) = \lambda\varrho(L_1)$ and $\varrho((1-\lambda)L_2) = (1-\lambda)\varrho(L_2)$. This proves axiom 7 (C) ensues from axioms 5 (S) and 6 (PH), hence a coherent risk measure is a convex risk measure. However, the converse does not hold.

Axiom 7 (C) practically stipulates that the risk of a diversified position $\lambda L_1 + (1-\lambda)L_2$ is smaller or equal to the weighted average of the individual risks λL_1 and $(1-\lambda)L_2$. This ensures diversification will never increase a portfolio's risk. The idea whereby diversification reduces risk is inherent to the justification of the convexity's equation (2.9). Since convexity essentially encourages diversification, and it is possible to find convex risk measures penalizing risk concentration, that is $\varrho(\lambda X) > \lambda\varrho(X)$ for large λ's, this class of risk measures seems to be an adequate response to the liquidity issues overlooked by positive homogeneity (Denuit et al., 2006; Embrechts et al., 2005; Wüthrich & Merz, 2013). It should be noted, however, that Acerbi and Scandolo (2008) expressed a skeptical viewpoint on this larger space of convex risk measures that allow for violations of positive homogeneity and subadditivity. In their paper on *Liquidity Risk Theory*, they argue the four axioms of coherence and liquidity risk are in fact fully compatible. Their assertion is based on the non-linear relation between portfolios and portfolio values in the presence of liquidity risk. This led them to propose a new formalism for liquidity risk which is perfectly compatible with coherent measures of risk. Basically, they recognised liquidity risk requires adjustments to classic portfolio valuation and risk measurement. Their paper was later used by Stange and Kaserer (2010) and Weber et al. (2013) as a starting point to integrate liquidity risk in risk measure frameworks. We will not elaborate further, as a thorough treatment of liquidity theory is beyond the scope of this thesis.

2.3 Objectivity or law-invariance

While coherent and convex risk measures have been introduced by Artzner et al. (1999) and extensively analysed by Foellmer and Schied (2002), a further axiom of law invariance was first characterised by Kusuoka (2001).

Definition 3 (Law-invariance)**.** A risk measure ϱ is law invariant when it depends only on the distribution of losses. As opposed to model free risk measures that solely depend on the possible scenarios, law invariant risk measures are model dependent (Chen, 2018; Devolder, 2021).

Value-at-risk, expected shortfall, and expectiles all fulfill this definition of law invariant risk measures. As noted by Emmer et al. (2015, p. 35), "risk measures that depend only on the distribution of losses are of special interest, because their values can be estimated from loss observations only (i.e. no additional information such as stress scenarios is needed)." Law invariant risk measures will then report equivalent risk, that is, "assign the same value to two risky positions" that are identically distributed (Frittelli & Rosazza Gianin, 2005, p. 3). Formally, whenever $L_1 \overset{\mathrm{d}}{=} L_2$, it holds that

$$\varrho(L_1) = \varrho(L_2) \tag{2.10}$$

When one considers two insurance portfolios whose total claim amounts are identically distributed, equation (2.10) then implies the two portfolios will systematically require the same risk capital when using law invariant risk measures. However, one could argue that two identically distributed asset portfolios may still differ depending on their correlation with the economy as a whole.

2.4 Comonotonic additivity

The concept of comonotonicity or perfect positive dependence has been extensively discussed by many authors and was introduced in Yaari's dual theory of choice (1987) and Schmeidler's Choquet expected utility theorem (1989). In their paper on comonotonic risks, Song and Yan (2006) proposed the use of comonotonic risks to relax the (contentious) subadditivity axiom, as well as the convexity alternative. Because the concept of comonotonicity embodies the idea of undiversifiable risks, the determination of risk-based capital will be strongly influenced by it.

The definition of comonotonic risks directly derives from the word comonotonicity; it stems from the combination of common and monotonicity. In a nutshell, comonotonic risks L_i are *monotonic* increasing functions h_i of a *common* risk factor Z (a real-valued random variable).

Definition 4 (Comonotonicity)**.** Any real-valued random variables L_1 and L_2 are said to be comonotonic, or perfectly dependent, when non-decreasing functions h_1 and h_2 of the same source of uncertainty Z exist. The functions h_1 and h_2 thence map the same underlying risk factor Z onto L_1 and L_2 (Emmer et al., 2015; McNeil et al., 2015) i.e.

$$L_i \overset{\mathrm{d}}{=} h_i(Z) \text{ for } i \in (1, 2) \tag{2.11}$$

In other words, if two random variables L_1 and L_2 are comonotonic, then $L_1(\omega)$ and $L_2(\omega)$ always move in the same direction as the state $\omega \in \Omega$ changes - there is no pair $\omega_1, \omega_2 \in \Omega$ such that $L_1(\omega_1) < L_1(\omega_2)$ and $L_2(\omega_1) > L_2(\omega_2)$ (Heyde et al., 2008, p.4). Formally,

$$\left(L_1(\omega_1) - L_1(\omega_2)\right)\left(L_2(\omega_1) - L_2(\omega_2)\right) \geq 0 \tag{2.12}$$

The definition of comonotone additivite risk measures ensues (Emmer et al., 2015),

Definition 5 (Comonotone additivity)**.** A risk measure ϱ is comonotonic additive, if for any comonotonic random variables L_1 and L_2, the riskiness of the situation remains the same when pooling the comonotonic risks, i.e.

$$\varrho(L_1 + L_2) = \varrho(L_1) + \varrho(L_2) \tag{2.13}$$

Remark 3. As mentioned in Remark 2, comonotonic additivity switches the inequality sign in the subadditivity axiom equation (2.6) to equality.

The common source of hazard Z justifies the total lack of risk-reducing effect when pooling comonotonic risks together. Since losses only occur from the same source of uncertainty, these two risks are perfectly dependent - i.e. the outcomes of L_1 and L_2 always move in the same direction. There are thus no hedge or diversification benefits to be found, which led to the comonotonic additivity property of risk measures (Denuit et al., 2006; Heyde et al., 2008; Wang et al., 1997).

The comonotonic additivity property of risk measures is particularly attractive to complement the subadditivity property. While subadditivity rewards diversification, comonotonic additivity will not attribute any diversification benefits whatsoever to comonotonic risks since they depend on the same hazard Z (Embrechts et al., 2002; Emmer et al., 2015). Lack of comonotonic additivity in a risk measure is then considered as a very serious drawback as it does not allow for a proper representation of aggregation and diversification effects. As discussed later in section 4, although expectiles are recognised for addressing the deficiencies of value-at-risk and expected shortfall, their lack of comonotonic additivity marks the strongest argument against them as a risk measure.

In their paper, Dhaene et al. (2002, p.135) present several applications of comonotonicity in the fields of actuarial sciences and finance. The main result to retain from these examples is the use of financial hedging techniques to reduce the aggregate risk of a portfolio of risks with a comonotonic dependence structure - or at least mutually dependent risks. Because the risks L_i of such a portfolio are all non-increasing functions of the same random variable Z (e.g. a cat nat, a real estate index, etc.), increasing the number of policies will not enable the insurance company to cope with the volatility of the average risk. In the presence of positive dependencies, one may then underestimate the probability of having large aggregate claims and the technique of risk pooling might not be as effective as expected. Whether by writing call options on the CAT-index[1] in case of hurricane and earthquake insurance, or buying put options on the real estate index in case on an insurance which protects house-owners against the depreciation of their property, hedging techniques all aim to reduce a portfolio's risk.

We refer to Dhaene et al. (2002), Heyde et al. (2008), and Song and Yan (2006) for a more comprehensive analysis of comonotonicity and its applications in actuarial sciences and finance.

2.5 Robustness

Another important qualitative property of risk measures is robustness. By definition, a risk measure is said to be robust if it abides by the following two conditions:

1. The risk measure should have the ability to "accommodate model misspecification" (Kou et al., 2013, p.400).
2. The risk measure should be insensitive to small changes in the data and deviations from the underlying model assumptions (Huber & Ronchetti, 2011; Kou et al., 2013).

Because value-at-risk, expected shortfall, and expectiles are law invariant - i.e. depend only on the distribution of losses - the analysis of their respective sensitivity to misspecification errors in the loss distribution is of utmost importance. Robustness with respect to the underlying model and data is even more essential when one considers regulatory risk measures (e.g. value-at-risk and expected shortfall). If the risk measure does not yield consistent and stable results, different firms may reach very different conclusions and give inconsistent or divergent reports depending on the model and method used to implement the risk measure, and on the updating of the firm's historical database. (Heyde et al., 2008)

Robustness is commonly measured in terms of continuity with respect to the weak topology - as any small measurement errors in the loss distribution are known to have huge undesired consequences on the risk measure estimate. Since both expected shortfall and expectiles - and volatility as well - are discontinuous with respect to the weak topology (see e.g. Huber and Ronchetti (2011, chap.2) and Kiesel et al. (2012)), two other schools of thought stood out when investigating the robustness of risk measures: Cont et al.'s concept of robustness which focuses more on the sensitivity to outliers and the Wasserstein distance which focuses on small measurement errors.

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[1] Index of Catastrophe Losses of the Chicago Board of Trade

The robust estimation procedure developed by Cont et al. (2010) differentiates between different degrees of robustness. By taking the estimation procedure into account, they base their robustness analysis on the sensitivity (relative change) of the risk measure estimate when a new observation is added to the data set.

Ironically, while expected shortfall was precisely introduced to address the lack of tail-risk sensitivity of value-at-risk, its tail-sensitivity requires a larger amount of data to backtest it compared to value-at-risk at the same level of certainty. (Bellini & Bignozzi, 2015)

Besides, Emmer et al. (2015) also questioned the adequacy of Cont et al.'s concept of robustness in finance and insurance. In both fields, "large values (...) are not outliers or measurement errors, but facts that are a part of the observed process itself". Notably "in (re)insurance," one could quite rightly consider that "large claims are actually more accurately monitored than small ones, and their values are better estimated" (Emmer et al., 2015, p.7).

In light of this, the stronger notion of convergence that is the Wasserstein distance has been suggested by Kiesel et al. (2016) as a more advisable choice when investigating robustness.

While Cont et al.'s concept of robustness focuses more on the sensitivity to outliers, the investigation of robustness in terms on continuity with respect to weak topology or Wasserstein distance focuses on small measurement errors.

Definition 6. Let $P_n, n \geq 1$, and P be probability measures, and let $X_n \sim P_n, n \geq 1$, and $P \sim X$. A risk measure ϱ is called continuous at X with respect to the Wasserstein distance if (Emmer et al., 2015, p.38)

$$\underset{n \longrightarrow \infty}{lim} d_W(X_n, X) = 0 \Rightarrow \underset{n \longrightarrow \infty}{lim} \| \varrho(X_n) - \varrho(X) \| = 0 \tag{2.14}$$

Where the Wasserstein distance of two probability measures P and Q is defined as the smallest expected difference between random variables with these distributions, i.e. (Bellini et al., 2014, p.46)

$$d_W(P, Q) := \inf\{E(|X - Y|) : X \sim P, Y \sim Q\} \tag{2.15}$$

We refer to Kiesel et al. paper for the proof of the continuity of value-at-risk and expected shortfall with respect to the Wasserstein distance. Bellini et al. (2014) also showed that expectiles are continuous with respect to the Wasserstein distance since they are Lipschitz-continuous with respect to the Wasserstein distance with constant $K = max\{\frac{\alpha}{1-\alpha}; \frac{1-\alpha}{\alpha}\}$.

2.6 Elicitability

Risk measures are *historical* predictors, and are thence estimated on historical data. The ability to perform backtesting[2], forecast verification, and performance comparison of competing estimation procedures is therefore key. Risk measures with such capacity are called elicitable (Gneiting, 2011).

The term elicitable functional was first introduced by Osband and Reichelstein (1985), and later developed by Lambert et al. (2008) and Gneiting (2011) in the evaluation of optimal point forecasts. The latter paper is considered fundamental to quantitative risk management as it introduced to many the concept of elicitability. The determination of optimal point forecasts consists in trying to find "the best possible point estimate of" a functional for any time point in the future, given our current knowledge (Ziegel, 2016, p.2–4). 1-day ahead or 10-days ahead value-at-risk are some of the most common examples of point forecasts. Based on that observation - estimating risk measures "is a special case (. . .) of estimating statistics of (. . .) a forecasting distribution" - the underlying ideas in the forecasting literature were naturally transposed to the backtesting approach of risk measures (McNeil et al., 2015, p.355).

The definition of elicitability given by Gneiting (2011, p.8); Emmer et al. (2015, p.5); McNeil et al. (2015) is linked to that of strictly consistent scoring functions.

Definition 7 (Scoring function)**.** A scoring function is any mapping $s : \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$ that assigns a numerical score $s(x, l)$ to a single-valued point forecast x based on the predictive point and realisation l i.e.

$$(x, l) \longrightarrow s(x, l) \tag{2.16}$$

For any $x, l \in \mathbb{R}$ the scoring function $s(x, l)$ is

(i) continuous in x;

(ii) increasing for $x > l$ and decreasing for $x < l$;

(iii) ≥ 0 and $s(x, l) = 0$ if and only if $x = l$;

Definition 8 (Consistency)**.** A scoring function $s : \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$ is **consistent** for the functional ν relative to the class of probability measures $\mathscr{P}$ if and only if, for any loss random variable L with distribution P, for all $P \in \mathscr{P}$, and all $t \in \nu(P)$, any $x \in \mathbb{R}$ is an optimal point forecast under s (Gneiting, 2011, p.12), i.e.

$$\mathbb{E}_P[s(t, L)] \leq \mathbb{E}_P[s(x, L)] \tag{2.17}$$

Definition 9 (Strict consistency)**.** The scoring function s is **strictly** consistent if it is consistent and the equality of the expectations implies that $x \in \mathbb{R}$ (Gneiting, 2011, p.12), i.e.

$$\mathbb{E}_P[s(t, L)] = \mathbb{E}_P[s(x, L)] \Rightarrow x \in \nu(P) \tag{2.18}$$

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[2] Backtesting refers to "validating a given estimation procedure for a risk measure on historical data" (Ziegel, 2016, p.2)

In their paper on standard measures comparison, Emmer et al. (2015, p.36) gave some examples of strictly consistent scoring functions. Among them the weighted absolute error and the weighted squared error can be find, which are both of special interest since quantiles (value-at-risk) are elicited by the weighted absolute error, and expectiles by the weighted squared error (Newey & Powell, 1987).

$$\begin{aligned}
&s(x,l) = (x-l)^2, \text{ squared error}\\
&s(x,l) = \left(\mathbb{1}_{\{x\geq l\}} - \alpha\right)(x-l)^2 \operatorname{sgn}(x-l), 0<\alpha<1, \text{ weighted squared error}\\
&s(x,l) = |x-l|\,, \text{ absolute error}\\
&s(x,l) = \left|\mathbb{1}_{\{x\geq l\}} - \alpha\right| |x-l|\,, 0<\alpha<1, \text{ weighted absolute error}
\end{aligned}$$

Definition 10 (Elicitability)**.** A functional ν is said to be **elicitable** relative to the class $\mathscr{P}$ if a scoring function s that is strictly consistent for it exists - for ν relative to $\mathscr{P}$ - such that it minimises the expected value of the scoring function. The optimal forecast $\hat{x}$ for $\nu(P)$ is then given by

$$\underbrace{\nu(P)}_{=\hat{x}} = \underset{x\in R}{arg\,min}\, \mathbb{E}[s(x,L)] = \underset{x\in R}{arg\,min} \int_{\mathbb{R}} s(x,l)dF_L(l)$$

where L is a real-valued random variable with loss CDF F_L. (Gneiting, 2011, p.8; Emmer et al., 2015, p.36; Acerbi and Szekely, 2014, p.1; McNeil et al., 2015, p.356)

In other words, the "class of (strictly) consistent scoring functions for a functional is identical to the class of functions under which (only) the functional is an optimal point forecast" (Emmer et al., 2015, p.36)

The performance of "competing forecast procedures[3] for ν", which result in different sets of estimates, can then be ranked "by their average scores (. . .) using the scoring function s":

$$\bar{s}^{(k)} = \frac{1}{n}\sum_{i=1}^{n} s\left(x_i^{(k)}, y_i\right) \tag{2.19}$$

where $x_i^{(k)}$ are the n point forecasts, y_i are the realizing observations, and k is the number of competing forecast procedures (Ziegel, 2016, p.2–3).

However, not all functionals are such. Gneiting (2011) showed that the existence of convex level sets is a necessary condition for the elicitability of a risk measure.

Definition 11 (Convex level sets)**.** If a functional is elicitable then its level sets are convex in the following sense: If $P_0 \in \mathscr{P}, P_1 \in \mathscr{P}$ and $p \in (0,1)$ are such that $P_p = (1-p)P_0 + pP_1 \in \mathscr{P}$, then $\mathbf{t} \in T(P_0)$ and $\mathbf{t} \in T(P_1)$ imply $\mathbf{t} \in T(P_p)$. (Gneiting, 2011, p.752)

Quantile-based risk measures such as value-at-risk and expectiles are elicitable. Whereas expected shortfall does not have convex level sets, and is thence not elicitable relative to any class $\mathscr{P}$ of probability distributions. Note that the variance is not elicitable either.

While expected shortfall can be perceived as a better alternative to value-at-risk with regards to its coherence and its tail-sensitivity, its non-elicitability causes problems when

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[3] 'There are numerous choices concerning models, methods and parameters that have to be made to come up with predictions.' (Ziegel, 2016, p.2–3)

it comes to forecasting and backtesting. Indeed, lack of elicitability, hence the *apparent* "inability to perform forecast verification or comparison, has often been pointed out as a partial explanation for the difficulties with robust estimation and backtesting". (Ziegel, 2016)

However, in their paper entitled *Backtesting Expected Shortfall*, Acerbi and Szekely (2014, p.76) disprove the belief that the backtestability of expected shortfall was made impossible by its non-elicitability. This erroneous belief was notably fueled by a consultation paper from the Basel Committee published in 2013 in which expected shortfall was intended to replace value-at-risk for determining capital requirements, but VaR would go on being used as the measure of choice for backtesting.

One of the points raised in Acerbi and Szekely (2014)'s argument against this alleged impossibility to backtest expected shortfall is that elicitability has actually to do with "model selection and not with model testing, and is therefore irrelevant for the choice of a regulatory risk standard". Therefore, compliance (or not) with the elicitability condition should not interfere with the regulatory debate in any respect whatsoever. Following Acerbi and Szekely (2014)'s result that expected shortfall is jointly elicitable with value-at-risk, Ziegel (2016) also proposed the use of Diebold-Mariano tests for backtesting expected shortfall.

Other authors, especially Emmer et al. (2015), put forward the property of second-order elicitability, also referred to as conditional elicitability, to cope with the risk measures's deficiencies related to their non-elicitability.

2.6.1 Conditional elicitability

According to Emmer et al. (2015, p.37), some useful risk measures like value-at-risk and variance are not elicitable but rather *second-order* elicitable.

Definition 12. A functional ν of $\mathscr{P}$ is called conditionally elicitable if functionals $\tilde{\gamma}$ and $\gamma exist : \mathscr{D} \longrightarrow 2^{\mathbb{R}}$ with $\mathscr{D} \subset \mathscr{P} \times 2^{\mathbb{R}}$ such that

(i) $\tilde{\gamma}$ is elicitable relative to $\mathscr{P}$,

(ii) $(P, \tilde{\gamma}(P)) \in \mathscr{D}$ for all $P \in \mathscr{P}$,

(iii) for all $c \in \tilde{\gamma}(\mathscr{P})$, the functional $\gamma_c : \mathscr{P}_c \longrightarrow 2^{\mathbb{R}}, P \mapsto \gamma(P, c) \subset \mathbb{R}$ is elicitable relative to $\mathscr{P}_c = \{P \in \mathscr{P} : (P, c) \in \mathscr{D}\}$,

(iv) $\nu(P) = \gamma(P, \tilde{\gamma}(P))$ for all $P \in \mathscr{P}$

Forecasting is then possible for risk measures lacking elicitability, including expected shortfall, provided that they are *second-order* elicitable. Indeed, Emmer et al. (2015, p.37) explains that, "due to the elicitability of $\tilde{\gamma}$, we can first forecast $\tilde{\gamma}(P)$ and then, in a second step, take the result for $\tilde{\gamma}(P)$ as fixed and forecast $\gamma(P, c)$, due to the elicitability of γ_c."

Remark 4. Every elicitable functional is conditionally elicitable.

We refer to Acerbi and Szekely, 2014; Bellini and Bignozzi, 2015; Gneiting, 2011; Nolde and Ziegel, 2017 for further information about elicitability and its relevance in finance and insurance.

Chapter 3

Classical risk measures

The present chapter investigates whether the variance, value-at-risk, and expected shortfall meet the properties deemed as essential to any good risk measures in the previous chapter. Section 1 gives an overview of the (variance and standard deviation) premium principles. The next two sections analyse value-at-risk and expected shortfall as the two leading risk measures in financial and insurance regulations.

3.1 Premium principles

Some of the early risk measures in actuarial science were built on the development of premium principles (Hardy, 2006). As their name suggests, these premium principles are used for determining the insurance premiums, that is, the price reflecting the dangerousness of the risk insured borne by the insurer. In other words, insurance premiums are measures of the risk insured i.e. of the potential insured claims. Among those, we reckon the standard deviation (volatility) and variance premium principles.

Definition 13 (Variance and standard deviation principles)**.** Let $\mathbb{V}ar[L]$ denote the variance of the loss random variable L and $\mathrm{sd}[L] = \sqrt{\mathbb{V}ar[L]}$ its volatility. The risk measures based on the variance and standard deviation principles are, for some constant $\alpha \geq 0$, respectively:

$$\varrho(L) = \mathbb{E}[L] + \alpha \mathbb{V}ar[L]$$
$$\varrho(L) = \mathbb{E}[L] + \alpha \ \mathrm{sd}[L]$$

Provided that $\alpha > 0$, both risk measures clearly generate a premium which is bigger than the expected loss. The premium loading is directly related to the variability of the loss (its variance or volatility) and thereupon provides a buffer against adverse effects whose total loss would exceed the expected value $\mathbb{E}[L]$. (Hardy, 2006)

The appeal for these variance based principles can certainly be attributed to the modern portfolio theory of Markowitz (1952). This mathematical framework was considered as a breakthrough in risk related fields and played a significant role in the extensive use of variance as a risk measurement tool.

