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Introduction

An important aspect of group theory is the study of group actions, and in particular group actions by linear transformations of vector spaces, which we call linear representations. A theme which I find particularly interesting is the study of the relationship between properties of representations and the algebraic structure of groups.

The representation theory of finite groups finds its origins in the work of Frobenius, at the start of the 20^{th} century, and it has since received a lot of attention. As a result, the general theory is relatively mature and well understood. For instance, the classification of complex linear representations of finite groups reduces to the classification of so-called irreducible representations, about which much can be said. It is in fact in this way that I was seriously introduced to group theory, and that I grew interest for the subject. On the other hand, infinite groups can display a variety of different pathological behaviors, which suggests that a general theory for linear representations of infinite groups can be very intricate. My first step towards an understanding of the representation theory of discrete infinite groups involves the study of Kazhdan's property (T), which is the central concept of my thesis.

Kazhdan's property (T) is a rigidity property of unitary representations of groups, ie. linear representations which additionally preserve some Hilbert space structure. More precisely, we say that a group has property (T) if, whenever a unitary representation almost has invariant vectors, then it has non-trivial invariant vectors. This property was introduced by Kazhdan in [9] as a tool to study lattices in real Lie groups, but revealed to be a fundamental concept in group theory. Indeed, property (T) has strong implications on the structure of groups. For instance, for discrete groups, property (T) implies finite generation. Trying to understand which finitely generated groups have property (T) or not has been my main source of motivation for this project.

In Chapter 1, we will review some necessary background for subsequent chapters. First, we discuss group presentations, and the construction of the Cayley graph. Then, we introduce unitary representations, along with two fundamental constructions : direct sums of representations and permutation representations. Then, we consider unitary representations of finite groups, and we state some classical results from the general theory. Finally, we describe the construction of induced representations, which will be useful later.

In Chapter 2, we introduce the definition of Kazhdan's property (T). We will see that all finite groups have property (T), but not the infinite group $\mathbb{Z}$. Then, we will give an equivalent definition of Kazhdan's property, which involves the notion of Kazhdan constants. An important consequence of property (T) on the structure of discrete groups is finite generation. We will also see that property (T) is stable under quotients and inherits to finite index subgroups. These necessary conditions yield several examples of infinite groups which do not have property (T). The search for infinite groups with property (T) will be an additional motivation for the following chapters.

In Chapter 3, we will describe two criteria for property (T). First, we discuss a spectral criterion, which involves the spectral gap of the Laplace operator on the Cayley graph. Second, we give a geometric criterion, which applies to groups generated by a finite number of finite subgroups, and involves the computation of the representation angle between subgroups. It is remarkable that, in this case, the local information about the representation theory of finite subgroups contains information about representations of the whole (possibly infinite) group. This suggests that the computation of the representation angle for finite groups can provide many examples of infinite groups with property (T).

In Chapter 4, we present the result of some research that has been done in the context of this thesis. It involves the computation of the representation angle in a specific family of finite p-groups. Our estimates improve those which can be found in the current published literature, and we will see, using the aforementioned geometric criterion, that they can be used to provide new examples of infinite groups with property (T).

My personal contributions are multiple. First, I have extracted the most essential ideas, in such a way that the first three chapters should be accessible to any graduate student in mathematics. Namely, I have chosen to avoid any topological considerations, although it should be noted that property (T) is relevant in the study of locally compact groups. In particular, the proofs have been adapted to the discrete case. I have also included many examples and illustrations which I find helpful. And, naturally, I have contributed to solving the research problem, which has been an exciting experience.

It is with great pleasure that I acknowledge my advisor Pierre-Emmanuel Caprace, and both their direct and indirect contributions to this work. I am very grateful for their support and their availability, both in person and online, all throughout this complicated year. I have learned a lot from our interactions, and I look forward to having more conversations soon (hopefully without masks !).

Contents

Chapter 1

Preliminaries

In this chapter, we will give a brief review of some fundamental concepts in group theory and the representation theory of groups. We will not systematically give proofs of all the results which we present here, but we refer to [13] for more detail on free groups and group presentations, and to [14] for an introduction to the representation theory of finite groups.

1.1 Group presentations

Group presentations constitute one way to construct groups, and we will use them all throughout this text to give examples. We start by introducing free groups.

Definition 1.1.1. Let F be a group and $X \subset F$ a subset. We say that F is a *free group* on X if it satisfies the following universal property : for any group G and any function $f : X \to G$, there exists a unique extension $f : F \to G$ which is a group homomorphism.

Example 1.1.2. The group $\mathbb{Z}$ is free on $X = \{1\}$. Indeed, for any group G and any assignment $f(1) = g \in G$, there is a unique extension :

$$f(n) = g^n,$$

which defines a homomorphism $f : \mathbb{Z} \to G$.

We describe a standard construction which yields free groups. More detail can be found in [13, Chapter 12].

Definition 1.1.3. Let $X = \{x_1, \dots, x_n\}$ be a finite set. Denote by W the set of finite words with letters in $X \cup X^{-1}$, where $X^{-1} = \left\{x_1^{-1}, \dots, x_n^{-1}\right\}$ denotes a copy of the set X which contains formal inverses. We denote by 1 the empty word. We define an equivalence relation on W, which identifies two words $w, w' \in W$ whenever one can obtain w' from w by a sequence of simplifications of the form : $x_i x_i^{-1} = 1$ or $x_i^{-1} x_i = 1$.

We denote by $F(X)$ the set of equivalence classes of words in W. Concatenation of words in W induces a group structure on $F(X)$.

Proposition 1.1.4. *With the same notations as in Definition 1.1.3, the group $F(X)$ is free on X.*

Free groups really capture the idea of being generated by a family of generators which do not have any other relations than those required to make a group. This allows us to introduce group presentations as follows.

Definition 1.1.5. Let $X = \{x_1, \ldots, x_n\}$ and $R \subset F(X)$ a finite subset. We define a group $G = F(X)/\langle\langle R\rangle\rangle$, where $\langle\langle R\rangle\rangle$ denotes the smallest normal subgroup of $F(X)$ which contains R. We say that G is given by a *group presentation* with generators X and relations R. We denote this situation by $G = \langle X|R\rangle$.

Example 1.1.6. Consider the group presentation $G = \langle a, b|aba^{-1}b^{-1}\rangle$. It should be understood as follows : it is the quotient of the free group on $X = \{a, b\}$ (as constructed in Definition 1.1.3) by the normal closure of the set $R = \{aba^{-1}b^{-1}\}$. In other words, an element $g \in G$ is a word in which we can simplify any occurrence of $aba^{-1}b^{-1}$. For instance, we can compute that in G, we have :

$$ab = aba^{-1}b^{-1}ba = ba.$$

This implies that G is an abelian group.

Similarly to free groups, group presentations satisfy some kind of "universal" property which translates the idea that they are obtained from free groups in the least obstructive way.

Proposition 1.1.7. *Let $G = \langle X|R\rangle$ be a group presentation. Then G satisfies the following property : for any group H and any function $f : X \to H$ such that for all $r = x_{i_1}^{\epsilon_1}x_{i