Definition 14 (Variance)**.** The variance of a random variable L, illustrated in figure 3.1.1, is defined as the expected value of the squared deviation from its mean $\mathbb{E}[L]$:

$$\mathbb{V}ar[L] = \mathbb{E}\left[(L - \mathbb{E}[L])^2\right] \tag{3.1}$$

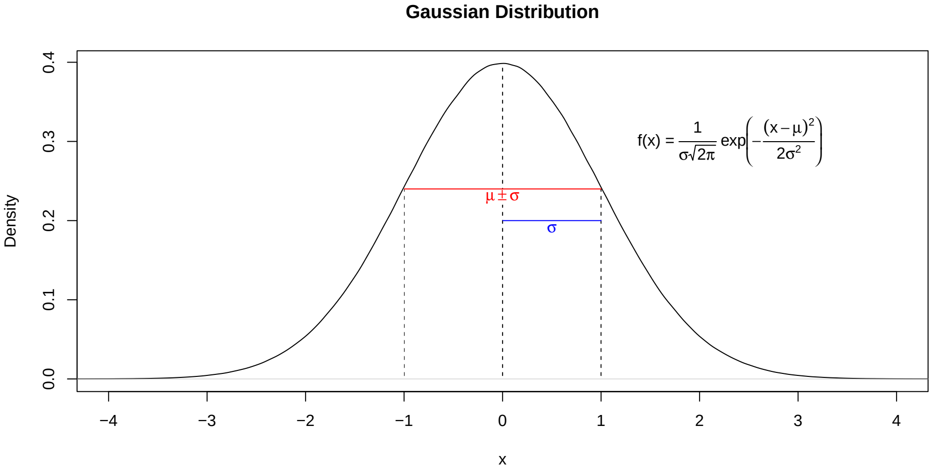

Figure 3.1.1: Standard deviation of standard normal distribution

3.1.1 Properties

By definition, the variance measures variability by squaring the deviation of each observation from the mean, and by weighting each of those squared values by its respective probability of occurrence. The smaller the variance, the closer the actual data points are distributed around the expected value of all possible outcomes. A low (high) variance thence indicates a lower (higher) degree of risk. This definition of the variance of L as the average dispersion around its expected value sheds light on some of its flaws as a risk measure candidate.

First, variance falls shorts of our definition of a risk measure as it is expressed in squared units (e.g. if the outcomes are measured in euros, then the variance is expressed in euros-squared). Hence the appeal of standard deviation (volatility) to revert to the units of the original observations.

Second, it makes no distinction between positive and negative deviations from the mean, and thus attributes the same weight to large gains or losses. While this makes variance a good risk measure for distributions that are (approximately) symmetric around the mean (e.g. the normal distribution, or a Student-t distribution with finite variance), it is no longer appropriate when one deals with right-skewed distributions (McNeil et al., 2015), as is often the case in practice. Indeed, since large negative co-movements are more likely in reality, variance would systematically underestimates such risk.

We could also argue that variance focuses only on one aspect of the notion of risk, that is the deviation from the mean, and does not pay attention to tail-risk measurement contrary to measures like expected shortfall (Righi, 2019). Variance then fails to account for tail-risk of fat tailed distribution, while any risk analyst's main concern is precisely to be able to quantify the upper right-tail of the distribution (where the large losses are).

The presence of outliers is another pitfall to variance. Since outliers are by definition far from the mean, they will weigh more heavily and could skew the data.

On the theoretical side, working with variance also requires that the second moment of the loss distribution is finite. Certain areas of non-life insurance could theoretically breach such requirement.

Some authors attempted to combine both concepts in alternative risk measures using semi-deviations (see Fischer (2003) and Chen and Wang (2008)) or by extending the concept of expected shortfall and adding penalties (see Righi and Ceretta (2016) and Furùan et al. (2017)). However, those moment based risk measures did not give rise to a general approach, as they are not monotone and thus cannot be used to make a sound comparison between the riskiness of different insurance contracts (see section 3.1.1). We refer to (Righi, 2019) for further details.

Besides that, variance also shows some serious shortcomings with regard to the axiomatic defined in chapter 2. Although variance does belong to the class of law invariant risk measures, it is neither positive homogeneous nor subadditive nor translation invariant as demonstrated below. It also lacks elicitability, as showed by Lambert et al. (2008).

Positive homogeneity

Assume a constant a so that $L = aX$. From the properties of variance (not linear), it follows that $\mathbb{V}[L] = a^2\mathbb{V}ar[X] \neq a\mathbb{V}ar[X]$ which is not consistent with the positive homogeneity axiom. Whereas by definition, the standard deviation of L is given by the square root of its variance, hence $\mathrm{sd}[aX] = \sqrt{\mathbb{V}ar[aX]} = \sqrt{a^2\mathbb{V}ar[X]} = a\ \mathrm{sd}[X]$. This shows that variance does not satisfy the positive homogeneity axiom 6, while standard deviation does. Similarly, the variance principle also fails positive homogeneity, while standard principle satisfies it:

$$\begin{aligned}\varrho(c\,L) &= E[c\,L] + \alpha\mathbb{V}ar[c\,L]\\ &= c\,E[L] + \alpha\,c^2\,\mathbb{V}ar[L]\\ &\neq c\,\varrho(L) = c\,E[L] + \alpha c\mathbb{V}ar[L]\end{aligned}$$

$$\begin{aligned}\varrho(c\,L) &= E[c\,L] + \alpha\sqrt{\mathbb{V}ar[c\,L]}\\ &= c\,E[L] + \alpha\,\sqrt{c^2\,\mathbb{V}ar[L]}\\ &= c\,E[L] + \alpha\,c\,\mathrm{sd}[L] = c\,\varrho(L)\end{aligned}$$

Translation invariance (consistency)

Based on the properties of variance, adding a scalar c to a loss random variable L does not add randomness i.e. $\mathbb{V}ar[L+c] = \mathbb{V}ar[L]$ and $sd[L+c] = sd[L]$. The null variance of the constant c thence proves the variance and standard deviation are not translation invariant either. On the other hand however, consistency is satisfied by both the variance and standard deviation principles:

$$\begin{aligned}\varrho(L+c) &= E[L+c] + \alpha\mathbb{V}ar[L+c]\\ &= E[L] + c + \alpha\mathbb{V}ar[L]\\ &= E[L] + \alpha\mathbb{V}ar[L] + c\\ &= \varrho(L) + c\end{aligned}$$

$$\begin{aligned}\varrho(L+c) &= E[L+c] + \alpha\sqrt{\mathbb{V}ar[L+c]}\\ &= E[L] + c + \alpha\sqrt{\mathbb{V}ar[L]}\\ &= \varrho(L) + c\end{aligned}$$

Monotonicity

Assume two correlated loss random variables L_1 and L_2 with correlation $\rho_{1,2} \neq \pm 1$. Note that this condition on $\rho_{1,2}$ ensures we will not encounter comonotonicity ($\rho_{1,2} = 1$) nor countercomonotonicity ($\rho_{1,2} = -1$). If L_1 is considered riskier than L_2 so that $L_1 > L_2$ and $L = L_1 - L_2 > 0$, it follows that $\mathbb{E}[L] = \mathbb{E}[L_1] - \mathbb{E}[L_2] > 0$. The variance of L is then given by

$$\mathbb{V}ar[L] = \mathbb{E}[(L - \mathbb{E}[L])(L - \mathbb{E}[L])] = \mathbb{V}ar[L_1] + \mathbb{V}ar[L_2] + 2\mathbb{C}ov(L_1, L_2) > 0 \tag{3.2}$$

or equivalently $\text{sd}[L_1] > \text{sd}[L_2]$. In other words, L_1 is more volatile than L_2 and both variance and standard deviation comply with the monotonicity axiom *provided that* $\rho_{1,2} \neq \pm 1$. Nevertheless, in general, we cannot ensure that $\mathbb{V}ar[L_1] > \mathbb{V}ar[L_2]$ since, by definition, the variance simply measures the average dispersion around the mean.

Moreover, we can also show that neither the standard deviation principle, nor the variance premium principle satisfy the monotonicity axiom. As a matter of fact, we can always find some random variables L_1 and L_2 so that $L_1 \leq L_2$ but $\varrho(L_1) > \varrho(L_2)$. For example, assume

$$L_1 = \begin{cases} 10 \text{ with probability } 0.5 \\ 90 \text{ with probability } 0.5 \end{cases} \qquad L_2 = 100 \text{ with probability } 1 \tag{3.3}$$

Clearly, $L_1 < L_2$. However, we can show that $\varrho(L_1) > \varrho(L_2)$ for both the variance and standard deviation premium principles:

$$\begin{aligned} \mathbb{E}[L_1] &= 0.5 \times 10 + 0.5 \times 90 = 50 \\ \mathbb{E}[L_1^2] &= 0.5 \times 10^2 + 0.5 \times 90^2 = 4100 \\ \mathbb{V}ar[L_1] &= 4100 - 50^2 = 1600 \\ sd[L_1] &= \sqrt{1600} = 40 \end{aligned}$$

Then, for $\alpha = 1.4$, we have

$$\begin{aligned} E[L_1] + \alpha \mathbb{V}ar[L_1] &= 50 + 1.4 \times 1600 = 2290 > E[L_2] + \alpha \mathbb{V}ar[L_2] = 100 \\ E[L_1] + \alpha\, sd[L_1] &= 50 + 1.4 \times 40 = 106 > E[L_2] + \alpha sd[L_2] = 100 \end{aligned}$$

Subadditivity

From the properties of variance, we know that

$$\mathbb{V}ar[L_1 + L_2] = \mathbb{V}ar[L_1] + \mathbb{V}ar[L_2] + 2\rho(L_1, L_2)\text{sd}[L_1]\text{sd}[L_2] \tag{3.4}$$

where $\rho(L_1, L_2)$ denotes the Pearson's correlation between L_1 and L_2. The subadditivity of the variance depends on the correlation coefficient. Unless L_1 and L_2 are linearly independent or negatively correlated with $\rho(L_1, L_2) \leq 0$, variance is not subadditive. Indeed, when $\rho(L_1, L_2) > 0$, the variance of the sum is always bigger than the sum of the variances, therefore $\mathbb{V}ar[L_1 + L_2] > \mathbb{V}ar[L_1] + \mathbb{V}ar[L_2]$. Similarly, we can show that variance does not respect the property of comonotonic additivity either. When $\rho(L_1, L_2) = 1$, $\mathbb{V}ar[L_1 + L_2] = \mathbb{V}ar[L_1] + \mathbb{V}ar[L_2] + 2\text{sd}[L_1]\text{sd}[L_2] \neq \mathbb{V}ar[L_1] + \mathbb{V}ar[L_2]$. The variance is actually super-additive.

It can also be demonstrated that combining risks that are positively correlated does not bring any risk reduction benefit when the risk measure is based on the variance principle:

$$\begin{aligned}\varrho(L_1+L_2) &= \mathbb{E}[L_1+L_2] + \alpha\mathbb{V}ar[L_1+L_2]\\ &= \mathbb{E}[L_1]+\mathbb{E}[L_2]+\alpha\left(\mathbb{V}ar[L_1]+\mathbb{V}ar[L_2]+2\mathbb{C}ov[L_1,L_2]\right)\\ &= (\mathbb{E}[L_1]+\alpha\mathbb{V}ar[L_1]) + (\mathbb{E}[L_2]+\alpha\mathbb{V}ar[L_2]) + 2\alpha\mathbb{C}ov[L_1,L_2]\\ &= \varrho(L_1)+\varrho(L_2)+2\alpha\mathbb{C}ov[L_1,L_2]\end{aligned}$$

When the risks are positively correlated $\rho(L_1, L_2) > 0$, the risk measure based on the variance principle is also super-additive.

Based on the equation (3.4), we can prove that, contrary to variance, standard deviation is always subadditive. Since $\rho(L_1, L_2) \in [-1, 1]$, $\mathbb{V}ar[L_1 + L_2]$ reaches its maximum when L_1 and L_2 are perfectly positively linearly dependent. We can then write the following equality based on equation (3.4) with $\rho(L_1, L_2) = 1$

$$\mathbb{V}ar[L_1+L_2] \leq \mathbb{V}ar[L_1]+\mathbb{V}ar[L_2]+2\mathrm{sd}[L_1]\mathrm{sd}[L_2]$$

Which can be rewritten as

$$\mathbb{V}ar[L_1+L_2] \leq \left(\mathrm{sd}[L_1]+\mathrm{sd}[L_2]\right)^2$$

By taking the square root on both sides, we get

$$\mathrm{sd}[L_1+L_2] \leq \mathrm{sd}[L_1]+\mathrm{sd}[L_2] \tag{3.5}$$

This proves the subadditivity of standard deviation. Similarly, we can prove risk measures based on standard deviation principle are also subadditive.

$$\begin{aligned}\varrho(L_1+L_2) &= \mathbb{E}[L_1+L_2] + \alpha\,\mathrm{sd}[L_1+L_2]\\ &= \mathbb{E}[L_1+L_2] + \alpha\sqrt{\mathbb{V}ar[L_1+L_2]}\\ &\leq \mathbb{E}[L_1]+\mathbb{E}[L_2]+\alpha\left(\mathrm{sd}[L_1]+\mathrm{sd}[L_2]\right) \text{ from equation 3.5}\\ &= \varrho(L_1)+\varrho(L_2)\end{aligned}$$

Overall, variance lacks (too) many desirable properties to be considered as a good risk measure. More developed risk measures, like value-at-risk and expected shortfall address most of the variance shortcomings and are widely used for determining economic (and regulatory) capital.

3.2 Value-at-Risk

This section studies the value-at-risk as the regulatory risk measure for capital requirements under Solvency II. Its failure to satisfy the coherence axiomatic and its inability to capture the tail-risk beyond the specified confidence level α (typically 99.5%) will be extensively discussed.

3.2.1 Motivation and definition

The benchmark risk measure value-at-risk (VaR) is based on a simple, yet interesting, quantile approach. The appeal for quantile-based approaches may be partially attributed to the simplicity of their interpretation. As a matter of fact, the quantile measure VaR summarises in a single number the minimum loss we can expect with a probabilistic threshold $1-\alpha$ (generally 1% or 0.5%) within a given time horizon (e.g. 1-day VaR). In broader terms, VaR at confidence level α represents the "maximum loss in all but $1-\alpha$% of cases" (Osband, 2011, p. 73), that is, the loss that will not be exceeded with probability $1-\alpha$.

Definition 15 (Value-at-risk)**.** Let F_L be the cumulative distribution function of a risk L, so that $F_L(\ell) = P(L \leq \ell)$. Given some confidence level $\alpha \in [0,1]$, the corresponding VaR of the real-valued random variable L, denoted by $VaR_\alpha(L)$, is defined as the α-quantile of L, that is, the smallest real number ℓ such that the probability that the loss L exceeds the given value ℓ is no larger than $1-\alpha$. (Embrechts et al., 2005)

$$VaR_\alpha(L) = q_\alpha = \inf\{\ell \in \mathbb{R} : P(L > \ell) \leq 1-\alpha\} = \inf\{\ell \in \mathbb{R} : F_L(\ell) \geq \alpha\} \tag{3.6}$$

Remark 5. From the above definition, it is manifest that VaR is a law invariant risk measure as its estimation solely requires the loss distribution.

Example 3.2.1. Assume the following discrete loss random variable:

$$L = \begin{cases} 0 \text{ with probability } 0.8 \\ 4 \text{ with probability } 0.13 \\ 20 \text{ with probability } 0.05 \\ 50 \text{ with probability } 0.02 \end{cases} \tag{3.7}$$

The 90% and 95% quantile risk measures for this discrete loss distribution are 4 and 20 respectively as illustrated in figure 3.2.1.

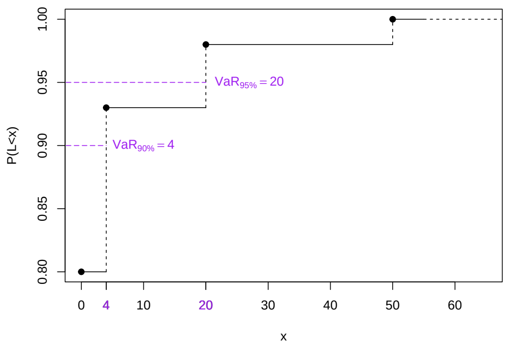

Figure 3.2.1: Cumulative probability distribution of the discreet random variable L

For a random variable L with a continuous and strictly increasing distribution F, the equation 3.6 merely becomes the inverse cumulative distribution function $VaR_\alpha(L) = F^{-1}(\alpha)$ where $F^{-1}(\alpha) = q_\alpha(F)$, that is, the quantile function of the loss distribution L.

The VaR at confidence level $\alpha = 95\%$ is illustrated in figure 3.2.2 with respect to four continuous loss distributions (standard normal, log-normal, uniform, and Pareto) established by Monte-Carlo simulation. As explained above, the continuity of these distributions means the value-at-risk is simply given by the α-quantile.

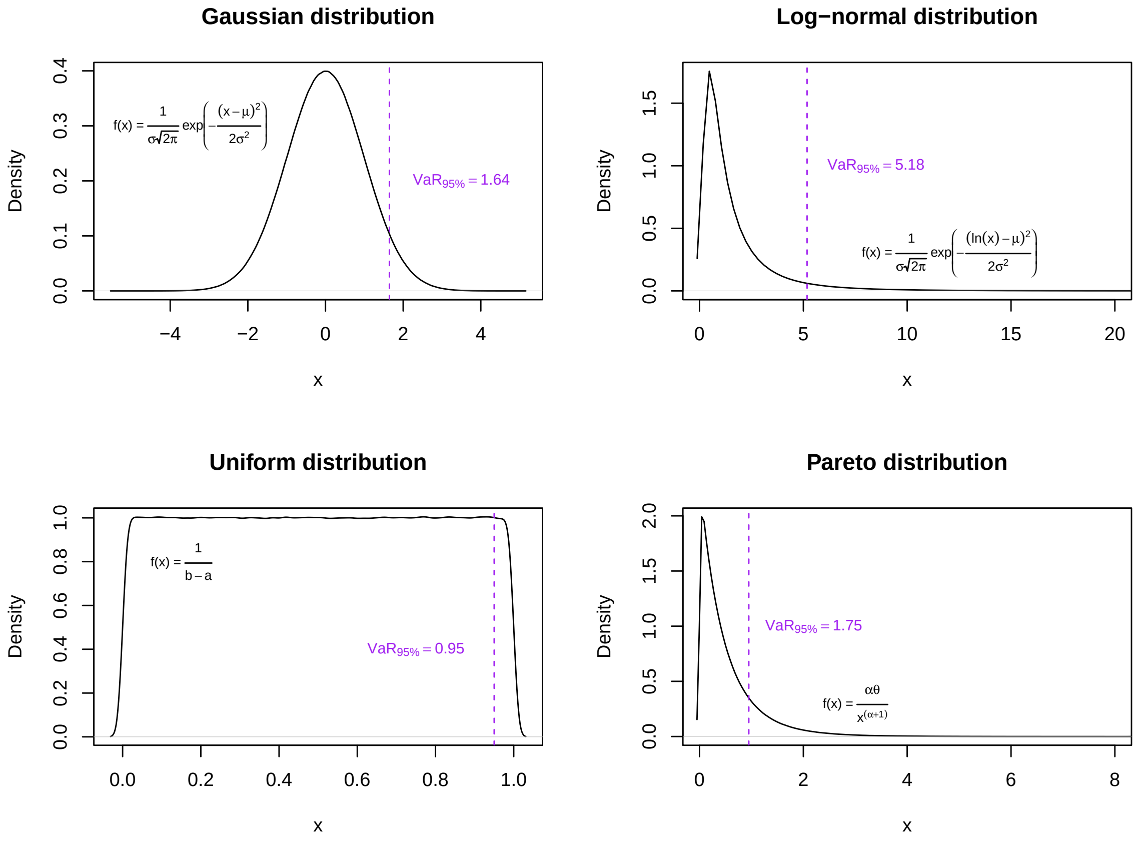

Figure 3.2.2: VaR at level 95% for the continuous random variables $W \sim \mathcal{N}(0,1)$, $X \sim log\mathcal{N}(0,1)$, $Y \sim \mathcal{U}(0,1)$ and $Z \sim \mathcal{P}(10,5)$

3.2.2 Properties

Following the 1996 amendment to incorporate market risk to the Basel I Accord, VaR at 99 percentile became the prime risk measurement tool in banking regulations and internal risk management systems (e.g. to rank investment choices or impose limits on traders). In this sense, regulatory committees used the measure to determine capital requirements against market risk exposure. In Solvency II, VaR at 99.5 percentile is also prescribed for measuring capital requirements to ensure insurance companies can still remain solvent in a catastrophic scenario. While VaR has also made its way into the Basel III framework and is still, to this day, the regulatory measure of choice in Solvency II, it has often been criticised for its lack of subadditivity and its failure to account for the severity of losses in the far tail of the distribution.

In his book *Pandora's Risk*, Osband (2011, p. 73) goes as far as to say "using standard VaR to tame financial risk is like using cigarette filters to tame cancer risk." In the following section, we seek to explain the implications of its drawbacks and shed lights on some misinterpretations.

Elicitability

As already mentioned in the section 2.6 introducing the notion of elicitability, quantiles (value-at-risk) are elicited by the weighted absolute error.

Proposition 3.2.2. *The statistical functional $VaR_\alpha(L) = F_L^{\leftarrow}(\alpha)$ is elicitable on the set of strictly increasing distribution functions with finite mean. The scoring function that is the weighted absolute error*

$$s(x,l) = \left|\mathbb{1}_{\{x\geq l\}} - \alpha\right| |x-l|\,, \text{ for any } 0<\alpha<1$$

is strictly consistent for the generalized inverse at α. (McNeil et al., 2015, p.357)

In their reference book on quantitative risk management, McNeil et al. (2015, p.357) proved the weighted absolute error is indeed strictly consistent for VaR and can be minimised by taking the derivative of its expected value.

Proof 1. For any real-valued random variable L with distribution function F_L, the expected score $\mathbb{E}\left[s(x,L)\right]$ is a continuous function that is differentiable at all points of continuity of F_L. The derivative is

$$\begin{aligned}
\mathbb{E}\left[s(x,L)\right] &= \frac{d}{dx}\int_{-\infty}^{+\infty} \left|\mathbb{1}_{\{x\geq l\}} - \alpha\right| |x-l|\, dF_L(l) \\
&= \frac{d}{dx}\int_{-\infty}^{x} (1-\alpha)\,(x-l)\, dF_L(l) + \frac{d}{dx}\int_{x}^{+\infty} \alpha\;(x-l)\, dF_L(l) \\
&= (1-\alpha)\int_{-\infty}^{x} dF_L(l) - \alpha\int_{x}^{+\infty} dF_L(l) \\
&= (1-\alpha)\left(F_L(x) - \underbrace{F_L(-\infty)}_{=0}\right) - \alpha\left(\underbrace{F_L(+\infty)}_{=1} - F_L(x)\right) \\
&= F_L(x) - \alpha
\end{aligned}$$

There are two cases to consider. If a point x exists such that $F_L(x) = \alpha$, then $x = F_L^{\leftarrow}(\alpha) = VaR_\alpha(L)$ and x clearly minimises $\mathbb{E}\left[s(x,L)\right]$. If, on the other hand, the set $\{x : F_L(x) = \alpha\}$ is empty, then it must be the case that there is a point x at which the distribution function F_L jumps and $F_L(y) - \alpha < 0$ for $l < x$ and $F_L(y) - \alpha > 0$ for $l > x$. It follows again that $x = F_L^{\leftarrow}(\alpha)$ and that x is the unique minimizer of $\mathbb{E}\left[s(x,L)\right]$ (McNeil et al., 2015, p.357).

Comonotonic additivity

The quantile risk measure VaR is additive for sums of comonotonic risks.

Proof 2. It is well know from the probability integral transform that any continuous random variable L is equal in distribution to its quantile function applied to a standard uniformly distributed random variable $U \sim \mathcal{U}(0,1)$:

$$F_L \overset{d}{=} F_L^{-1} \circ U$$

Assume - for simplicity - only two comonotonic continuous random variables L_1^c and L_2^c with marginal distribution functions F_1 and F_2. For some $U \sim \mathcal{U}(0,1)$, we have:

$$\begin{aligned}(U_1, U_2) &\stackrel{d}{=} (F_1(L_1), F_2(L_2)) \\ (L_1, L_2) &\stackrel{d}{=} \left(F_1^{-1}(U), F_2^{-1}(U)\right)\end{aligned}$$

Let $T(U)$ denote an increasing left-continuous function such that

$$T(U) = L_1 + L_2 = F_1^{-1}(U) + F_2^{-1}(U)$$

The comonotone additivity of quantiles follows:

$$\begin{aligned}\varrho\,(L_1 + L_2) &= VaR_\alpha(L_1 + L_2) = F_{L_1+L_2}^{-1}(\alpha) = F_{T(U)}^{-1}(\alpha) = T(U) \circ F^{-1}(\alpha) = T\left(F_U^{-1}(\alpha)\right) \\ &= T(\alpha) = F_{L_1}^{-1}(\alpha) + F_{L_2}^{-1}(\alpha) = VaR_\alpha(L_1) + VaR_\alpha(L_1) \\ &= \varrho\,(L_1) + \varrho\,(L_2)\end{aligned}$$

This proves the $\alpha-$quantiles of the sum $L_1^c + L_2^c$ “is equal to the sum of the” $\alpha-$quantiles of L_1^c and L_2^c (McNeil et al., 2015, p.258). The quantile risk measure VaR is henceforth additive for sums of comonotonic risks.

Not coherent

Value-at-risk meets three of the four axioms of coherence developed in section 2.1. While it enjoys the properties of monotonicity, translation invariance, and positive homogeneity, it infringes the subadditivity axiom.

Axiom 3 (TI):

$$\begin{aligned}\varrho(L + c) = VaR_\alpha(L + c) &= \inf\{\ell : P(L + c \leq \ell) \geq \alpha\} \\ &= c + \inf\{\ell - c : P(L \leq \ell - c) \geq \alpha\} \\ &= c + VaR_\alpha(L) = \varrho(L) + c\end{aligned}$$

Axiom 4 (M): For any loss random variables $L_1 \leq L_2$ or equivalently $P(L_1 > \ell) \leq P(L_2 > \ell)$,

$$\begin{aligned}\inf\{\ell : P(L_1 > \ell) \leq 1 - \alpha\} &\leq \inf\{\ell : P(L_2 > \ell) \leq 1 - \alpha \\ &\Leftrightarrow\ VaR_\alpha(L_1)\&leq VaR_\alpha(L_2)\end{aligned}$$

Axiom 6 (PH):

$$\begin{aligned}\varrho(\lambda L) = VaR_\alpha(\lambda L) &= \inf\{\ell : P(\lambda L \leq \ell) \geq \alpha\} \\ &= \lambda \inf\left\{\frac{\ell}{\lambda} : P\left(L \leq \frac{\ell}{\lambda}\right) \geq \alpha\right\} \\ &= \lambda VaR_\alpha(L) = \lambda\varrho(L)\end{aligned}$$

Axiom 5 (S): The following example shows VaR’s failure to fulfill the subadditivity axiom.

Example 3.2.3. Assume two iid discrete random variables L_1 and L_2, with

$$L_1 \overset{\text{d}}{=} L_2 = \begin{cases} 0 \text{ with probability } p \\ 100 \text{ with probability } 1-p \end{cases}$$

Given X and Y are independent, the aggregate $X+Y$ is given by,

$$L_1 + L_2 = \begin{cases} 0 \text{ with probability } p^2 \\ 100 \text{ with probability } 2p(1-p) \\ 200 \text{ with probability } (1-p)^2 \end{cases}$$

With $p = 0.96$ and $\alpha = 0.95$, we find $VaR_\alpha(L_1) + VaR_\alpha(L_2) = 0 < VaR_\alpha(L_1 + L_2) = 100$

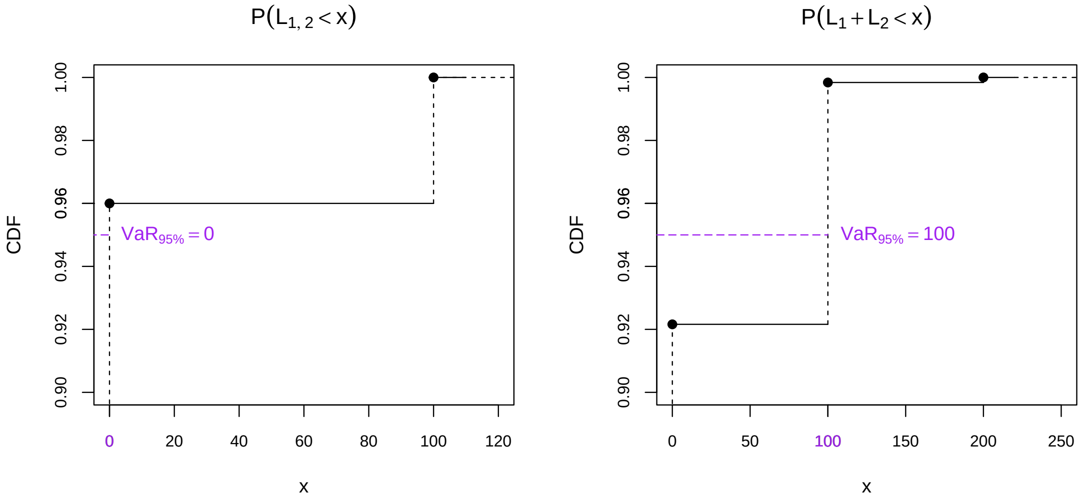

Figure 3.2.3

Non-subadditivity brings about the unintended property that the VaR of a sum (the merged risks) $q_\alpha(L_1 + L_2)$ can be greater than the sum of VaRs of the individual risks $q_\alpha(L_1) + q_\alpha(L_2)$. The failure of VaR “to properly measure the benefits of diversification” means “diversification will” deceptively “lead to more risk being reported” (Chen, 2018, p. 4).

Remark 6. Because of its non-subadditivity, VaR also flunks the convexity axiom 7.

The ongoing debate surrounding the practical relevance of the non-subadditivity of VaR has resulted in a host of comprehensive research papers including numerous examples in which value-at-risk fails to comply with the subadditivity axiom 5 (see Danielsson et al. (2005); McNeil et al. (2015) Examples 2.25, 7.30, 8.39-40; Embrechts et al. (2002) Examples 6-7), as well as its implications for portfolio optimisation (see Danielsson et al., 2005; Danielsson et al., 2012; Denuit et al., 2006). The points standing out from the literature are the following:

First, value-at-risk is known for its lack of subadditivity (axiom 5), except for elliptically distributed risk factors. In fact, assume a multivariate normal n-dimensional random vector $Z = (Z_1, \dots, Z_n)$. Any linear combination of the variables is then also normally distributed and the subadditivity $VaR_\alpha(L_1 + L_2) \leq VaR_\alpha(L_1) + VaR_\alpha(L_2)$ holds for any linear portfolio $L_1 = \sum_{i=1}^n a_i Z_i$ and $L_2 = \sum_{i=1}^n b_i Z_i$ (see Embrechts et al. (2005)).

However, it is well known, at least since Mandelbrot (1963) and Fama (1965), asset returns exhibit fat tails. The distributions of most insurance portfolios also showcase right heavy-tails.

Among others, Ibragimov (2005), Danielsson et al. (2005), Garcia et al. (2007) as well as Marshall et al. (2011) demonstrated VaR is still subadditive in the tails for such heavy fat–tailed (and dependent) distributions provided that the probability levels are sufficiently low and the tails are not too fat.

Although we may see violations of VaR subadditivity for probability levels in the interior of the distributions, risk analysts generally focus on the upper tail probabilities - especially when performing stress tests. As a consequence, worries about non-subadditivity in the center of the loss distributions is not likely to be a good enough reason to favour another risk measure over VaR in risk management applications. The condition on sufficiently low probability levels is thus not binding as these levels are actually most relevant for practical applications.

Besides, not too fat tails distribution refer within this framework to risk factors having jointly regularly varying non-degenerate tails (thus allowing for dependence - see Danıelsson et al. (2012) for an explicit example of interdependent asset returns) with tail index α larger than one such that the mean in finite. Such distribution "approximately follow a multivariate power law" like the Pareto or Cauchy distributions (Danıelsson et al., 2012, p.284).

The subadditivity of VaR in "the tails of all fat tailed distributions, provided the tails are not" so fat that the first moment is not defined "further implies that diversification will not work for super fat tails" with tail index $\alpha < 1$ as "already established by Fama and Miller (1972, page 270)" (Danielsson et al., 2005, p.8).

"Data falling into this category would" most likely "be characterized by a large number of very small outcomes interdispersed with very large outcomes. While such assets do exist" and are easy to identify, "they are hard to find. Such anomalous cases (...) require special treatment in risk models" as "subadditivity violations" in such cases "are likely to be a matter of serious concern." (Danielsson et al., 2005, p.3–8)

Following their evidence that VaR is subadditive "for most practical applications", Danielsson et al. (2005, p.1) claimed "there is no reason to" consider other "risk measure than VaR, solely for reasons of" coherence.

Moreover, while *unrecognised* violations of subadditivity can still cause a number of (serious) problems discussed in axiom 5 for financial institutions, it was also pointed out that a merger may actually increase risk - which runs counter to the principle of subbaditivity argued by Artzner et al. In this instance, we explained it was preferable to keep the risks apart to confine the enormous losses - to prevent dragging along the merged portfolio towards bankruptcy. According to Heyde et al. (2008, p.7), "the collapse of Britain's Barings Bank in 1995 due to the" actions "of a single trader clearly indicates that merger may increase risk. Had Barings Bank set up separate" subsidiaries, the bankruptcy in "its Singapore unit (...) would not have sink the entire bank". Further arguments against subadditivity will be put forward whilst analysing the coherent expected shortfall measure.

Tail sensitivity

Its violation of the subadditivity axiom 5 is not the only theoretical deficiency of value-at-risk as a risk management tool. As mentioned in its very definition, VaR is a quantile of a loss distribution. This implies VaR does not provide any information about the potential severity of losses, reflected by the thickness of the upper right-tail of the loss distribution. It rather acts as “an indicator linked to the probability of ruin”, that is, how often “losses greater than the VaR” (Devolder, 2021) can be expected. Simply put, VaR is not an *expected* value but a *frequency* measure giving a probability of occurrence of heavy losses. This is considered as a major weakness since regulators, shareholders and management should be at least as much concerned with the size of losses beyond the confidence level α (*how bad is bad ?*) as with its frequency (Denuit et al., 2006).

The figure 3.2.4 illustrates such a situation where VaR fails to identify the thickness of the tail beyond the probabilistic threshold of 5% - as it depends only on the frequency of the extreme events, and not their magnitude. The value of the 95% VaR is, for both distributions, equal to 0.0148. The equality in VaR despite the differences in the upper right-tails behaviour clearly illustrates the tail-insensitivity of VaR.

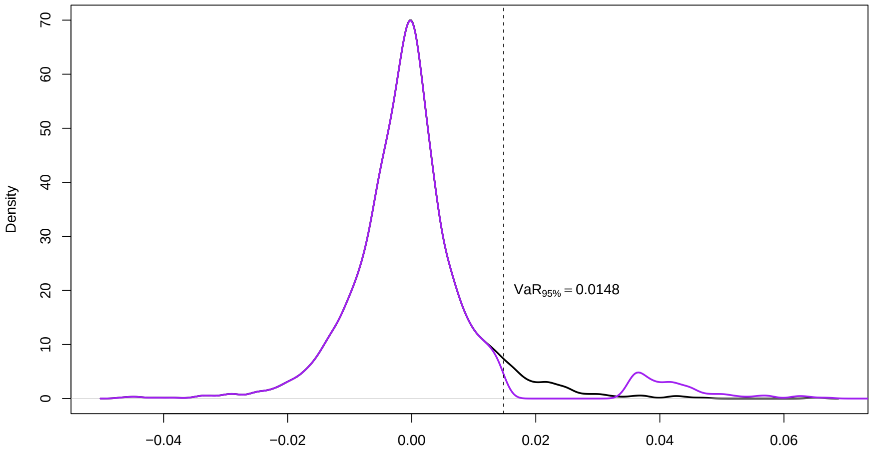

Figure 3.2.4: Tail-insensitivity of value-at-risk at a 95% confidence level

Ignoring the value-at-risk inability to account for the potential severity of tail-risk can lead to dangerous understandings of VaR numbers. The most common mistake is to understand the value given by a daily VaR at security level $\alpha = 99.5\%$ as: “the loss will be no more than this amount in the 0.5% worst cases”. This interpretation is all the more misleading given market liquidity and more broadly model risk. For, at least, both these reasons, the VaR number should be understood as the potential *minimum* loss in the $(1-\alpha)\%$ worst cases rather than a value we don’t expect to exceed. A financial analogy would consist in construing VaR as a floor rather than a cap. (Osband, 2011)

The first reason, market liquidity, was brought up by Lawrence and Robinson (1995). In a nutshell, a market is termed liquid when its assets can be quickly traded (bought or sold) without having to cut its price to make it attractive. Conversely, when there is no counterparty willing to take on the other side of the trade, or when selling or purchasing causes drastic changes in the asset’s current market price, the market is termed illiquid. Illiquidity of markets must be taken into account by including the costs of liquidation, which can be significant, to the VaR number, which is not an easy task. This difficulty comes from a number of reasons, including the time frame to close position and the uncertainty around some elusive market factors. For further literature on liquidity adjusted VaR, we refer to Angelidis and Benos

(2006) and Garleanu and Pedersen (2007).

The above interpretation of VaR also neglects any consequences of model risk, that is incurred losses due to models misspecification or because of the use of assumptions which did not came about in practice. For instance, working with a Gaussian copula market model to price CDOs (collateralised debt obligation) is a bad idea since it neglects any tail-risk. Problems arising from the failure to recognise - and model - the real distribution as heavy-tailed, or tail dependence or even the presence of volatility clustering are, of course, even more pronounced when one estimates VaR at a higher confidence level, such as $\alpha = 99.5\%$, as it is required for determining the capital requirements under Solvency II. Moreover it should be borne in mind that any risk-management model is by definition a simplified and thus flawed representation of the global economic system. This suggests model risk affects all models. The near-collapse of the hedge fund LTCM in 1998 is a testament to model risk in VaR-based risk-management systems (see Embrechts et al. (2005, p.78) and Jorion (2000) for a detailed explanation).

Overall, while the value-at-risk lack of subadditivity is debatable, its failure to quantify the upper tail-risk marks the harshest argument against VaR as a risk measure. While value-at-risk at 99.5% confidence level remains the Solvency II regulatory measure for determining the solvency capital requirements (SCR[1]) of insurance companies, Basel IV addressed the shortcomings of VaR by shifting the quantitative risk metrics system from value-at-risk to expected shortfall, and decreasing the confidence level from 99% to 97.5% (Basel Committee on Banking Supervision, 2014). This revised market risk framework is essentially a response to VaR's inability to capture tail risk. Remember that, by neglecting tail-risk, VaR disregards any loss beyond the VaR level. In their press release, the Basel Committee on Banking Supervision (2016) states that the use of ES is a mean "to ensure prudent capture of tail risk and so maintain capital adequacy during periods of significant market stress". According to Basel Committee on Banking Supervision (2014, p. 18), moving to a confidence level of 97.5% "will provide a broadly similar level of risk capture as the existing 99th percentile VaR threshold" as well as "a number of benefits, including (...) more stable model output and (...) less sensitivity to extreme" outliers. This prudential regulation gave rise to an increased use of ES, instead of VaR, in many institutions and, in particular, for capital allocation.

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[1] The SCR defines the capital to be hold by insurance companies to ascertain they are in capacity to pay out their policyholders' claims.

3.3 Expected Shortfall

This section investigates the law invariant expected shortfall as an alternative to value-at-risk. While expected shortfall appeared increasingly preferred by regulators - probably swayed by its theoretical advantages over VaR, its subadditivity and tail-sensitivity are also directly tied to one of its own theoretical flaws: its lack of robustness. We will further review its failure to be elicitable and related concerns involving its backtestability.

3.3.1 Motivation and definition

The expected shortfall (ES), also referred to as the tail-VaR (TVaR) for *continuous* distributions, is often deemed superior to value-at-risk as it addresses its drawbacks. First, because its adherence to all 4 axioms is verified, ES is coherent. Be that as it may, given Danielsson et al. (2005)'s evidence that VaR is subadditive in most actual applications, one could argue the relevancy of such an argument. Second, being an expectation, it provides information about the severity of losses beyond value-at-risk, contrary to the VaR itself. Yet, this sought out property will prove to hinder the robustness of expected shortfall (see section 3.3.2).

Definition 16 (Expected Shortfall)**.** Given a risk L with continuous distribution function F_L and inverse function $F_L^{-1}(\alpha) = q_\alpha(L)$, for some confidence level $\alpha \in$ (0,1) we have:

$$\mathrm{TVaR}_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 \underbrace{VaR_\xi(L)}_{=q_\alpha(L)}\, d\xi \tag{3.8}$$

Remark 7. ES can be defined more generally as a stop-loss premium with retention level equal to the VaR:

$$ES_\alpha(L) = \mathbb{E}\left[(L - VaR_\alpha(L))_+\right] \tag{3.9}$$

Formally, TVaR is closely related to VaR as it is defined as a weighted sum of all VaRs above the security level α. By taking on more information than just one VaR, we look further into the tail of the loss distribution and $ES_\alpha \geq VaR_\alpha$. The figure 3.3.1 illustrates the relationship between expected shortfall and VaR at a confidence level of 95%. While $VaR_{95\%}$ is the best possible case of the worst 5% losses, $ES_{95\%}$ can be interpreted as the average of the 5% worst losses. In other words, the value-at-risk does not give any indication of the size of the loss within the remaining $1-\alpha$%, unlike TVaR (Anderson, 2012).

Remark 8. For continuous loss distributions, the CTE and TVaR are equal (Gschöpf, 2014),

$$TVaR_\alpha = CTE_\alpha = \mathbb{E}\{L|L > \underbrace{VaR_\alpha}_{F^{-1}(\alpha)=q_\alpha}\},\ \alpha \in (0,1) \tag{3.10}$$

Expected shortfall can then be interpreted as the *expected* loss that is incurred conditional on the fact that we are in the tail of the distribution beyond the VaR, hence the more intuitive name *tail*-VaR. For management purposes, expected shortfall's interpretation might be less prone to misinterpretation than Value-at-Risk.

Remark 9. When L is not continuous, the $\alpha-$CTE is:

$$CTE_\alpha = \frac{(\bar{\alpha}-\alpha)\, VaR_\alpha + (1-\bar{\alpha})\, E\left(L|L > VaR_\alpha\right)}{1-\alpha} \tag{3.11}$$

where $\bar{\alpha} = P(L \leq VaR_\alpha)$.

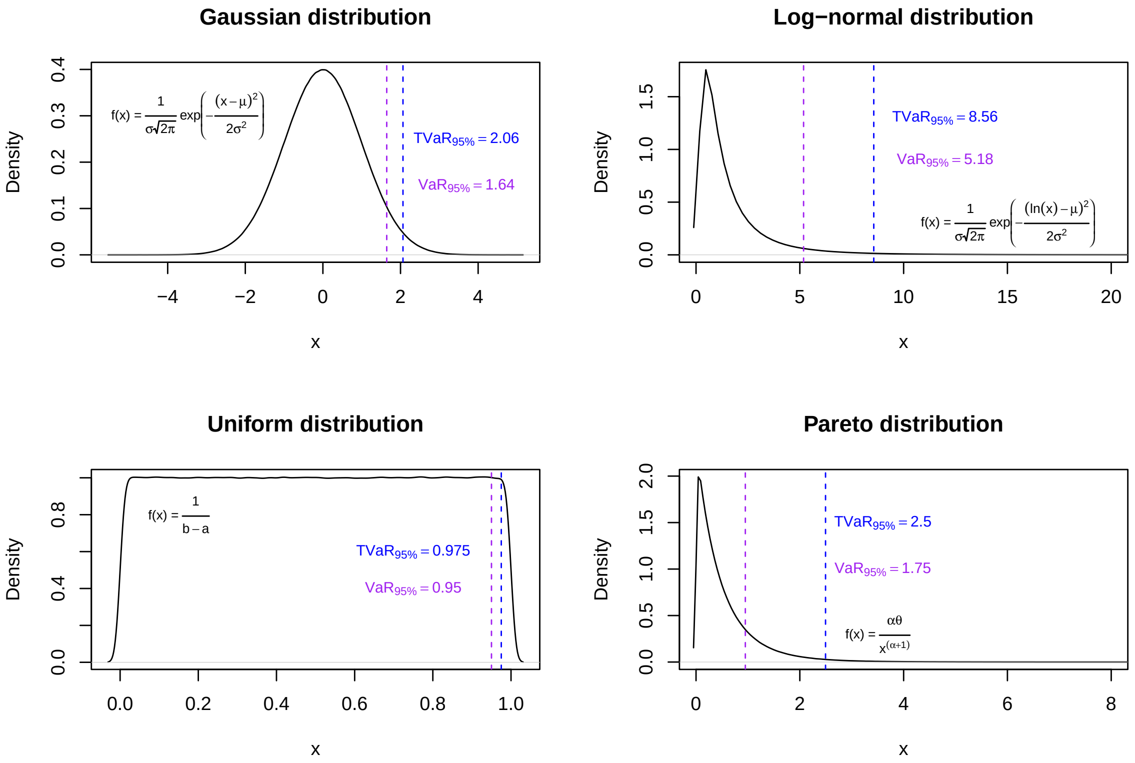

Figure 3.3.1: VaR and TVaR at level 95% for the continuous random variables $W \sim \mathcal{N}(0,1)$, $X \sim log\mathcal{N}(0,1)$, $Y \sim \mathcal{U}(0,1)$ and $Z \sim \mathcal{P}(10,5)$

Example 3.3.1. In our example 3.2.1 L was not continuous. For $\alpha = 0.95$, we thus have:

$$\begin{aligned}
TVaR_{0.95} &= \frac{1}{1-0.95}\int_{0.95}^{1} VaR_u du \\
&= \frac{1}{0.05}\left(\int_{0.95}^{0.98} 20\,du + \int_{0.98}^{1} 50\,du\right) \\
&= 32 > VaR_{0.95} = 20 \\
E\left(L|L > VaR_{0.95} = 20\right) &= \frac{50 \times 0.02}{1-0.98} = 50 \\
CTE_{0.95} &= \frac{20(0.98-0.95) + 50(1-0.98)}{1-0.95} = 32
\end{aligned} \tag{3.12}$$

3.3.2 Properties

Coherence

Contrary to value-at-risk, expected shortfall is a coherent risk measure *at all times*. Its translation invariance, monotonicity, comonotonic additivity, and positive homogeneity stem directly from the corresponding properties for quantiles.

Axiom 3 (TI):

$$\begin{aligned}\varrho(L+c) = \mathrm{TVaR}_\alpha(L+c) &= \frac{1}{1-\alpha}\int_\alpha^1 F_u^{-1}(L+c)du \\ &= \frac{1}{1-\alpha}\int_\alpha^1 F_u^{-1}(L)+c\,du \\ &= \mathrm{TVaR}_\alpha(L)+c = \varrho(L)+c\end{aligned}$$

Axiom 4 (M): For any loss random variables $L_1 \leq L_2$ and some $\alpha \in (0,1)$:

$$\begin{aligned}&VaR_\alpha(L_1) \leq VaR_\alpha(L_2) \\ \Rightarrow\ &TVaR_\alpha(L_1) \leq TVaR_\alpha(L_2)\end{aligned}$$

Axiom 5 (S): For any random variables L_1 and L_2 and some $\alpha \in (0,1)$, ES_α is subadditive:

$$ES_\alpha(L_1+L_2) \leq ES_\alpha(L_1) + ES_\alpha(L_2)$$

We refer to the paper by Embrechts and Wang (2015) where they developed seven proofs of the subadditivity of ES by which different properties of expected shortfall are brought to the forefront.

Axiom 6 (PH):

$$\begin{aligned}\varrho(\lambda L) = \mathrm{TVaR}_\alpha(\lambda L) &= \frac{1}{1-\alpha}\int_\alpha^1 F_u^{-1}(\lambda L)du \\ &= \frac{1}{1-\alpha}\int_\alpha^1 \lambda F_u^{-1}(L)du \\ &= \lambda\,\mathrm{TVaR}_\alpha(L) = \lambda\varrho(L)\end{aligned}$$

Remark 10. Given the tail-VaR is coherent, the convexity follows. For any (dependent) real-valued random variables L_1 and L_2, for some $\lambda \in (0,1)$ we have:

$$\begin{aligned}&TVaR_\alpha\left(\lambda L_1 + (1-\lambda)L_2\right) \\ &\leq \lambda TVaR_\alpha\left(L_1\right) + (1-\lambda)\, TVaR_\alpha\left(L_2\right)\end{aligned}$$

Superadditivity

In their paper where they explored the superadditivity of coherent risk measures, Dhaene et al. (2008) showed that the suggestion whereby a merger of two insurance portfolios L_1 and L_2 should decrease the shortfall of the portfolio, i.e. $(L_1+L_2-\varrho(L_1+L_2))^+ \leq (L_1-\varrho(L_1))^+ + (L_2-\varrho(L_2))^+$, is violated by coherent risk measures, including tail conditional expectation (CTE), as this condition actually implies superadditivity rather than subadditivity (Heyde et al., 2008, p.7). As a result, if one were to deem the intent of this condition as a requisite for a proper determination of the SCR, the 99.5% value-at-risk imposed by Solvency II should prevail over the 99% tail-VaR of the Swiss Solvency Test.

Tail-sensitivity and robustness

The tail-insensitivity of VaR was easily explained in section 3.2.2 by the fact that VaR is a quantile, and not a measure of tail expectation unlike ES. The shape of the tail beyond the confidence level had thus no leverage whatsoever on the VaR number. This observation was covered as the main drawback of VaR as a regulatory risk measure. However, another shortcoming arises when one deals with coherent risk measures, like expected shortfall, that measure by definition tail expectations (see figure 3.3.2). As foreseen in the section discussing robustness, coherent risk measures may indeed satisfy subadditivity, but at the expense of their robustness to outliers and underlying model assumptions.

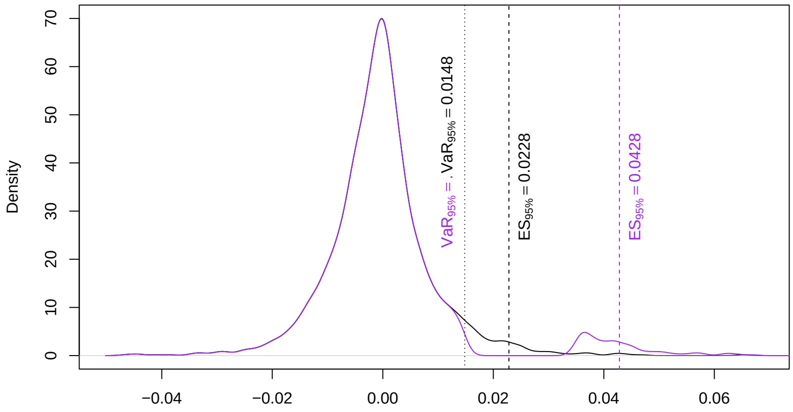

Figure 3.3.2: Tail-sensitivity of expected shortfall at a 95% confidence level.

As expected shortfall gives added weights on larger observations, the (subjective) assumptions made on the tail behaviour of the loss distribution (e.g. its heaviness) hinders the robustness of expected shortfall. All the more so as the behaviour of the far tail is harder to pin down from the data (smaller amount of data points in the far tails) (Heyde et al., 2008).

Besides this consideration, the tail-sensitivity of expected shortfall may also be problematic to the extent that a too conservative measurement of capital adequacy is not ideal either. It so happens that if the measure ϱ used to determine the total amount of capital available is too prudent, the institution will set aside an excess of cash. That is money that could be allocated more efficiently to other assets. Moreover, one can actually show value-at-risk is an optimal way to determine the solvency capital when the point of view of both the insurance company and the regulator are taken into consideration.

The regulator's main concern is the risk of insolvency of the insurance company, that is, the risk the potential aggregate loss L of the insurance company exceeds the total available capital $\varrho(L)$, i.e. the shortfall $\max(0, L - \varrho(L)) = (L - \varrho(L))^+$.

Remark 11. The total available capital $\varrho(L)$ is defined as the available capital V, plus the solvency buffer $\varrho(L) - V$ where V are the technical provisions available to cover future liabilities.

To quantify this risk of insolvency, the regulator applies a risk measure, denoted ϕ, on $(L - \varrho[L])^+$. The larger the total capital available $\varrho[L]$, the smaller the risk of insolvency, the happier the regulator is. Be that as it may, holding an extra amount of solvency capital $\varrho[L] - V$ comes with a cost for the insurance company, denoted by ε. This cost of capital ε is the return on investment demanded by the investors. As a result, the larger the $\varrho[L]$, the more costly the extra solvency buffer is for the insurance company. In short, while the regulator favours a larger amount of capital to protect the policyholders, the company will prefer a lower amount because of the cost of capital. Clearly, the best interests of the regulator and insurance company are not fully aligned. In order to balance these two conflicting point of views, a trade-off that is acceptable for both parties should be find.

The two diverging points of views of the regulator and the insurance company can be combined in a cost function in order to find the "optimal" amount of capital an insurance company has to hold K such that the cost function $C(L, K)$ is minimal.

Theorem 3.3.2. *Take $0 < \epsilon < 1$ and take as the regulator's risk measure ϕ the expected value. The cost function $C(x, \varrho[L])$ defined by*

$$C(X, \varrho[L]) = \mathbb{E}\left[(L - \varrho[L])_+\right] + (\varrho[L] - V)\,\epsilon$$

reaches its minimum if

$$\varrho[L] = VaR_{1-\epsilon}[L]$$

Proof 3. Assuming the cumulative distribution function of L is continuous and strictly increasing and $K = \varrho[L]$, the proof of Theorem 3.3.2 follows:

$$\begin{aligned}
&\frac{d}{dK}C(X, \varrho[L]) = \frac{d}{dK}\mathbb{E}\left[(L-K)_+\right] + \frac{d}{dK}(K - V)\varepsilon = 0\\
\Leftrightarrow\ &\frac{d}{dK}\int_K^{+\infty}(1 - F_L(x))dx + \frac{d}{dK}(K - V)\varepsilon = 0\\
\Leftrightarrow\ &F_L(K) - 1 + \varepsilon = 0\\
\Leftrightarrow\ &F_L^{-1}(F_L(K)) = F_L^{-1}(1 - \varepsilon)\\
\Leftrightarrow\ &K = F_L^{-1}(1 - \varepsilon) = VaR_{1-\varepsilon}\\
\Leftrightarrow\ &\varrho[L] = VaR_{1-\varepsilon}
\end{aligned}$$

What this result shows is that the value-at-risk is an optimal way to determine the solvency capital, not because it gives a very clear picture of the upper tail, like the TVaR, but because it balances both the insurance company and the regulator point of views.

Non-elicitability: problems with backtesting

Despite its coherence and ability to quantify tail risk, Gneiting (2011) showed that expected shortfall falls short of the elicitability criterion. As already mentioned in section 2.6, many authors, including Chen (2018), deem the criterion of elicitability as "essential to effective backtesting". The (erroneous) belief that ES could not be backtested aroused a number of criticisms against the introduction of ES in spite of VaR in the Basel Committee on Banking Supervision (2019). This lead others to question the relevance of elicitability in the issues related to backtesting. Among them, Acerbi and Szekely (2014), Fissler et al. (2015), and Nolde and Ziegel (2017) introduced alternative back-test methodologies. In this respect, Acerbi and Szekely (2014) and Nolde and Ziegel (2017, p.183) affirm that the mathematical property of elicitability “has to do with model selection and not model testing, making it almost irrelevant for choosing a regulatory risk standard”.

Ultimately, this means backtesting for expected shrortfall “is less straightforward than (. . .) backtesting for VaR” (Emmer et al., 2015). In short, while it is possible to “compare predicted quantiles to actual observed returns” with VaR, “it is harder to backtest ES since one would need to compare a prediction of an expectation to some estimated expectation” (Danielsson & Zhou, 2016). With that in mind, Fissler et al. (2015) showed that expected shortfall is jointly elicitable with VaR. Note that while the joint elicitability of ES with VaR increases “the estimation error” as “ES is estimated conditional on VaR” and results in “much more model risk in ES backtesting”, Danielsson and Zhou (2016) also argue it may actually lead to a more precise estimation “because it smooths out the tails” - which is in line with the thinking underlying the lack of robustness of ES due do the tail behaviour assumptions. More recently, Embrechts et al. (2021) also introduced the notions of Bayes risk measures as a counterpart of elicitable risk measures.

For a more detailed insight into backtesting methods, we refer to Acerbi and Szekely (2014), Fissler et al. (2015), and Nolde and Ziegel (2017), and especially the research paper by Emmer et al. (2015) whereby they proposed an empirical approach consisting in replacing ES by a set of four quantiles in order to backtest VaR. Chen (2018) also stressed that traditional backtesting frameworks such as those prescribed by the regulation is not sufficient on its own - although necessary. Incidentally, Acerbi and Szekely (2014) suggested non-parametric backtesting methodologies for ES which are a priori considered as more efficient than those required by applicable Basel regulation.

Chapter 4

Expectile, an alternative to VaR and TVaR

The lack of coherence and elicitability of value-at-risk and expected shortfall respectively heightened interest in new risk measures candidates sharing both properties. Against that backdrop, it has been shown in several papers that expectiles are the only law invariant coherent and elicitable risk measures, albeit for $\tau \geq 0.5$, thereby offsetting the weaknesses of both value-at-risk and expected shortfall. The aim of this chapter is to provide a compendium of expectiles as a new risk measurement tool and to explore its characteristics as a potentially more suitable risk measure than VaR and ES. In that regard, Section 1 introduces expectile as a special class of quantiles and Section 2 develops its mathematical properties including its shortcomings.

4.1 Motivation and definition

Despite the quite renewed academic interest in expectiles, the term expectile was first coined by Newey and Powell in 1987. By comparing the equation (4.1) suggested by A. Goldberger to the analogous relation (4.2) for quantiles, Newey and Powell (1987, p.823) showed that "expectiles are determined by tail expectations in the same way that quantiles are determined by the distribution function".

A. Goldberger suggested to Newey and Powell (1987, p.823) the following equation,

$$\frac{\tau}{(1-\tau)} = \left[\int_{(-\infty,e(\tau))} (e(\tau) - y)\, dF(y)\right] \cdot \left[\int_{(e(\tau),\infty)} (y - e(\tau))\, dF(y)\right]^{-1} \tag{4.1}$$

whose analogous relation for quantiles is:

$$\frac{\tau}{(1-\tau)} = \left[F\left(\eta(\theta)\right)\right] \cdot \left[1 - F\left(\eta(\theta)\right)\right]^{-1} \tag{4.2}$$

From the above, Gneiting (2011, p.755) defines expectile as follows.

Definition 17 (Expectile). *The τ-expectile functional ($0 < \tau < 1$) of a probability measure F with finite mean as the unique solution $x = e_\tau$ to the equation*

$$\tau \int_x^\infty (y - x) dF(y) = (1-\tau) \int_{-\infty}^x (x - y) dF(y) \tag{4.3}$$

At the heart of quantile regression lies the property whereby the left and right quantiles of a random variable L can be defined as the minimizers of an asymmetrically weighted (piecewise linear) loss function for $\alpha \in (0,1)$,

$$q_\alpha(L) = \underset{\ell \in R}{\text{arg min}} \ \ \alpha \times \mathbb{E}\left[(L-\ell)^-\right] + (1-\alpha) \times \mathbb{E}\left[(L-\ell)^+\right] \tag{4.4}$$

Similarly, Newey and Powell (1987) defined expectiles as the minimizers of an asymmetrically weighted squared (piecewise-quadratic) loss function:

Definition 18 (Expectile)**.** For $0 < \tau < 1$ and square integrable L, the expectile $e_\tau(L)$ of order τ is defined as

$$e_\tau(L) = \underset{\ell \in R}{arg\,min}\, \tau \times \mathbb{E}\left[\max(L-\ell,0)^2\right] + (1-\tau) \times \mathbb{E}\left[\max(\ell-L,0)^2\right] \tag{4.5}$$

Remark 12. The notation $\text{EVaR}_\tau(L)$ is equivalent to that of $e_\tau(L)$.

Note that the above definition 4.8 requires that the random variable L is square integrable i.e. has a well-defined finite mean and, in addition, a finite variance. In their paper about tail expectiles, Daouia et al. (2018, p.5) proposed an equivalent equation to that of (4.5) to emphasise the square integrability of L with the presence of terms L_+^2 and L_-^2:

$$e_\tau(L) = \underset{\ell \in R}{arg\,min} \{\tau \mathbb{E}\left[(L-\ell)_+^2 - L_+^2\right] + (1-\tau)\mathbb{E}\left[(L-\ell)_-^2 - L_-^2\right]\} \tag{4.6}$$

The first-order necessary condition for optimality can then be written as

$$e_\tau(L) - \mathbb{E}[L] = \frac{2\tau - 1}{1-\tau}\mathbb{E}\left[(L - e_\tau(L))^+\right] \tag{4.7}$$

Provided that the first moment of the distribution function F_L is finite, $\mathbb{E}(L) < \infty$, Daouia et al. (2018, p.5) showed that this equation (4.7) has a unique solution. We can then conclude that, indeed, the presence of L_+^2 and L_-^2 makes this equation well-defined for expectiles when the mean is finite $\mathbb{E}(L) < \infty$.

The preceding definitions both require the square integrability of the random variable, and are therefore not the most general. In that sense, Emmer et al. (2015, p.8) suggested a more general but less intuitive definition based on the observations of Bellini et al. (2014) and Newey and Powell (1987) in which expectiles can be uniquely defined in terms of their first-order condition where positive deviations are weighted by τ and negative deviations using $(1-\tau)$.

Lemma 1. *If L is an integrable random variable, then $e_\tau(L)$ is the unique solution ℓ of*

$$\begin{gathered} \tau\mathbb{E}[\max(L-\ell,0)] = (1-\tau)\mathbb{E}[\max(\ell-L,0)] \\ \tau\mathbb{E}[(L-\ell)^+] = (1-\tau)\mathbb{E}[(L-\ell)^-] \end{gathered} \tag{4.8}$$

Consequently, $e_\tau(L)$ satisfies

$$e_\tau(L) = \frac{\tau\mathbb{E}[L\mathbb{1}_{\{L \geq e_\tau(L)\}}] + (1-\tau)\mathbb{E}[L\mathbb{1}_{\{L < e_\tau(L)\}}]}{\tau P[L \geq e_\tau(L)] + (1-\tau)P[L < e_\tau(L)]} \tag{4.9}$$

Expectiles not only provide information about the symmetry of the distribution by weighting the positive and negative deviations, but are also more sensitive - than quantiles - to the magnitude of extreme values due to the *quadratic* form of the loss function (Daouia et al., 2018, p.5–6).

Proof 4. As suggested by McNeil et al. (2015, p.290), equation 4.8 is easily proven by minimising the expected value of the scoring function $s(x, L) = \left|\mathbb{1}_{\{x \geq L\}} - \tau\right| (L - x)^2$ that yields the expectile:

$$\begin{aligned}
\frac{d}{dx}\mathbb{E}\left[s(x, L)\right] &= \frac{d}{dx}\int_{-\infty}^{+\infty} \left|\mathbb{1}_{\{x \geq y\}} - \tau\right| (y - x)^2 \, dF_L(y) \\
&= \frac{d}{dx}\int_{-\infty}^{x} (1 - \tau)\, (y - x)^2 \, dF_L(y) + \frac{d}{dx}\int_{x}^{\infty} \tau \, (y - x)^2 \, dF_L(y) \\
&= 2\, (1 - \tau) \int_{-\infty}^{x} (y - x) \, dF_L(y) + 2\tau \int_{x}^{\infty} (y - x) \, dF_L(y) \\
&= 2\tau\mathbb{E}[(L - \ell)^+] - 2(1 - \tau)\mathbb{E}[(L - \ell)^-]
\end{aligned}$$

Setting it to 0 yields the definition of expectiles in equation 4.8.

4.2 Properties

As already mentioned, the interest around expectiles lies in the fact that they are indeed the only law invariant coherent and elicitable risk measures. The elicitability proof is given by Gneiting (2011, p.755, Theorem 10). The properties defined by Bellini et al. (2014) are summarized in the following proposition by Emmer et al. (2015, p.8-9, Proposition 2.1).

Proposition 4.2.1. *Expectiles have the following properties.*

(i) *For* $0 < \tau < 1$*, expectiles are homogeneous and law invariant. As a consequence, expectiles are additive for linearly dependent random variables, ie,*

$$corr[L_1, L_2] = 1 \Rightarrow e_\tau(L_1 + L_2) = e_\tau(L_1) + e_\tau(L_2) \tag{4.10}$$

(ii) *For* $0.5 \leq \tau < 1$*, expectiles are subadditive (and hence coherent), whereas, for* $0.5 \geq \tau > 0$*, they are superadditive.*

Daouia et al. (2018, p.5–6) further added Lipschitzianity w.r.t. the Wasserstein distance:

(iii) *For all* $Y, \tilde{Y}$ *with* $\mathbb{E}(Y) < \infty$ *and* $\mathbb{E}(\tilde{Y}) < \infty$ *and* $\tau \in (0, 1)$*, it holds that* $\|\xi_{Y,\tau} - \xi_{\tilde{Y},\tau}\| \leq \tilde{\tau} \cdot d_W(Y, \tilde{Y})$*, where* $\tilde{\tau} = \max\{\frac{\tau}{1-\tau}, \frac{1-\tau}{\tau}\}$ *and*

$$d_W(Y, \tilde{Y}) = \int_{-\infty}^{\infty} \|F_Y(y) - F_{\tilde{Y}}(y)\| dy = \int_{0}^{1} \|F_Y^{-1}(t) - F_{\tilde{Y}}^{-1}(t)\| dt \tag{4.11}$$

4.2.1 Lack of comonotonic additivity

From the foregoing, it might seem like expectiles are better than value-at-risk and expected shortfall. However, their lack of comonotonic additivity has to be recognised as a serious deficiency - as they might fail to recognize risk concentration due to non-linear dependencies.

Proposition 4.2.2 (. *emmer] For* $0.5 < \tau < 1$, *expectiles are not comonotonically additive. If* e_τ *were comonotonically additive, then, by Tasche (2002, Theorem 3.6), it would be a so-called spectral risk measure. But then, from their so-called Kusuoka representation given, for example, in Ziegel (2014, Corollary 4.3), it would not be elicitable, in contradiction to Gneiting (2011,Theorem 10).*

While the absence of elicitability should not influence the regulatory risk standard as it did not prove to be a major issue, the absence of comonotonic additivity certainly is. For comonotonic positions with less than perfect linear correlation, there is a clear danger that risk measures like standard deviation and expectiles underestimate the lack of diversification. In their paper *Backtesting Expected Shortfall*, Acerbi and Szekely (2014, p.10) provided an example that highlights how the use of expectiles could deceive someone into taking unwise actions: “An expectile ϱ will tell you that a long position in a call option c is partially hedged by a long (yes, long) position in the underlying stock S: $\varrho(C+S) < \varrho(C) + \varrho(S)$”.

4.2.2 Expectiles versus quantiles

As its name suggests, the expectile is a blending of expectation and quantile. Remember that expectiles were defined as “the minimizer of an asymmetric least squares criterion, making it a weighted average” (Philipps, 2021, p.1). In other words, expectiles are a measure of the expected value of both tails of the distribution. When $\tau = 0.5$, the weights in equation 4.8 are symmetric, thus the expected positive and negative deviations from $\ell = e_\tau(L)$ are equal and $e_\tau(L) = \mathbb{E}(L)$. Expectiles may then be considered as an asymmetric generalization of the mean, just as quantiles generalize the median (Bellini & Di Bernardino, 2017; Daouia et al., 2018). Daouia et al. (2018, p.263) further add that “both expectiles and quantiles are useful descriptors of the higher and lower regions of the data points in the same way as the mean and median are related to their central behavior”. Although “expectiles summarize the distribution function in much the same way that the quantiles $q_\alpha := F_L^{-1}(\alpha) = \inf\{\ell \in \mathbb{R} : F_L(\ell) \geq \tau\}$ do” (Daouia et al., 2018, p.5), there is no such explicit formula for expectiles. Bellini and Di Bernardino (2017) suggested the use of the asymptotic first order approximation (4.7). The equation (4.8) can then be explicitly solved by means of the so called Lambert W function for Weibull-type distributions (see Bellini and Di Bernardino (2017)).

In figure 4.2.1 it can be observed the expectile function $e(\tau)$ has “a smaller slope than the” corresponding quantile function $q(\alpha)$ “near $\tau = 0.5$ and a larger slope (...) near $\tau = 0$ or $\tau = 1$” for both the Uniform and Gaussian distributions (Newey & Powell, 1987, p.824).

The table 4.2.1 contains the corresponding α of $\tau = 0.99$ for the $\mathcal{N}(0,1)$, $log\mathcal{N}(0,1)$, $\mathcal{U}(0,1)$ and $\mathcal{P}(\alpha = 2, \theta = 1)$ distributions. We observe the tail probability associated with the expectile changes with the underlying distribution, and thence depend on it, as explained by Kuan et al. (2009, p.269)

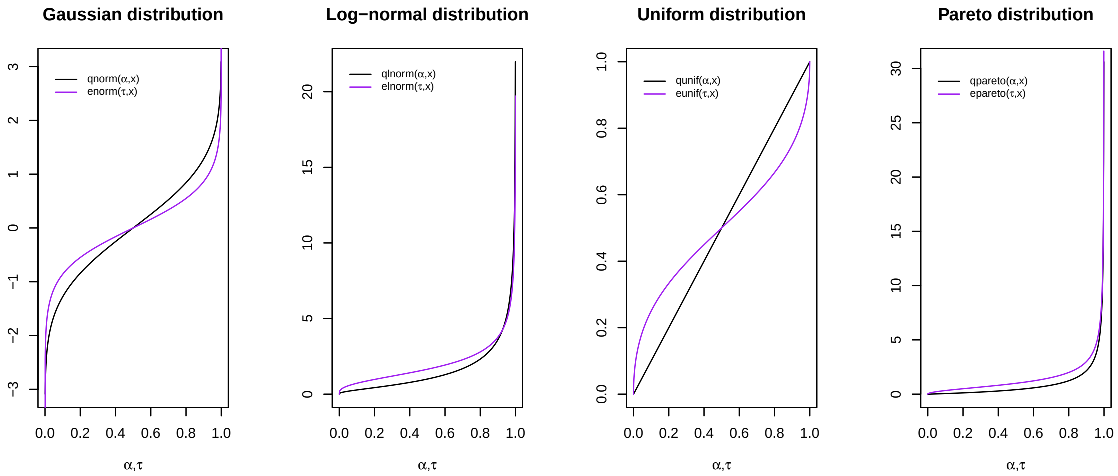

Figure 4.2.1: Quantile and expectile functions on the unit interval for the continuous random variables $W \sim \mathcal{N}(0,1)$, $X \sim log\mathcal{N}(0,1)$, $Y \sim \mathcal{U}(0,1)$ and $Z \sim \mathcal{P}(\alpha = 2, \theta = 1)$.

Distribution	τ	α
$\mathcal{N}(0,1)$	0.99	0.957
$log\mathcal{N}(0,1)$	0.99	0.984
$\mathcal{U}(0,1)$	0.99	0.909
$\mathcal{P}(\alpha = 2, \theta = 1)$	0.99	0.992

Table 4.2.1: Corresponding α of $\tau = 0.99$ for the $\mathcal{N}(0,1)$, $log\mathcal{N}(0,1)$, $\mathcal{U}(0,1)$ and $\mathcal{P}(\alpha = 2, \theta = 1)$.

4.2.3 Assessment of both tails

As already mentioned, the quantile-VaR is tail-insensitive, making it robust to extreme values but increasing its odds of underestimating the risk. On the other hand, ES is “only” sensitive to the magnitude of the losses in the upper right-tail, allowing it to account for the average severity of exceedances which are of interest (e.g. when examining flood risks), but making it “too” conservative. By contrast, expectile produces a more comprehensive evaluation of the risk since it depends on both tails of the distribution and their probability. Based on that observation, expectiles should be “more alert than quantiles to the magnitude of infrequent catastrophic losses” according to Daouia et al. (2018, p.2).

To assess the expectile alertness to the magnitude of extremes relative to that of value-at-risk and tail-VaR, we exaimne a data set on hurricane losses compiled by the American Insurance Association, and presented by Hogg and Klugman (1984, p.128, Table 4.1). It contains 35 hurricanes for which total losses[1] exceeded \$5,000,000. Each loss is treated as the realization of one of the 35 hurricanes - the catastrophic events - represented by iid loss random variables $L_1, \ldots, L_{35}$. The corresponding sample mean is \$199 900, and the sample standard deviation is \$325 807.

The middle panel of Figure 4.2.2 clearly illustrates the “lack of smoothness and stability of both sample” quantile, in red, and expected shortfall, in green. According to Daouia et al. (2019, p.1377), “their discreteness as piecewise constant functions of the” security level α “is a

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[1] The amounts are inflation-adjusted (using the U.S. Residential Construction Index)

serious defect, especially in the important upper tail. Indeed, a small change in α can trigger a (severe) jump in the values of the estimated VaR and ES. Moreover, the fact that the steps result in the same or similar measures for significantly different risk levels is itself risky". All the more so as we mostly focus on low probability levels where the size of the jumps appears to increase. By contrast, expectile, in blue, "have the benefit to be very stable and to change continuously and increasingly without recourse to any smoothness procedure." (Daouia et al., 2019, p.1378)

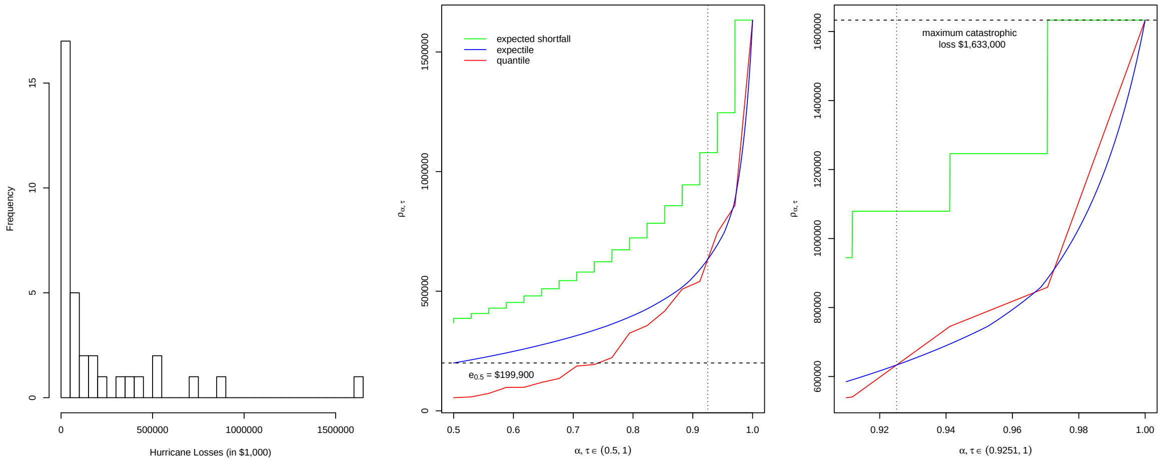

Figure 4.2.2: Histogram from 35 hurricane losses in excess of $5 million (in units of $1,000) and empirical quantile, expected shortfall, and expectile at confidence level $\alpha, \tau \in [0.5, 1]$. Adapted from Daouia et al. (2019, p.1377, Figure 4)

At the same certainty level, the TVaR estimation is much larger as it focuses on the most extremes outcomes - and is therefore more conservative. In contrast, the estimated expectile produces a less bleak picture than ES, while still remaining more alert to extremes than quantile until it breaks down at 0.9251. Thenceforth, "although the expectile exhibits a smooth evolution, it diverges from" value-at-risk "in the region" $\tau \in [0.9251, 1]$ (see right panel of Figure 4.2.2) "and becomes less alert to infrequent disasters." (Daouia et al., 2019, p.1377–1378)

The conclusion is similar when one compares the empirical quantile, ES and expectile on the two loss distributions from figure 3.3.2, one of which involved a thicker right tail.

The left graph 4.2.3 is based on the loss random variable illustrated in black in figure 3.3.2, while the right one is based on the one with thicker tails. In both situations, the expectile becomes less alert to infrequent large values in the upper tail of interest and the disparity is even more pronounced in the case of the thicker right-tail.

4.2.4 Less intuitive

While the underlying idea of both value-at-risk and expected shortfall is quite intuitive (VaR is nothing but a quantile and ES is the average above that quantile), it gets a bit more complicated when one tries to establish how expectiles should be understood. Their opacity and lack of intuitive appeal are often cited as the most glaring reasons to discourage their use as a risk measure (Chen, 2018, p.2). Nonetheless, the issue of interpretability has been addressed by many authors. Among them, a more intuitive definition of expectiles "in terms of VaR, expected

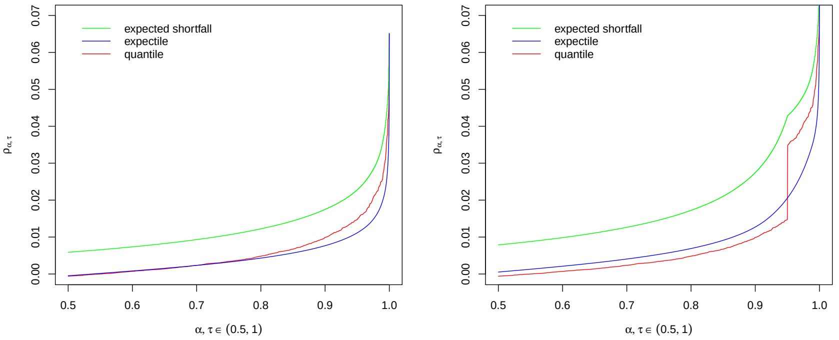

Figure 4.2.3: Empirical quantile, expected shortfall, and expectile at confidence level $\alpha, \tau \in [0.5, 1]$ from figure 3.3.2

shortfall, and the thresholds at which those competing risk measures are enforced" (Chen, 2018, p.1, 18) has been proposed "for ease of regulatory implementation" (Chen, 2018, p.1, 18). The asymmetry parameter τ as a function of value-at-risk and expected shortfall is such that:

$$VaR_\alpha = \Phi^{-1}(\alpha) \text{ and } ES_\alpha = \frac{1}{\alpha\sqrt{2\pi}} e^{-\frac{1}{2}VaR_\alpha^2}$$

$$\tau = \frac{\alpha(ES_\alpha - VaR_\alpha)}{VaR_\alpha + 2\alpha(ES_\alpha - VaR_\alpha)} \tag{4.12}$$

Moreover, by isolating ES_α in equation (4.12), the α-expected shortfall can be expressed as the product of the corresponding value of VaR and a term greater than 1 for $\tau \in (0, 0.5)$:

$$ES_\alpha = VaR_\alpha \left(1 + \frac{\tau}{\alpha(1 - 2\tau)}\right) \tag{4.13}$$

Similarly, Kuan et al. (2009) expressed the expectile's argument τ as the index of prudentiality. By taking e_τ as a capital requirement (margin), Kuan et al. (2009, p.262) showed τ is "the relative cost of the expected margin shortfall" from equation (4.3):

$$\tau = \frac{\int_{-\infty}^{e_\tau}(y - e_\tau)dF(y)}{\int_{-\infty}^{e_\tau}(y - e_\tau)dF(y) + \int_{e_\tau}^{\infty}(y - e_\tau)dF(y)} = \frac{\int_{-\infty}^{e_\tau}(y - e_\tau)dF(y)}{\int_{-\infty}^{+\infty}(y - e_\tau)dF(y)} \tag{4.14}$$

where $\int_{-\infty}^{e_\tau}(y - e_\tau)dF(y)$ is "the expected margin shortfall" and $\int_{e_\tau}^{\infty}(y - e_\tau)dF(y)$ is "the opportunity cost due to the expected margin overcharge". The sum $\int_{-\infty}^{+\infty}(y - e_\tau)dF(y)$ is "thus the expected total cost of holding the capital requirement" $e(\tau)$. The higher the risk aversion, the higher the level of prudentiality, the larger e_τ, the smaller the expected capital requirement shortfall. (Kuan et al., 2009, p.263)

Chapter 5

Expectile and solvency capital

As previously mentioned, risk measures are commonly used to measure the capital to be hold by an insurance company. Specifically, the solvency capital requirements (SCR) is defined as the minimum level of initial available capital prescribed by regulators on insurance companies. The underlying idea behind the SCR is the protection of the policyholders from the insurance company's insolvency.

Under Solvency II, the SCR is calculated so that the probability of ruin on one year or less is equal to 0.5% (that is, one default accepted every 200 years). In the standard formula, the SCR is given by the VaR at 99.5% of the Net Asset Value (NAV) variation on 1 year. This minimum amount of solvency capital (equity) should then guarantee the viability of the company in 99.5% of cases (Devolder, 2021).

5.1 Empirical analysis

To carry out this empirical analysis, we consider an insurance company established at time $t = 0$ with an initial equity E_0. On top of the own funds, a cohort of single premiums L_0 (provisions) is paid. We assume no more cash inflows (closed fund).

Assets	Liabilities
$A_0 = \$50,000$	$L_0 = aA_0 = \$35,000$
	$E_0 = (1-a)A_0 = \$15,000$

Table 5.1.1: Initial balance sheet of the life insurer.

The total liabilities (equity included) will be allocated in an investment fund $(A_t)_{t\geq 0}$ whose underlying asset dynamic follows a geometric Brownian motion. Under the real-world probability measure $\mathcal{P}$, the asset process has a constant mean return (drift) $\mu = 6\%$ and a constant volatility of 20%. Formally,

$$\begin{aligned} dB_t &= rB_t dt \text{ (the risk-free asset)} \\ dA_t &= \mu A_t dt + \sigma A_t dW_t^{\mathcal{P}} \\ A_t &= A_0\, e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t^{\mathcal{P}}} \end{aligned}$$

Using Euler's discretisation scheme, paths of the asset process can be simulated (see figure 6.0.1):

$$A_{t+\Delta t} = A_t \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_t\right)$$

where the noise ε_t is standard normally distributed $\mathcal{N}(0,1)$.

Under the risk-neutral measure Q (with the deposit B_t as the numeraire and market risk-free rate $r = 3\%$), the discounted asset process is a martingale.

$$\begin{aligned} dA_t &= rA_t dt + \sigma A_t dW_t^{\mathcal{Q}} \text{ with } dW_t^{\mathcal{Q}} = dW_t^{\mathcal{P}} + \frac{\mu - r}{\sigma} dt \\ A_t &= A_0\, e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^{\mathcal{Q}}} \end{aligned}$$

where $W_t^{\mathcal{Q}}$ and $W_t^{\mathcal{P}}$ are standard Brownian motions under the risk measures $\mathcal{Q}$ and $\mathcal{P}$ and are normally distributed $\mathcal{N}(0,t)$. We further assume a completed market free of arbitrage and zero transaction costs.

We consider a pure endowment insurance (payment of a lump sump C only in case of survival at maturity T in 20 years) with single premium, issued at age $x = 45$. The single premium is given by:

$$P = L_0 = C\,{}_TE_x = C\,{}_Tp_x \frac{1}{(1+i)^T} = \$35{,}000$$

where the survival probability ${}_Tp_x$ and the technical interest rate i (guaranteed discounted rate) are chosen by the product managers of the insurance company.

The life insurer offers its policyholder a participation in profit in the form of a terminal bonus (maturity guarantee) on top of the minimum guaranteed interest rate $r_G = 1\%$ (branch 21 life insurance). The guaranteed amount at $T = 20$ is equal to:

$$L_G = L_0 e^{r_G T} = \$42{,}749.1$$

The terminal bonus (in the event of positive results) is path independent (only depends on the final value of the asset process) and is given by:

$$\beta \max\left(aA_T - L_G, 0\right) = \beta\, a\left(A_T - \frac{L_G}{a}\right)^+$$

where $e^{-rT}\left(A_T - \frac{L_G}{a}\right)^+$ is the price $c\left(A_T, T, \frac{L_G}{a}\right)$ of a long position in a European call on the asset process with maturity T and strike $K = \frac{L_G}{a} = \$61{,}070.14$:

$$c\left(A_t, t, \frac{L_T^G}{a}\right) = A_t \Phi\left(d_1\right) - Ke^{-r(T-t)}\Phi\left(d_2\right)$$

where

$$\begin{aligned} d_1 &= \frac{\left(r + \frac{1}{2}\sigma^2\right)(T-t) + \ln\left(\frac{A_t}{K}\right)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

The final (stochastic) cash-flow (life capital and bonus) is given by:

$$C = \frac{1}{{}_T p_x} P \left((1+i)^T + \left(\beta g_T - \left((1+i)^T - 1 \right) \right)^+ \right)$$

where β is the participation level, $g_T = \frac{A_T}{A_0} - 1$ the real return on $(x, x+T)$ and $(1+i)^T - 1$ the technical return on $(x, x+T)$.

At time t, the fair value of the liabilities is given by the risk neutral expectation of the future cash-flows discounted using the risk-free interest rate (market number):

$$\begin{aligned} FV(L_t) &= e^{-r(T-t)}\, \mathbb{E}^{\mathcal{Q}} \left[L_G + \beta\, a \left(A_t - \frac{L_T^G}{a} \right)^+ \right] \\ &= e^{-r(T-t)}\, L_T^G + \beta\, a\, c \left(A_t, t, \frac{L_G}{a} \right) \\ FV(E_t) &= A_t - FV(L_t) \end{aligned}$$

Note that in the Black & Scholes framework we assume a constant market risk-free rate r.

The contract is said to be fair or equilibrated when its initial fair value is equal to the single premium paid by the policyholder:

$$\underbrace{a A_0}_{=L_0} = \underbrace{e^{-rT}\, L_G + \beta\, a\, c \left(A_0, 0, \frac{L_T^G}{a} \right)}_{=FV(L_0)} \tag{5.1}$$

Given the choice of parameters, the “fair” participation level obtained from the equilibrium equation (5.1) is equal to $\beta = 68.7846\%$:

$$\beta^* = \frac{L_0 - e^{-r\,T}\, L_G}{a\, c\, (A_0, 0, K)} \tag{5.2}$$

The choice of the time horizon (one year) of our analysis is guided by the current regulation from Solvency II, where value-at-risk estimates the risk over a one-year period. It is indeed a reasonable assumption since it reflects the standard time horizon over which insurance contracts are renewed (the insurance company is usually committed to hold its portfolio for one year). Under the Solvency II norm, the initial assets must be enough in order to have at time $t = 1$:

$$P(A_1 - L_1 \geq 0) \geq 99.5\%$$

This basic condition can be transformed as a value-at-risk on the variation of NAV:

$$\begin{aligned} & P(\underbrace{A_1 - L_1}_{=NAV_1} \geq 0) \geq 99.5\% \\ or\ & P \left(\frac{NAV_1}{1+r} - NAV_0 \geq -NAV_0 \right) \geq 99.5\% \end{aligned}$$

where r is the market risk-free rate, A_t denotes the assets value (where A_0 is the initial assets value). Similarly, L_t are the liabilities at time t and NAV_t the net asset values. Also, note that NAV_0 is certain $(t = 0)$, whereas NAV_1 is random.

The SCR can then be defined as the amount given by:

$$\begin{aligned} SCR_0 &= \inf\left\{V : P\left(\frac{NAV_1}{1+r} - NAV_0 \geq -V\right) \geq 99.5\%\right\} \\ &= \inf\left\{V : P\left(NAV_0 - \frac{NAV_1}{1+r} \leq V\right) \geq 99.5\%\right\} \\ &= VaR_{99.5\%}\left(-\Delta NAV\right) \end{aligned} \tag{5.3}$$

The solvency condition can also be expressed as follows $NAV_0 \geq SCR_0$ i.e. the initial available capital must be at least equal to the required capital (solvency test).

The initial net asset value and the asset value after one-year are respectively equal to:

$$\begin{aligned} &NAV_0 = E_0 = A_0 - L_0 = \$50,000 - \$35,000 = \$15,000 \\ &NAV_1 = E_1 = A_1 - L_1 = \$50,000\, e^{\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma W_1^{\mathcal{P}}} - \left(L_G\, e^{-r(T-1)} + \beta\, a\, c(A_1, 1, K)\right) \end{aligned}$$

with

$$\begin{aligned} c(A_1, 1, K) &= e^{\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma W_1^{\mathcal{P}}}\, \Phi\left(d_1\right) - \frac{L_G}{a} e^{-r(T-1)} \Phi\left(d_2\right) \\ d_1 &= \frac{\left(r + \frac{1}{2}\sigma^2\right)(T-1) + \ln\left(\frac{A_1}{K}\right)}{\sigma\sqrt{T-1}} \\ d_2 &= d_1 - \sigma\sqrt{T-1} \end{aligned}$$

The level of extra capital SC_t the insurance company has to hold (on top of the initial premium), in order to be able to pay its liabilities after t years, is equal to the stressed value of the own funds:

$$\begin{aligned} SCR_t &= E_t - E_t^{stressed} \\ SC_t &= SCR_t - NAV_t = E_t^{stressed} \end{aligned}$$

where $E_t^{stressed}$ is the value of the own funds under a stressed scenario. Since the investment fund is the only source of stochasticity in our problem, the stress test scenarios will be done with respect to the asset process. Given our choice of parameters, the 0.5%-quantile stress scenario results in an immediate decline of the assets by about 40%. While the guaranteed amount will not be affected by the stress on the assets, the terminal bonus will be negatively impacted:

$$A_t^{stressed} = A_t\left(e^{\mu - r - \frac{1}{2}\sigma^2 + \sigma q_{0.5\%}(\mathcal{N}(0,1))}\right) = A_t \times 0.6034055 \tag{5.4}$$

$$\begin{aligned} SCR_0 = \mathrm{VaR}_{0.5\%}\left(-\Delta\mathrm{NAV}\right) &= \$15,000 - \$50,000\, e^{\left(\lambda - \frac{1}{2}\sigma^2\right) + \sigma\, q_{0.5\%}^{\mathcal{N}(0,1)}} \\ &+ \left(L_G\, e^{-r(T-1)} + \beta\, a\, c\left(\$50,000\, e^{\left(\lambda - \frac{1}{2}\sigma^2\right) + \sigma\, q_{0.5\%}^{\mathcal{N}(0,1)}}, 1, K\right)\right) \text{ with } \lambda = \mu - r \\ &= E_0 - E_0^{stressed} = \$15,000 - \$2,186.507 = \$12,813.49 \end{aligned}$$

The insurance company is said to be solvent as long as its solvency ratio $\frac{E_t}{SCR_t}$ is greater than 100%. The 99.5%-quantile stress scenario resulted in a solvency ratio of 117% (the own funds are indeed greater than the capital requirements).

With regard to the choice of the expected shortfall and expectile's parameters α (confidence level) and τ (index of prudentiality), given we aim to calculate the amount of solvency capital, a high confidence level is surely called for as was the case for value-at-risk. Although we will not take into account the cost of capital in our analysis, note that even if a higher level may offer a greater protection to policyholders, it will also increase the level of capital to be remunerated. We find that the Solvency II $\mathrm{VaR}_{99.5\%}$ is essentially the same as $\simeq \mathrm{ES}_{98.7\%} \simeq e_{\tau=0.9994}$ for a normally distributed risk:

$$\begin{aligned}
\mathrm{VaR}_{99.5\%} &= \mu + \sigma\Phi^{-1}(\alpha) = \mu + \sigma\, 2.575829 \text{ (see formula (6.1))}\\
\mathrm{TVaR}_{98.70301\%} &= \mu + \sigma\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha} = \mu + \sigma\, 2.57583 \text{ (see formula (6.3))}\\
e_{99.9387123\%} &= \mu + \sigma\, 2.575828 \text{ (see formula (6.6))}
\end{aligned}$$

Similarly, Nolde and Ziegel (2017, p.1835) and Bellini and Di Bernardino (2017, p.496) suggested that $e_{\tau=99.855\%}$ "leads to a comparable magnitude of risk" as the Basel III $\mathrm{VaR}_{99\%}$ and Basel IV $\mathrm{ES}_{97.5\%}$. This suggestion is indeed consistent with our findings for a normally distributed random variable:

$$\begin{aligned}
\mathrm{VaR}_{99\%} &= \mu + \sigma\, 2.326348\\
\mathrm{TVaR}_{97.4232\%} &= \mu + \sigma\, 2.326347\\
e_{99.854759\%} &= \mu + \sigma\, 2.326349
\end{aligned}$$

Whereas, for a log-normally distributed variable $\log\mathcal{N}(\mu = 0, \sigma = 1)$, we found that $\mathrm{VaR}_{99.5\%} \simeq \mathrm{TVaR}_{98.45\%}$ and $\simeq \mathrm{e}_{99.75\%}$:

$$\begin{aligned}
\mathrm{VaR}_{99.5\%} &= \exp\left(\mu + \sigma\Phi^{-1}(\alpha)\right) = \exp\left(\mu + \sigma\, 2.575829\right) = 13.14221 \text{ (see formula (6.2))}\\
\mathrm{TVaR}_{98.44749\%} &= \frac{1}{1-\alpha}\exp\left(\mu + \frac{1}{2}\sigma^2\right)\left(1 - \Phi\left(\Phi^{-1}(\alpha) - \sigma\right)\right) = 13.14221 \text{ (see formula (6.4))}\\
e_{99.747709\%} &= 13.14221 \text{ (see formula (6.6))}
\end{aligned}$$

The 0.5%-EVaR stress scenario results in an immediate decline of the assets by about 30% and a solvency ratio of 157%:

$$\begin{aligned}
A_t^{stressed} &= A_t \times 0.6988105 \text{ with } \mu = \lambda - \frac{1}{2}\sigma^2 \text{ (see formula (6.6))}\\
SCR_0 &= \mathrm{e}_{0.5\%}\left(-\Delta\mathrm{NAV}\right) = \$15,000 - \$50,000 \times 0.6988105\\
&\quad + \left(L_G\, e^{-r(T-1)} + \beta\, a\, c\,(\$50,000 \times 0.6988105, 1, K)\right)\\
&= E_0 - E_0^{stressed} = \$15,000 - \$5,440.193 = \$9,559.807
\end{aligned}$$

Those results are consistent with what we already observed in section 4.2.3; the expectile is less alert than the VaR at the same level in the far upper-tail, which results in a smaller level of extra capital $SC_0 = E_0^{stressed}$.

Note that the expectile at a prudential level τ of 99.957855% results in a stress factor of 0.603406, like the one obtained with the VaR at a security level of 0.5% in equation 5.4:

$$\begin{aligned}
A_t^{stressed} &= A_t \times 0.603406 \text{ with } \mu = \lambda - \frac{1}{2}\sigma^2 \text{ and } 1-\tau \text{ (see formula (6.6))}\\
SCR_0 &= \mathrm{e}_{0.042145\%}\left(-\Delta\mathrm{NAV}\right) = \mathrm{VaR}_{0.5\%}\left(-\Delta\mathrm{NAV}\right) = \$12,813.48
\end{aligned}$$

Similarly, for the 0.5%-tail-VaR stress scenario, we have:

$$\begin{aligned} A_t^{stressed} &= A_t \left(\exp\left(\left(\lambda - \frac{1}{2}\sigma^2 \right) + \frac{1}{2}\sigma^2 \right) \frac{1}{0.5\%} \right) \left(\Phi\left(\Phi^{-1}(0.5\%) - \sigma \right) \right) = A_t \times 0.5673797 \\ SCR_0 &= \mathrm{TVaR}_{0.5\%}\left(-\Delta\mathrm{NAV}\right) = \$15,000 - \$50,000 \times 0.5673797 \\ &\quad + \left(L_G\, e^{-r(T-1)} + \beta\, a\, c\left(\$50,000 \times 0.5673797, 1, K\right) \right) \\ &\quad = E_0 - E_0^{stressed} = \$15,000 - \$919.6519 = \$14,080.35 \end{aligned}$$

While the EVaR stress factor was greater than that of VaR at 0.5%, the TVaR stress factor is smaller - and this is once again consistent with what we already observed (the TVaR is more conservative than VaR for a same confidence level). The 0.5%-tail-VaR stress scenario thus results in an immediate decline of the assets by about 43.2% (compared to 40% for the VaR and 30% for the EVaR). As a result, the own funds are barely sufficient and the solvency ratio drops to 106.53%.

The TVaR at a security level $1 - \alpha = 1 - 98.738965\%$ results in a stress factor of 0.603406, like the one obtained with the VaR at a security level of 0.5% in equation 5.4:

$$\begin{aligned} A_t^{stressed} &= A_t \left(\exp\left(\left(\lambda - \frac{1}{2}\sigma^2 \right) + \frac{1}{2}\sigma^2 \right) \frac{1}{1.261035\%} \right) \left(\Phi\left(\Phi^{-1}(1.261035\%) - \sigma \right) \right) = A_t \times 0.603406 \\ SCR_0 &= \mathrm{TVaR}_{1.261035\%}\left(-\Delta\mathrm{NAV}\right) = \mathrm{e}_{0.042145\%}\left(-\Delta\mathrm{NAV}\right) = \mathrm{VaR}_{0.5\%}\left(-\Delta\mathrm{NAV}\right) = \$12,813.49 \end{aligned}$$

The risk measures parameters α, τ in this last equation allow us to conclude, once again, that the TVaR is more conservative, and thence requires a smaller security level to produce the same SCR than VaR. On the contrary, expectile is less alert in the upper-tail region and therefore requires a higher prudential level τ to produce the same SCR.

5.2 Sensitivity analysis

To finalize this analysis, we will assess the sensitivity of the SCR - and the risk measures - to a change in the volatility σ of the asset process, to a change in the market risk free rate r (and thence in the assets mean return $\mu = r + \lambda$) and to a change in the guaranteed rate r_G. As reported in the next three subsections, the sensitivity analysis suggests that, provided that we use an equivalent symmetry (τ) and security (α) levels to the 0.5% value-at-risk, the level of solvency capital is not affected by the choice of the risk measure.

5.2.1 Volatility

The change in volatility first affects the participation level through the call price in equation 5.2 as illustrated in figure 6.0.2. The stress factors, computed in three different stressed scenarios with $\mathrm{TVaR}_{1.261035\%}$, $\mathrm{e}_{0.042145\%}$ and $\mathrm{VaR}_{0.5\%}$ are also negatively affected by the increase in the asset volatility as shown in figure 6.0.3. We can notice that, while the stress factors under the TVaR at 1.261035% are perfectly overlapping, the expectile at a prudential level 0.042145% slightly breaks away as the volatility increases. This might be partially explained by the use of the recursive formula (6.5) to approximate the expectile of the log-normal asset value. The sensitivities of β and the stressed factor to a change in the volatility are directly reflected in the stressed liabilities $L_0^{stressed}$ and equity $E_0^{stressed} = SC_0$ in figures 6.0.4 and 6.0.5. As expected, the increase in the volatility of the asset results in an increase of the required solvency capital - because the business' risk increases. Note that for $\sigma = 1\%$, the equation 5.2 produced a participation level of 100% ($\beta = 1$ (inaccuracy), see tables 6.0.2 and 6.0.3), which results in a negative solvency ratio (see figure 6.0.7) as the entire profit is given back to the policyholders. It is also noteworthy that the expectile stress scenario produced a smaller SCR, which is directly linked to the discrepancy observed between the stress factor of $\text{VaR}_{0.5\%}$ and $\text{e}_{0.042145\%}$. While this may be induced by the expectile's approximation, it still results in an expectile stress scenario that is (slightly) less alert to an increase in the volatility of the assets.

5.2.2 Market risk-free interest rate

Given the value of our guaranteed return rate $r_G = 1\%$, the sensibility to the market risk-free interest rate is analysed by taking rates between 1% and 10%. Taking for r a value smaller than 1% ($= r_G$) would lead to arbitrage (existence of two market risk-free rates). When $r_G = r = 1\%$, the equation (5.2) produces a participation of 0% as showcased in table 6.0.4 and figure 6.0.8. The increase of the level of generosity when the market rate increases is the result of the fairness principle; as r increases, the guarantee r_G becomes less attractive, hence the higher level of generosity to offset the discrepancy $r - r_G$.

The stress factors remain unaffected by the change in the market risk-free rate since we assumed a constant risk premium $\lambda = 3\%$. However, both the stressed liabilities and equity are impacted by this change in the market rate through the price of the call option and the discounting of the guaranteed amount (and the level of generosity β) as showed in figures 6.0.9 and 6.0.10. Overall, an increase of the market rate positively affects the solvency ratio (see figures 6.0.11 and 6.0.12) given the reduction of the liabilities.

5.2.3 Guaranteed rate

Applying the same reasoning regarding the absence of arbitrage (as set out above), we assessed the sensitivity to a change in the guaranteed rate r_G subject to $r_G \leq r = 3\%$. The change in the guaranteed rate first positively affects the guaranteed amount at maturity $L_G = \$35,000e^{20\times r_G}$ as showed in figure 6.0.13. As a result, both the strike and the participation level are impacted by the change in r_G through L_G. Analogously to what was said in the sensitivity to the market rate, the participation level decreases when the guaranteed rate increases as the guarantee r_G becomes more and more attractive with respect to the risk-free rate r. Overall, the solvency ratio illustrated in figure 6.0.15 declines as the guaranteed rate increases, as a direct consequence of the increase in liabilities (see figure 6.0.14).

In conclusion, for such branch 21 product, applying a stress solvency test with the 0.5% value-at-risk, the 1.261% tail-VaR or the 0.042% expectile produces the same solvency capital.

Chapter 6

Conclusion

Overall, the convenience in the quantile's interpretation as well as its ease-of-use are likely to ensure value-at-risk retains its predominance among risk practitioners. Yet, its popularity as an elicitable risk measure should not overshadow its deficiency in covering the tail risk - making it nonetheless robust to extreme observations. From the perspective of the axiomatic approach, the subadditive ES is clearly a better alternative to VaR. Although justified in principle, the concerns around ES's lack of elicitability and robustness should not outweigh its theoretical appeal as a coherent and tail sensitive risk measure. In spite of their promising definition as the only coherent and elicitable risk measures, expectiles fail to live up to expectations due to their lack of comonotonic additivity. Their potential failure to detect interdependence and concentration of risks should be a matter of great concern given the multivariate nature of risk. In brief, while the use of ES over VaR could become more prevalent in the internal operations of institutions to identify dependence between extreme outcomes, it is not likely to be superseded by expectiles. More sophisticated is, in this case, not necessarily synonym of better. Nevertheless, since each risk measure has its strengths and weaknesses, the choice of the *best* - or most appropriate - risk measure should depend on the intended use "subject to the reservation that no risk measure can achieve exactitude in regulation " (Chen, 2018). Understanding their theoretical properties - or lack thereof - is key to develop a set of best practices for the risk at hand. For instance, the observation by (Fissler et al., 2015) that the spectral risk measure ES is jointly elicitable with VaR may suggest the simultaneous application of multiple measures that complement one another.

Finally, note that extremiles deserve a thorough study as a reasonable alternative to quantiles and expectiles. As coherent, comonotonic additive and elicitable risk measures, they address the main drawbacks of value-at-risk (absence of subadditivity), expectile (lack of comonotonic additivity) and expected shortfall (non elicitable). Like expectiles, they depend on both tail realizations and their probability. Besides, they also benefit from a more straightforward interpretation and implementation thanks to their explicit formulations. Their appealing properties and conceptual simplicity are detailed in a research paper by Daouia et al. (2019) in which they stress their adequacy when the prime interest lies in the sensitivity to the magnitude of extremes.

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Formulas

The table 6.0.1 provides a summary of the respective properties of the risk measures.

Property	Variance	VaR	ES	EVaR ($\tau > 0.5$)
Coherence			✓	✓
Comonotonic Additivity		✓	✓	
Convexity			✓	✓
Robust, Weak Topology		✓		
Robust, Wasserstein	✓	✓	✓	✓
Elicitability		✓		✓
Conditional Elicitability	✓	✓	✓	✓

Table 6.0.1: Properties of variance, value-at-risk, expected shortfall and expectile

1. Value-at-risk

Normal distribution: For $X \sim log\mathcal{N}$ and $Y = \ln(X) \sim \mathcal{N}(\mu, \sigma)$

$$\begin{aligned} p &= F_Y\left(VaR_p(Y)\right) \\ &= P\left(\frac{Y-\mu}{\sigma} \leq \frac{VaR_p(Y)-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{VaR_p(Y)-\mu}{\sigma}\right) \\ VaR_p &= \mu + \sigma\Phi^{-1}(p) \end{aligned} \tag{6.1}$$

Log-normal distribution:

$$VaR_p = \exp\left(\mu + \sigma\Phi^{-1}(p)\right) \tag{6.2}$$

Uniform distribution: For $X \sim \mathcal{U}(a, b)$

$$\begin{aligned} F_X(x) &= \frac{x-a}{b-a} \text{ for } a \leq x \leq b \\ VaR_p &= F^{-1}(p) = a + p(b-a) \text{ for } 0 < p < 1 \end{aligned}$$

Pareto distribution: For $X \sim \mathcal{P}(\alpha, \theta)$

$$\begin{aligned} F_X(x) &= 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha} \\ VaR_p &= F^{-1}(p) = \theta\left((1-p)^{-1/\alpha} - 1\right) \end{aligned}$$

2. Expected shortfall

The average of the $(1-\alpha)$-worst outcomes of the normal probability distribution is given by:

$$\begin{aligned}
TVaR_\alpha(X) &= \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(X)du \\
&= \frac{1}{1-\alpha}\int_\alpha^1 \big(\mu + \sigma\,\Phi^{-1}(1-u)\big)\,du \\
&= \mu + \frac{1}{1-\alpha}\int_\alpha^1 \big(\sigma\,\Phi^{-1}(1-u)\big)\,du \\
&= \mu + \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\Phi^{-1}(1)} \left(\sigma\,\underbrace{\Phi^{-1}\left(1-\Phi(x)\right)}_{=\Phi^{-1}(\Phi(-x))=-x}\right)\phi(x)dx \text{ by use of the change of variable } u=\Phi(x) \\
&= \mu + \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\infty} \left(-x\,\sigma\,\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)\right)dx \\
&= \mu - \frac{1}{1-\alpha}\,\sigma\,\frac{1}{\sqrt{2\pi}}\left[\exp\left(-\frac{x^2}{2}\right)\right]_{\Phi^{-1}(\alpha)}^{\infty} \\
&= \mu - \frac{1}{1-\alpha}\,\sigma\,\frac{1}{\sqrt{2\pi}}\left[0-\exp\left(-\frac{\Phi^{-1}(\alpha)^2}{2}\right)\right] \\
&= \mu + \frac{1}{1-\alpha}\,\sigma\,\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\Phi^{-1}(\alpha)^2}{2}\right) \\
&= \mu + \frac{\sigma}{1-\alpha}\,\phi\left(\Phi^{-1}(\alpha)\right)
\end{aligned} \tag{6.3}$$

Similarly, the average of the $(1-\alpha)$-worst outcomes of the log-normal probability distribution is given by:

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\,e^{\mu+\frac{1}{2}\sigma^2}\left(1-\Phi\left(\Phi^{-1}(\alpha)-\sigma\right)\right) \tag{6.4}$$

3. Expectile

Standard normal distribution: From Lemma 1, we have:

$$e_\tau = \frac{\tau\int_{e_\tau}^{\infty} xdF(x) + (1-\tau)\int_{-\infty}^{e_\tau} xdF(x)}{\tau\int_{e_\tau}^{\infty} dF(x) + (1-\tau)\int_{-\infty}^{e_\tau} dF(x)} \tag{6.5}$$

For $X \sim \mathcal{N}(0,1)$, we then have:

$$\begin{aligned}
e_\tau(X) &= \frac{-\left(1-\tau\right)\phi\left(e_\tau\right)+\tau\phi\left(e_\tau\right)}{\left(1-\tau\right)\Phi\left(e_\tau\right)+\tau\left(1-\Phi\left(e_\tau\right)\right)} \\
&= \frac{2\tau\phi\left(e_\tau\right)-1}{\left(1-2\tau\right)\Phi\left(e_\tau\right)+\tau}
\end{aligned} \tag{6.6}$$

where

$$\int_{-\infty}^{e_\tau} xdF(x) = \int_{-\infty}^{e_\tau} xd\Phi(x) = \int_{-\infty}^{e_\tau} x\phi(x)dx = -\phi\left(e_\tau\right) \tag{6.7}$$

Moreover,

$$\mathbb{E}[(X-x)^+] \sim \frac{1-F_X(x)}{x} \sim \frac{f_X(x)}{x^2} \text{ since } m_X(x) \sim \frac{1-F_X(x)}{f_X(x)} \sim \frac{1}{x} \text{ for } X \sim \mathcal{N}(0,1)$$

Asymptotically, the first order condition for $X \sim \mathcal{N}(0,1)$ is thence given by

$$e_\tau(X)^3 \exp\left(\frac{1}{2}e_\tau(X)^2\right) \sim \frac{1}{\sqrt{2\pi}(1-\tau)}$$

and the explicit solution follows:

$$e_\tau(X) \sim \sqrt{-2\log(1-\tau)} \tag{6.8}$$

<u>Uniform distribution:</u>

$$\begin{aligned} F_X(x) &= \frac{x-a}{b-a} \text{ for } x \in [a,b] \\ f_X(x) &= \frac{1}{b-a} \text{ and } \mathbb{E}[X] = \int_a^b x f_x(x)dx = \frac{1}{2}(a+b) \\ \mathbb{E}[(X-x)^+] &= \int_x^b (1-F_X(x))\,dx = \frac{1}{2(b-a)}\left(x^2 - bx + b^2\right) \\ &= \frac{x^2-2x+1}{2} \text{ for } X \sim \mathcal{U}(0,1) \end{aligned}$$

The first order condition for $X \sim \mathcal{U}(0,1)$ is given by

$$e_\tau(X) - \underbrace{\frac{1}{2}}_{=\mathbb{E}[X]} = \frac{2\tau-1}{1-\tau}\left(\frac{e_\tau(X)^2 - 2e_\tau(X) + 1}{2}\right)$$

and the explicit solution follows:

$$e_\tau(X) = \frac{\tau - \sqrt{\tau-\tau^2}}{2\tau-1}$$

For $X \sim \mathcal{U}(a,b)$, we find:

$$e_\tau(X) = \frac{a\sqrt{1-\tau} + b\sqrt{\tau}}{\sqrt{\tau} + \sqrt{1-\tau}}$$

<u>Pareto distribution:</u>

$$\begin{aligned} F_X(x) &= 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha \text{ for } x \in [\theta,\infty) \\ f_X(x) &= \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \text{ and } \mathbb{E}[X] = \frac{\theta}{\alpha-1} \\ \mathbb{E}[(X-x)^+] &= \frac{\theta^\alpha\,(x+\theta)^{1-\alpha}}{\alpha-1} \end{aligned}$$

The first order condition is thence given by

$$e_\tau(X) - \frac{\theta}{\alpha-1} = \frac{2\tau-1}{1-\tau}\left(\frac{\theta^\alpha\,(e_\tau(X)+\theta)^{-\alpha+1}}{\alpha-1}\right)$$

For $\alpha = 2$, the explicit solution follows:

$$e_\tau(X) = \theta\frac{\sqrt{\tau}}{\sqrt{1-\tau}}$$

Figures

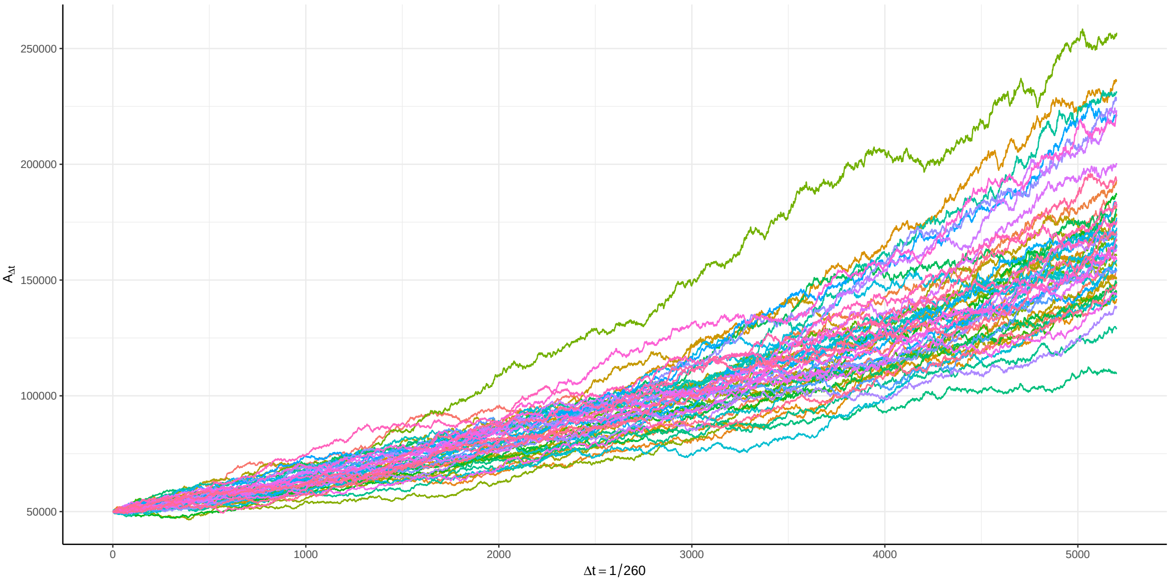

Figure 6.0.1: 50 simulations of the trajectories of the asset process $(A_t)_{t\geq 0}$ with a daily (business day) time step of $\Delta t = \frac{1}{260}$ and initial asset value of $A_0 = \$50,000$

Volatility sensitivity

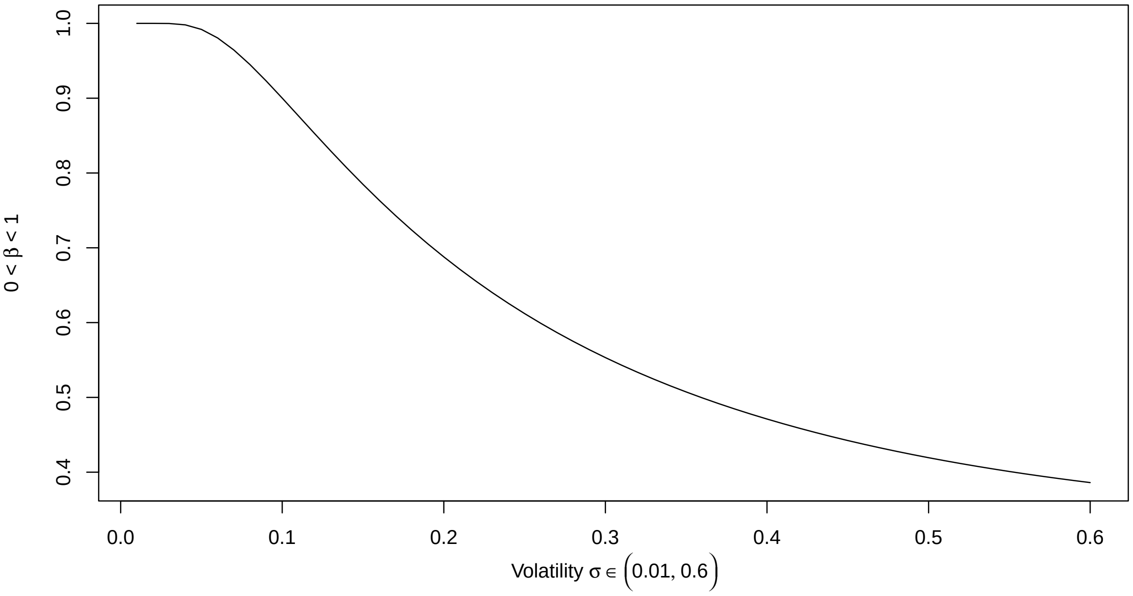

Figure 6.0.2: Sensitivity of the participation level β (see formula (5.2)) to a change in the volatility σ of the asset process.

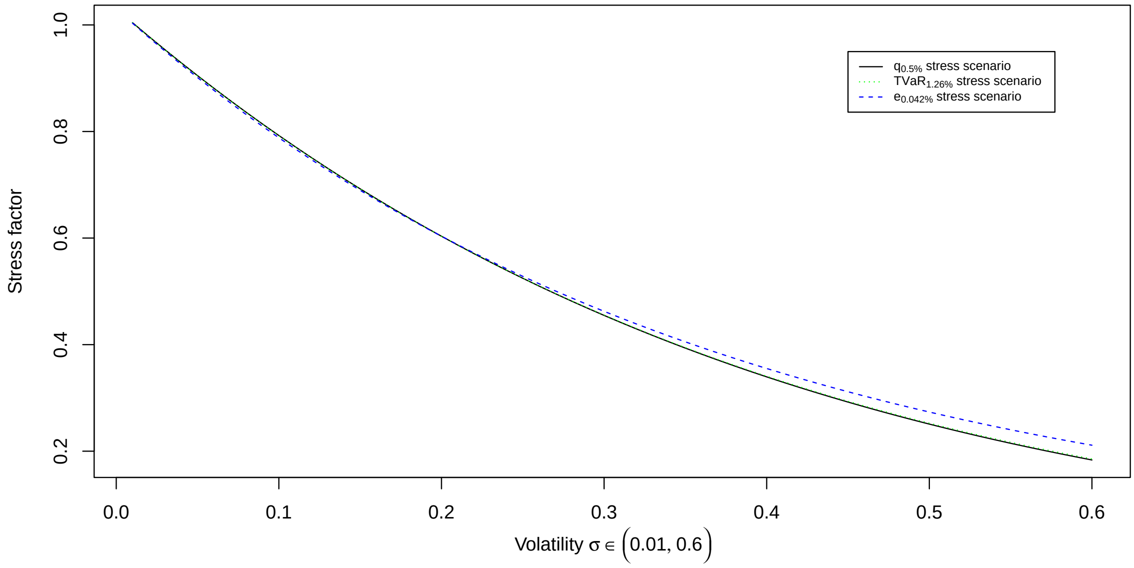

Figure 6.0.3: Sensitivity of the stress factors to a change in the volatility σ of the asset process.

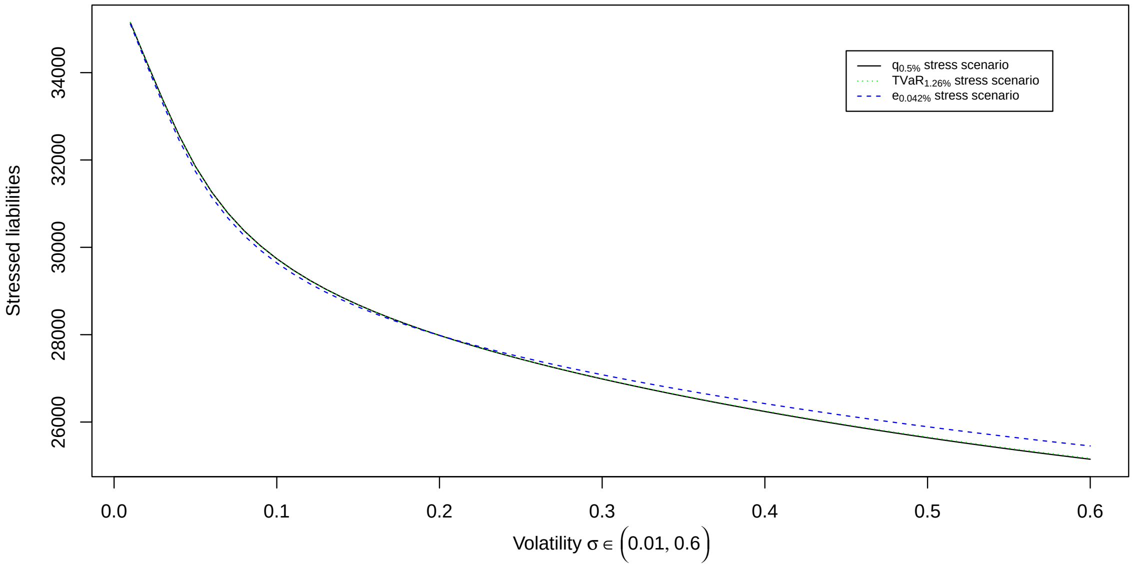

Figure 6.0.4: Sensitivity of the stressed liabilities $L_0^{stressed}$ to a change in the volatility σ of the asset process.

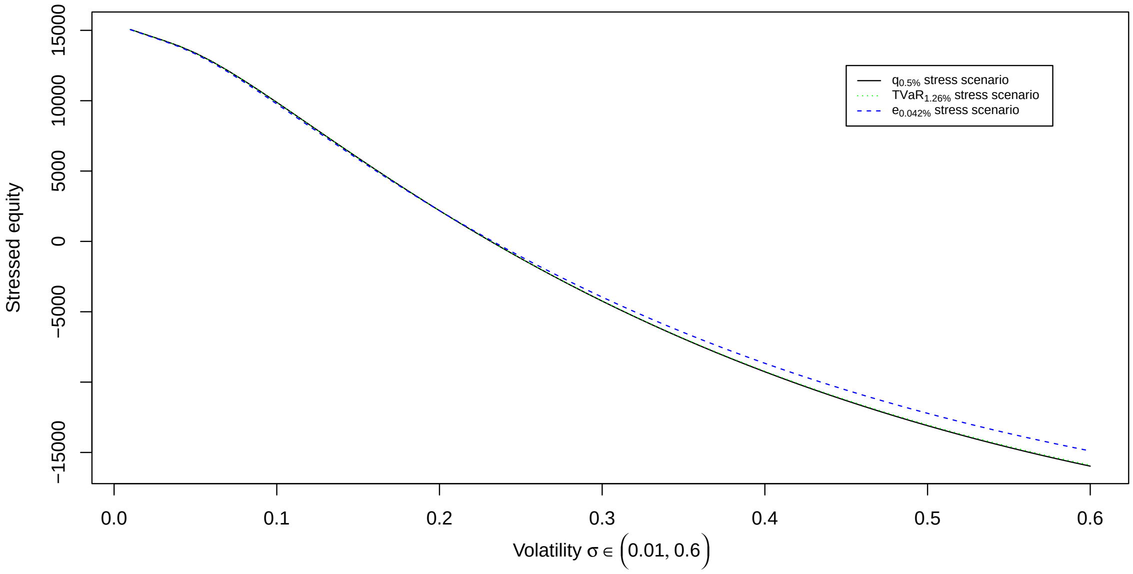

Figure 6.0.5: Sensitivity of the stressed equity $E_0^{stressed}$ to a change in the volatility σ of the asset process.

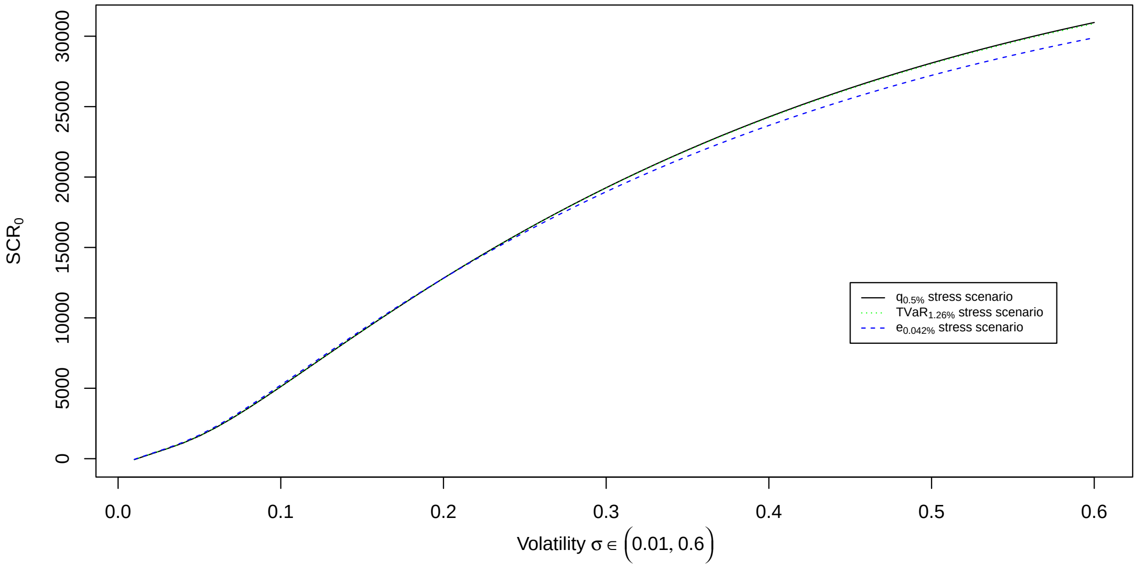

Figure 6.0.6: Sensitivity of the SCR to a change in the volatility σ of the asset process.

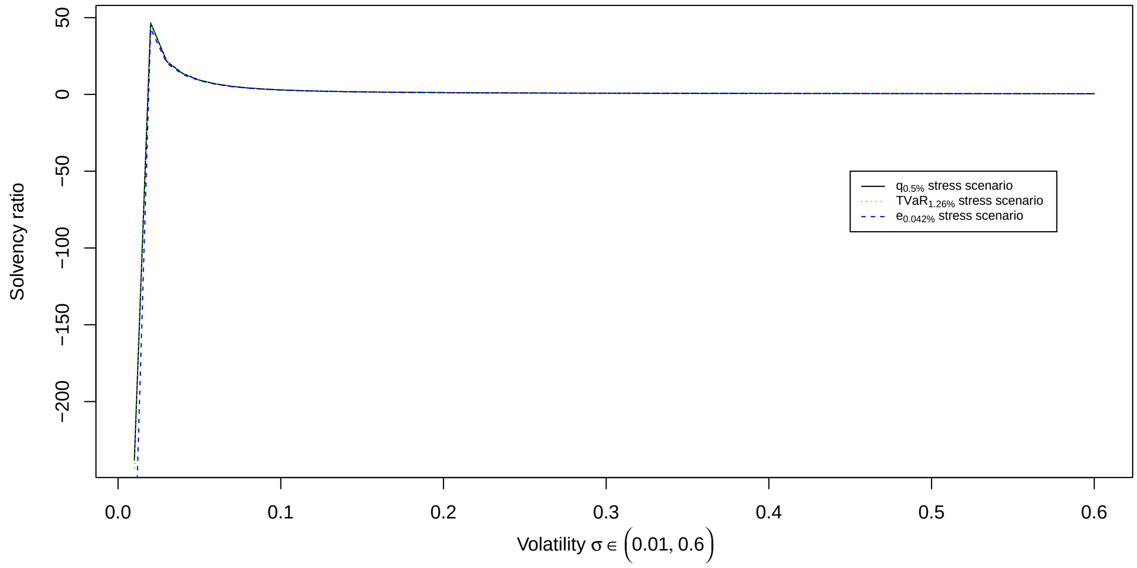

Figure 6.0.7: Sensitivity of the solvency ratio to a change in the volatility σ of the asset process.

Volatility σ	β	$\mu = \lambda - \frac{1}{2}\sigma^2$	stress factor $q_{0.5\%}$	stress factor $TVaR_{1.26\%}$	stress factor $e_{0.042\%}$	$E_0^{stressed}(q_{0.5\%})$	$E_0^{stressed}(TVaR_{1.26\%})$	$E_0^{stressed}(e_{0.042\%})$
0.01	1	0.02995	1.004201	1.004107	1.003202	15063.01	15061.61	15048.04
0.02	0.9999998	0.0298	0.978518	0.978346	0.976672	14677.76	14675.18	14650.07
0.03	0.999864	0.02955	0.953396	0.953159	0.950844	14297.23	14293.65	14258.61
0.04	0.998052	0.0292	0.928827	0.928538	0.925701	13883.87	13879.23	13833.66
0.05	0.991952	0.02875	0.9048	0.904471	0.901225	13387.28	13381.41	13323.43
0.06	0.980568	0.0282	0.881307	0.880948	0.8774	12798.57	12791.42	12720.43
0.07	0.964514	0.02755	0.858337	0.85796	0.854208	12134.55	12126.22	12043.23
0.08	0.944993	0.0268	0.835883	0.835496	0.831632	11416.33	11407.04	11314.13
0.09	0.923198	0.02595	0.813935	0.813547	0.809658	10661.85	10651.88	10551.75
0.1	0.90011	0.025	0.792484	0.792103	0.788269	9884.804	9874.448	9770.068
0.11	0.876468	0.02395	0.771521	0.771154	0.76745	9095.287	9084.851	8979.243
0.12	0.852802	0.0228	0.751038	0.750692	0.747187	8300.668	8290.445	8186.596
0.13	0.829478	0.02155	0.731025	0.730707	0.727464	7506.339	7496.612	7397.396
0.14	0.806738	0.0202	0.711474	0.711189	0.708268	6716.269	6707.302	6615.449
0.15	0.784741	0.01875	0.692377	0.69213	0.689585	5933.397	5925.433	5843.506
0.16	0.763577	0.0172	0.673725	0.67352	0.671402	5159.903	5153.167	5083.555
0.17	0.743298	0.01555	0.65551	0.655352	0.653705	4397.412	4392.107	4337.02
0.18	0.72392	0.0138	0.637724	0.637616	0.636481	3647.129	3643.439	3604.91
0.19	0.705442	0.01195	0.620359	0.620303	0.619719	2909.944	2908.035	2887.925
0.2	0.687846	0.01	0.603405	0.603406	0.603406	2186.507	2186.524	2186.525
0.21	0.671106	0.00795	0.586857	0.586916	0.58753	1477.279	1479.351	1500.991
0.22	0.65519	0.0058	0.570705	0.570824	0.57208	782.5758	786.8137	831.4695
0.23	0.640063	0.00355	0.554943	0.555124	0.557044	102.599	109.0989	177.9956
0.24	0.625688	0.0012	0.539561	0.539806	0.542411	-562.541	-553.698	-459.478
0.25	0.612027	-0.00125	0.524554	0.524863	0.528171	-1212.8	-1201.55	-1081.06
0.26	0.599044	-0.0038	0.509913	0.510287	0.514314	-1848.21	-1834.5	-1686.91
0.27	0.586703	-0.00645	0.495631	0.49607	0.500829	-2468.83	-2452.61	-2277.24
0.28	0.574969	-0.0092	0.481701	0.482206	0.487705	-3074.76	-3056.01	-2852.26
0.29	0.563809	-0.01205	0.468115	0.468687	0.474935	-3666.14	-3644.84	-3412.24
0.3	0.553192	-0.015	0.454868	0.455505	0.462507	-4243.13	-4219.27	-3957.44
0.31	0.543087	-0.01805	0.441951	0.442653	0.450414	-4805.9	-4779.49	-4488.15
0.32	0.533466	-0.0212	0.429357	0.430125	0.438645	-5354.65	-5325.69	-5004.67
0.33	0.524304	-0.02445	0.417081	0.417913	0.427193	-5889.59	-5858.1	-5507.29
0.34	0.515574	-0.0278	0.405116	0.406011	0.416049	-6410.93	-6376.95	-5996.32
0.35	0.507253	-0.03125	0.393454	0.394411	0.405204	-6918.91	-6882.45	-6472.06
0.36	0.49932	-0.0348	0.38209	0.383108	0.394651	-7413.75	-7374.86	-6934.83
0.37	0.491753	-0.03845	0.371017	0.372095	0.384382	-7895.69	-7854.42	-7384.94
0.38	0.484534	-0.0422	0.360229	0.361366	0.374389	-8364.97	-8321.37	-7822.66
0.39	0.477643	-0.04605	0.34972	0.350913	0.364664	-8821.84	-8775.96	-8248.34
0.4	0.471065	-0.05	0.339483	0.340732	0.355201	-9266.54	-9218.44	-8662.26
0.41	0.464782	-0.05405	0.329513	0.330816	0.345993	-9699.32	-9649.05	-9064.71
0.42	0.458781	-0.0582	0.319804	0.321159	0.337031	-10120.4	-10068.1	-9455.99
0.43	0.453047	-0.06245	0.310349	0.311755	0.328311	-10530.1	-10475.7	-9836.39
0.44	0.447566	-0.0668	0.301145	0.302599	0.319825	-10928.6	-10872.2	-10206.2
0.45	0.442326	-0.07125	0.292184	0.293685	0.311567	-11316.1	-11257.9	-10565.7
0.46	0.437316	-0.0758	0.283461	0.285007	0.303531	-11692.9	-11632.9	-10915.1
0.47	0.432525	-0.08045	0.274971	0.27656	0.29571	-12059.3	-11997.5	-11254.7
0.48	0.427941	-0.0852	0.266709	0.268339	0.2881	-12415.4	-12352	-11584.9
0.49	0.423556	-0.09005	0.258669	0.260338	0.280694	-12761.5	-12696.5	-11905.7
0.5	0.41936	-0.095	0.250846	0.252553	0.273486	-13097.9	-13031.4	-12217.6
0.51	0.415344	-0.10005	0.243236	0.244977	0.266472	-13424.7	-13356.8	-12520.7
0.52	0.4115	-0.1052	0.235833	0.237608	0.259646	-13742.2	-13673	-12815.2
0.53	0.407821	-0.11045	0.228632	0.230438	0.253003	-14050.7	-13980.2	-13101.5
0.54	0.404299	-0.1158	0.22163	0.223465	0.246538	-14350.2	-14278.6	-13379.6
0.55	0.400927	-0.12125	0.21482	0.216682	0.240246	-14641.1	-14568.4	-13650
0.56	0.397698	-0.1268	0.208198	0.210086	0.234123	-14923.6	-14849.8	-13912.7
0.57	0.394607	-0.13245	0.201761	0.203672	0.228163	-15197.8	-15123.1	-14168
0.58	0.391647	-0.1382	0.195503	0.197436	0.222363	-15464	-15388.4	-14416.1
0.59	0.388812	-0.14405	0.18942	0.191372	0.216718	-15722.4	-15646	-14657.2
0.6	0.386098	-0.15	0.183508	0.185478	0.211224	-15973.2	-15896.1	-14891.5

Table 6.0.2: Sensitivity to the volatility of the asset process (table 1).

Volatility σ	$L_0^{stressed}\,(\mathrm{q}_{0.5\%})$	$L_0^{stressed}\,(\mathrm{TVaR}_{1.26\%})$	$L_0^{stressed}\,(\mathrm{e}_{0.042\%})$	$\mathrm{SCR}(\mathrm{q}_{0.5\%})$	$\mathrm{SCR}(\mathrm{TVaR}_{1.26\%})$	$\mathrm{SCR}(\mathrm{e}_{0.042\%})$	S2 ratio $\mathrm{q}_{0.5\%}$	S2 ratio $\mathrm{TVaR}_{1.26\%}$	S2 ratio $\mathrm{e}_{0.042\%}$
0.01	35147.02	35143.75	35112.09	-63.0076	-61.6092	-48.0372	-238.067	-243.47	-312.258
0.02	34248.12	34242.1	34183.52	322.2417	324.819	349.9281	46.54891	46.17956	42.86594
0.03	33372.57	33364.31	33283.59	702.7669	706.3487	741.3854	21.3442	21.23597	20.23239
0.04	32557.46	32547.65	32451.39	1116.131	1120.767	1166.338	13.43928	13.38369	12.86076
0.05	31852.72	31842.12	31737.84	1612.723	1618.592	1676.569	9.301037	9.267314	8.946845
0.06	31266.75	31255.98	31149.57	2201.427	2208.581	2279.571	6.813763	6.791692	6.580186
0.07	30782.31	30771.76	30667.16	2865.445	2873.775	2956.773	5.234789	5.219615	5.073098
0.08	30377.83	30367.75	30267.48	3583.671	3592.96	3685.867	4.185652	4.174831	4.069599
0.09	30034.91	30025.46	29931.15	4338.153	4348.125	4448.248	3.457693	3.449763	3.372114
0.1	29739.4	29730.69	29643.38	5115.196	5125.552	5229.932	2.932439	2.926514	2.868106
0.11	29480.77	29472.87	29393.26	5904.713	5915.149	6020.757	2.540344	2.535862	2.491381
0.12	29251.21	29244.15	29172.73	6699.332	6709.555	6813.404	2.239029	2.235618	2.201543
0.13	29044.9	29038.71	28975.8	7493.661	7503.388	7602.604	2.001692	1.999097	1.973008
0.14	28857.44	28852.13	28797.95	8283.731	8292.698	8384.551	1.810778	1.80882	1.789005
0.15	28685.46	28681.05	28635.75	9066.603	9074.567	9156.494	1.654423	1.652971	1.638182
0.16	28526.37	28522.84	28486.53	9840.097	9846.833	9916.445	1.524375	1.523332	1.512639
0.17	28378.11	28375.48	28348.21	10602.59	10607.89	10662.98	1.414749	1.414041	1.406736
0.18	28239.08	28237.34	28219.15	11352.87	11356.56	11395.09	1.321252	1.320822	1.316356
0.19	28107.99	28107.12	28098.03	12090.06	12091.97	12112.08	1.240689	1.240493	1.238434
0.2	27983.77	27983.77	27983.77	12813.49	12813.48	12813.48	1.170641	1.170643	1.170643
0.21	27865.57	27866.44	27875.51	13522.72	13520.65	13499.01	1.109244	1.109414	1.111193
0.22	27752.69	27754.4	27772.52	14217.42	14213.19	14168.53	1.055043	1.055358	1.058684
0.23	27644.53	27647.08	27674.19	14897.4	14890.9	14822	1.006887	1.007327	1.012009
0.24	27540.6	27543.98	27580.04	15562.54	15553.7	15459.48	0.963853	0.964401	0.970279
0.25	27440.49	27444.68	27489.63	16212.8	16201.55	16081.06	0.925195	0.925837	0.932774
0.26	27343.85	27348.83	27402.61	16848.21	16834.5	16686.91	0.890302	0.891027	0.898908
0.27	27250.37	27256.13	27318.66	17468.83	17452.61	17277.24	0.858672	0.85947	0.868194
0.28	27159.8	27166.32	27237.53	18074.76	18056.01	17852.26	0.829886	0.830748	0.84023
0.29	27071.91	27079.18	27158.97	18666.14	18644.84	18412.24	0.803594	0.804512	0.814676
0.3	26986.51	26994.51	27082.8	19243.13	19219.27	18957.44	0.779499	0.780467	0.791246
0.31	26903.43	26912.15	27008.84	19805.9	19779.49	19488.15	0.75735	0.758361	0.769698
0.32	26822.52	26831.94	26936.93	20354.65	20325.69	20004.67	0.736932	0.737982	0.749825
0.33	26743.66	26753.76	26866.94	20889.59	20858.1	20507.29	0.718061	0.719145	0.731447
0.34	26666.73	26677.49	26798.76	21410.93	21376.95	20996.32	0.700577	0.701691	0.714411
0.35	26591.62	26603.03	26732.28	21918.91	21882.45	21472.06	0.684341	0.685481	0.698582
0.36	26518.26	26530.29	26667.39	22413.75	22374.86	21934.83	0.669232	0.670395	0.683844
0.37	26446.55	26459.19	26604.02	22895.69	22854.42	22384.94	0.655145	0.656328	0.670093
0.38	26376.43	26389.65	26542.1	23364.97	23321.37	22822.66	0.641987	0.643187	0.657241
0.39	26307.83	26321.62	26481.55	23821.84	23775.96	23248.34	0.629674	0.630889	0.645207
0.4	26240.69	26255.04	26422.31	24266.54	24218.44	23662.26	0.618135	0.619363	0.633921
0.41	26174.96	26189.84	26364.33	24699.32	24649.05	24064.71	0.607304	0.608543	0.623319
0.42	26110.6	26125.99	26307.56	25120.42	25068.05	24455.99	0.597124	0.598371	0.613347
0.43	26047.55	26063.44	26251.94	25530.08	25475.69	24836.39	0.587542	0.588797	0.603953
0.44	25985.78	26002.15	26197.44	25928.56	25872.21	25206.19	0.578513	0.579773	0.595092
0.45	25925.26	25942.09	26144.01	26316.08	26257.86	25565.66	0.569994	0.571258	0.586724
0.46	25865.95	25883.21	26091.63	26692.91	26632.87	25915.09	0.561947	0.563214	0.578813
0.47	25807.81	25825.49	26040.26	27059.26	26997.5	26254.73	0.554339	0.555607	0.571326
0.48	25750.82	25768.91	25989.86	27415.39	27351.97	26584.86	0.547138	0.548407	0.564231
0.49	25694.96	25713.43	25940.42	27761.52	27696.52	26905.72	0.540316	0.541584	0.557502
0.5	25640.2	25659.03	25891.89	28097.88	28031.39	27217.57	0.533848	0.535114	0.551115
0.51	25586.51	25605.68	25844.26	28424.71	28356.81	27520.66	0.52771	0.528973	0.545045
0.52	25533.88	25553.37	25797.52	28742.23	28672.99	27815.2	0.52188	0.52314	0.539274
0.53	25482.28	25502.08	25751.62	29050.66	28980.17	28101.45	0.516339	0.517595	0.53378
0.54	25431.7	25451.79	25706.56	29350.22	29278.55	28379.64	0.511069	0.51232	0.528548
0.55	25382.11	25402.47	25662.31	29641.12	29568.36	28649.99	0.506054	0.507299	0.52356
0.56	25333.5	25354.12	25618.86	29923.59	29849.81	28912.71	0.501277	0.502516	0.518803
0.57	25285.86	25306.71	25576.19	30197.82	30123.1	29168.02	0.496725	0.497957	0.514262
0.58	25239.16	25260.23	25534.29	30464.03	30388.45	29416.13	0.492384	0.493609	0.509924
0.59	25193.4	25214.66	25493.13	30722.41	30646.04	29657.23	0.488243	0.48946	0.505779
0.6	25148.56	25170	25452.71	30973.16	30896.08	29891.52	0.48429	0.485498	0.501815

Table 6.0.3: Sensitivity to the volatility of the asset process (table 2).

Market rate sensitivity

r	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
β	0	0.440082	0.687846	0.827465	0.905847	0.949476	0.97344	0.986371	0.993195	0.996704
$L_0^{stressed}\,(\mathrm{q}_{0.5\%})$	35000	30980.82	27983.77	25790.12	24218.61	23120	22372.77	21879.76	21565.09	21371.31
$L_0^{stressed}\,(\mathrm{TVaR}_{1.26\%})$	35000	30980.82	27983.77	25790.13	24218.62	23120.01	22372.79	21879.77	21565.11	21371.33
$L_0^{stressed}\,(\mathrm{e}_{0.042\%})$	35000	30980.82	27983.77	25790.13	24218.62	23120.01	22372.79	21879.77	21565.11	21371.33
$E_0^{stressed}\,(\mathrm{q}_{0.5\%})$	-4829.73	-810.545	2186.507	4380.159	5951.662	7050.276	7797.503	8290.518	8605.184	8798.962
$E_0^{stressed}\,(\mathrm{TVaR}_{1.26\%})$	-4829.7	-810.525	2186.524	4380.173	5951.674	7050.286	7797.513	8290.526	8605.192	8798.97
$E_0^{stressed}\,(\mathrm{e}_{0.042\%})$	-4829.7	-810.525	2186.525	4380.174	5951.674	7050.287	7797.513	8290.527	8605.192	8798.97
$\mathrm{SCR}(\mathrm{q}_{0.5\%})$	19829.73	15810.55	12813.49	10619.84	9048.338	7949.724	7202.497	6709.482	6394.816	6201.038
$\mathrm{SCR}(\mathrm{TVaR}_{1.26\%})$	19829.7	15810.53	12813.48	10619.83	9048.326	7949.714	7202.487	6709.474	6394.808	6201.03
$\mathrm{SCR}(\mathrm{e}_{0.042\%})$	19829.7	15810.52	12813.48	10619.83	9048.326	7949.713	7202.487	6709.473	6394.808	6201.03
S2 ratio $\mathrm{q}_{0.5\%}$	0.75644	0.948734	1.170641	1.412451	1.657763	1.886858	2.082611	2.235642	2.34565	2.41895
S2 ratio $\mathrm{TVaR}_{1.26\%}$	0.756441	0.948735	1.170643	1.412452	1.657765	1.88686	2.082614	2.235645	2.345653	2.418953
S2 ratio $\mathrm{e}_{0.042\%}$	0.756441	0.948735	1.170643	1.412452	1.657765	1.88686	2.082614	2.235645	2.345653	2.418953

Table 6.0.4: Sensitivity to the market risk-free interest rate.

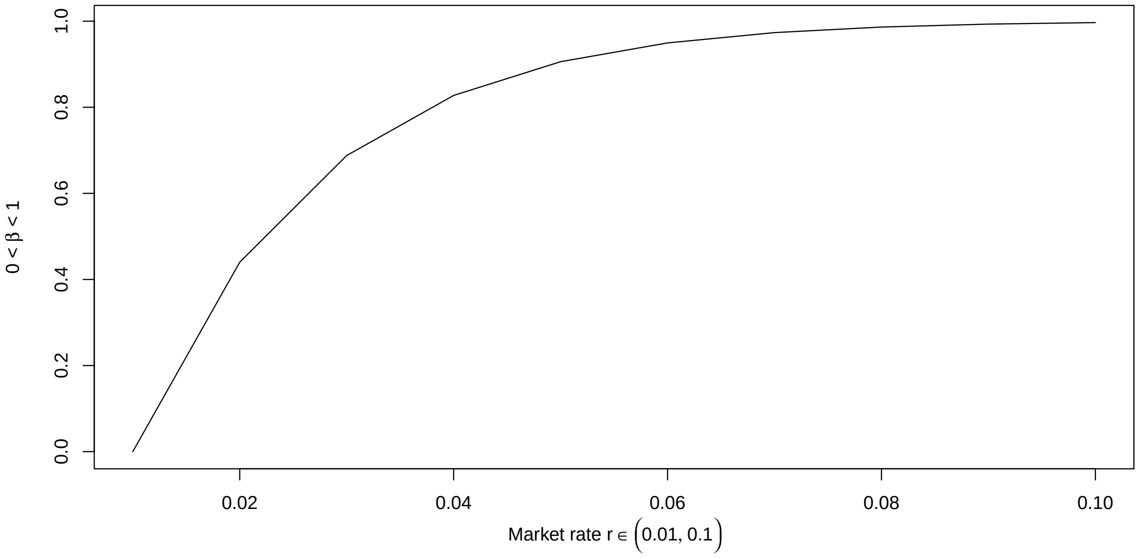

Figure 6.0.8: Sensitivity of the participation level β (see formula (5.2)) to a change in the market risk-free interest rate r.

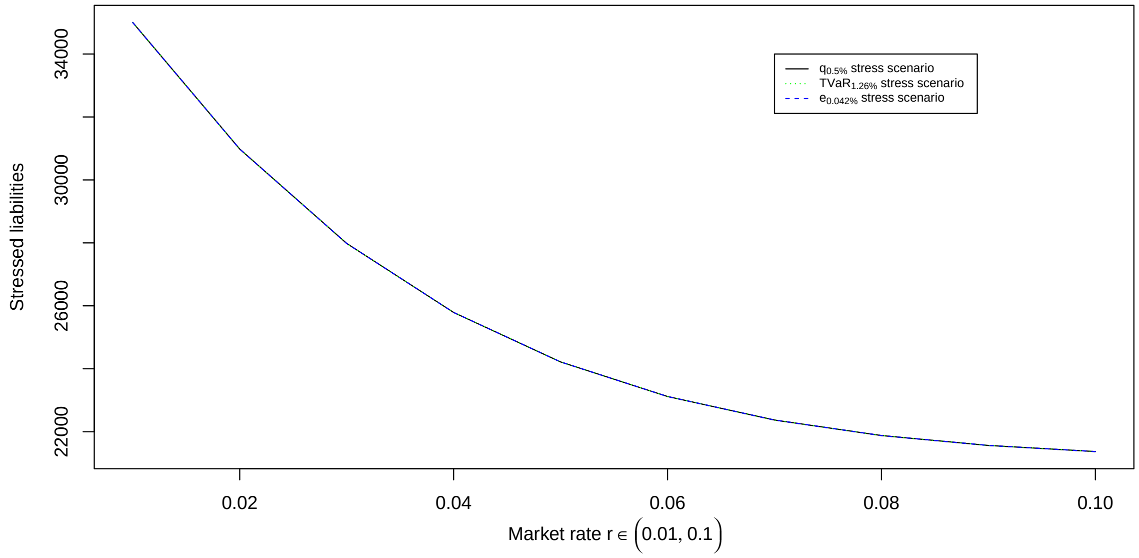

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Figure 6.0.9: Sensitivity of the stressed liabilities to a change in the market risk-free interest rate r.

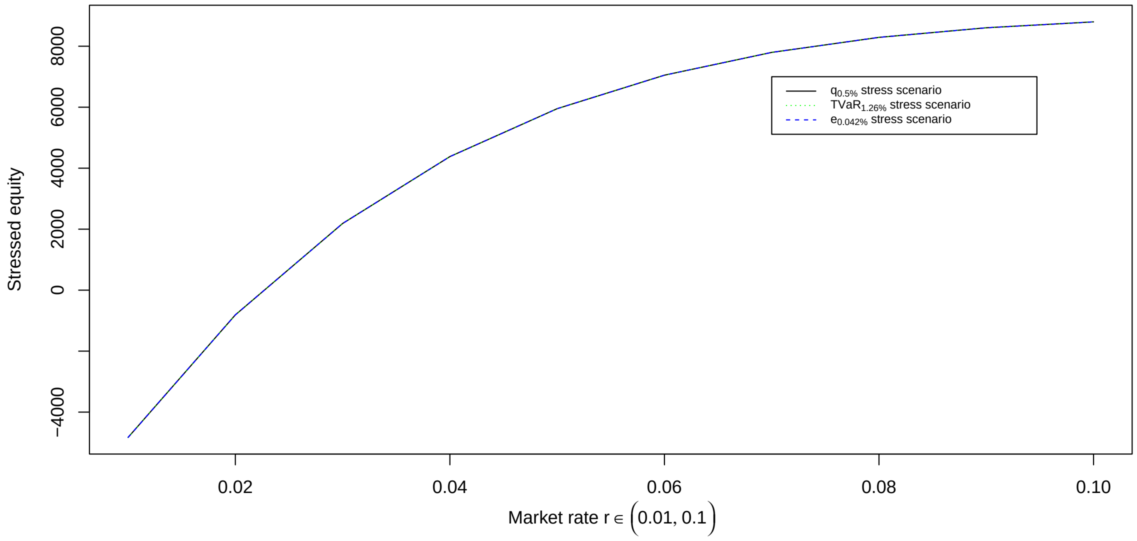

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Figure 6.0.10: Sensitivity of the stressed equity to a change in the market risk-free interest rate r.

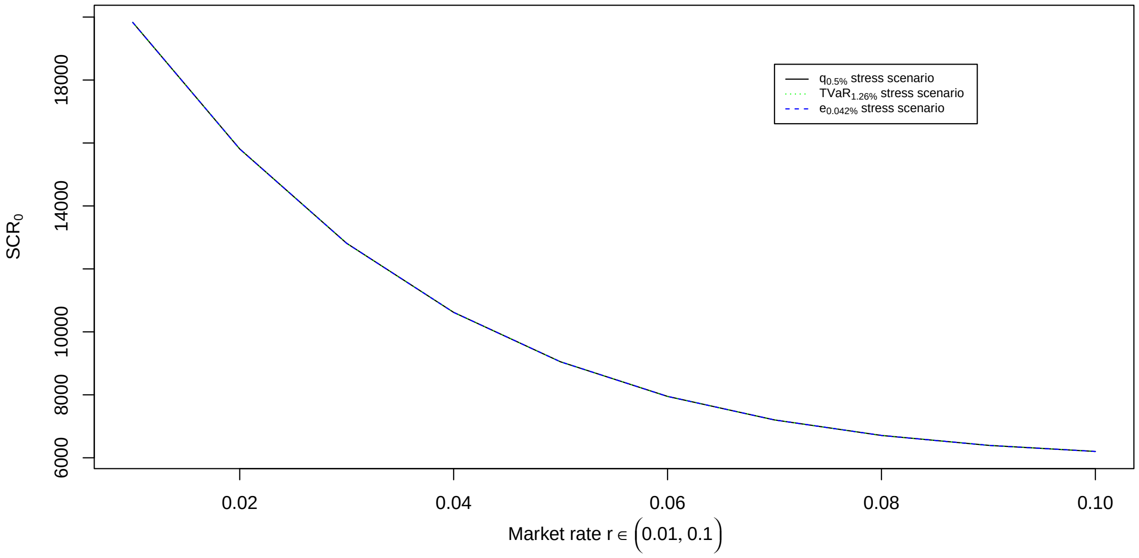

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Figure 6.0.11: Sensitivity of the SCR to a change in the market risk-free interest rate r.

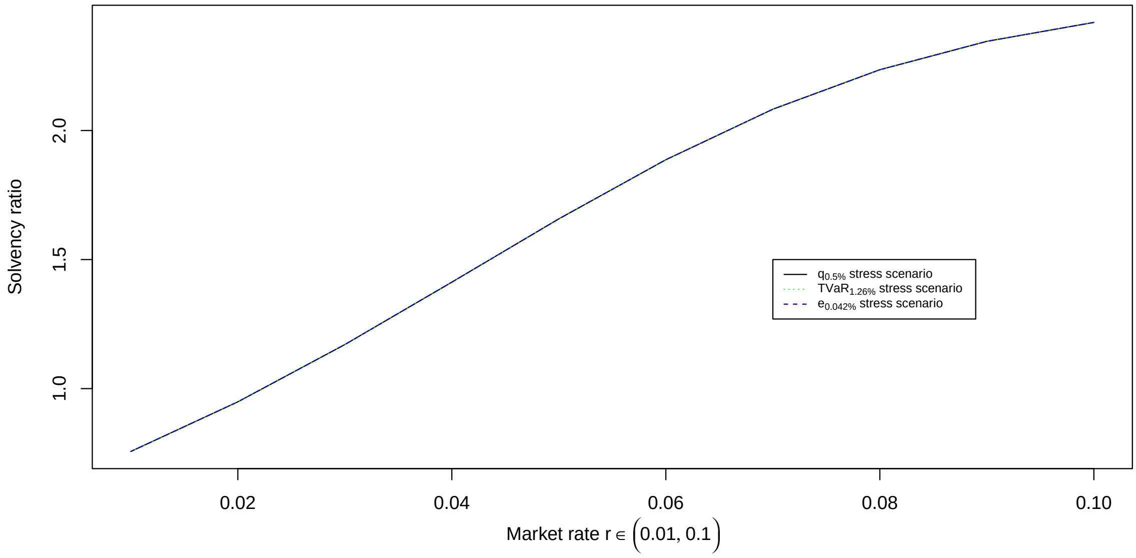

Figure 6.0.12: Sensitivity of the solvency ratio to a change in the market risk-free interest rate r.

Guaranteed rate sensitivity

r_G	0	0.005	0.01	0.015	0.02	0.025	0.03
L_G	35000	38680.98	42749.1	47245.06	52213.86	57705.24	63774.16
K	50000	55258.55	61070.14	67492.94	74591.23	82436.06	91105.94
β	0.827465	0.767617	0.687846	0.581593	0.440082	0.25152	6.02E-16
$L_0^{stressed}\,(\mathrm{q}_{0.5\%})$	25790.12	26798.65	27983.77	29369.15	30980.82	32847.33	35000
$L_0^{stressed}\,(\mathrm{TVaR}_{1.26\%})$	25790.13	26798.66	27983.77	29369.16	30980.82	32847.33	35000
$L_0^{stressed}\,(\mathrm{e}_{0.042\%})$	25790.13	26798.66	27983.77	29369.16	30980.82	32847.33	35000
$E_0^{stressed}\,(\mathrm{q}_{0.5\%})$	4380.159	3371.623	2186.507	801.1234	-810.545	-2677.05	-4829.73
$E_0^{stressed}\,(\mathrm{TVaR}_{1.26\%})$	4380.173	3371.639	2186.524	801.1417	-810.525	-2677.03	-4829.7
$E_0^{stressed}\,(\mathrm{e}_{0.042\%})$	4380.174	3371.639	2186.525	801.1423	-810.525	-2677.03	-4829.7
$\mathrm{SCR}(\mathrm{q}_{0.5\%})$	10619.84	11628.38	12813.49	14198.88	15810.55	17677.05	19829.73
$\mathrm{SCR}(\mathrm{TVaR}_{1.26\%})$	10619.83	11628.36	12813.48	14198.86	15810.53	17677.03	19829.7
$\mathrm{SCR}(\mathrm{e}_{0.042\%})$	10619.83	11628.36	12813.48	14198.86	15810.52	17677.03	19829.7
S2 ratio $\mathrm{q}_{0.5\%}$	1.412451	1.289948	1.170641	1.056422	0.948734	0.848558	0.75644
S2 ratio $\mathrm{TVaR}_{1.26\%}$	1.412452	1.28995	1.170643	1.056423	0.948735	0.848559	0.756441
S2 ratio $\mathrm{e}_{0.042\%}$	1.412452	1.28995	1.170643	1.056423	0.948735	0.848559	0.756441

Table 6.0.5: Sensitivity to the guaranteed rate r_G.

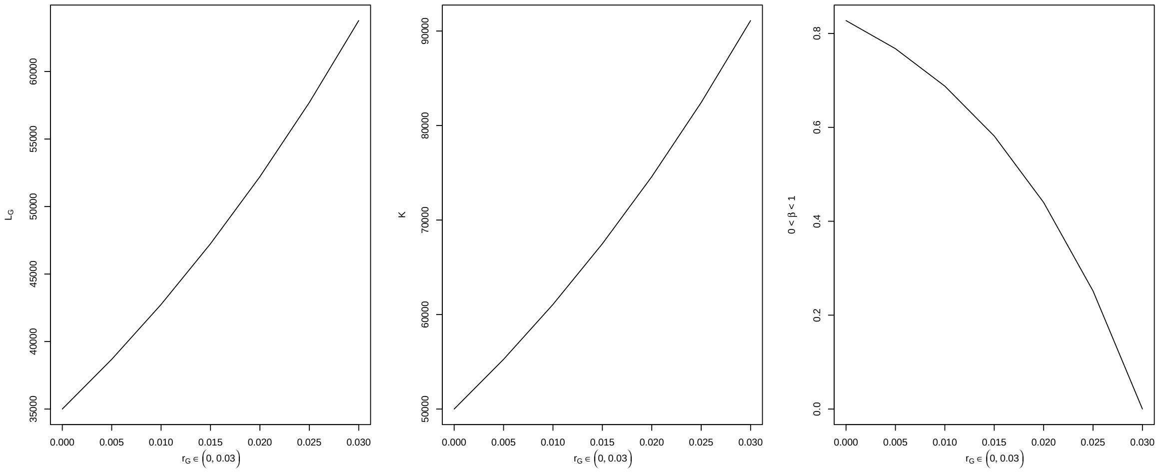

Figure 6.0.13: Sensitivity of guaranteed amount L_G, the strike $K = \frac{L_G}{a}$ and the participation level β to a change in the guaranteed interest rate r_G.

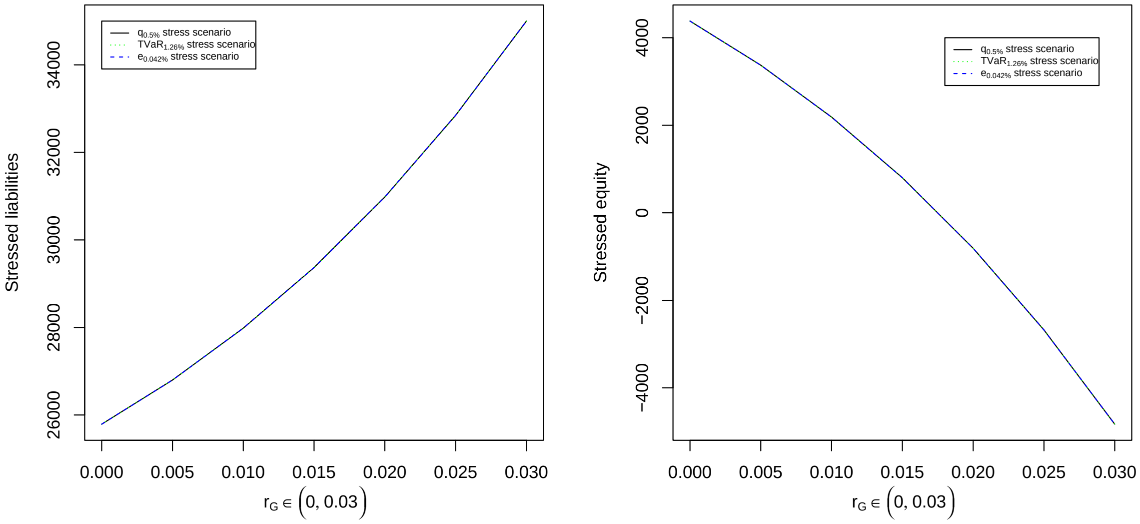

Figure 6.0.14: Sensitivity of the stressed liabilities $L_0^{stressed}$ and equity $E_0^{stressed}$ to a change in the guaranteed interest rate r_G.

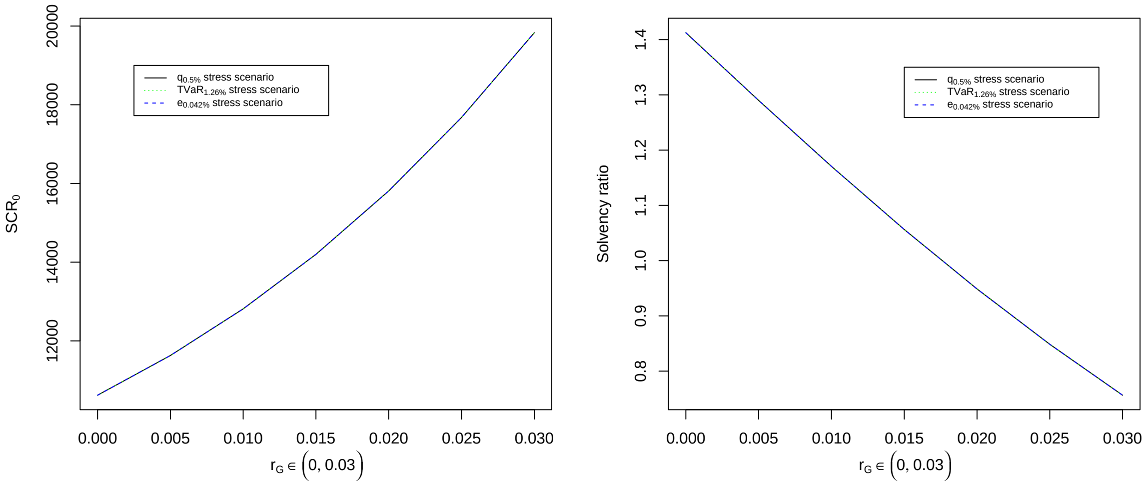

Figure 6.0.15: Sensitivity of the SCR and solvency ratio to a change in the guaranteed interest rate r_G.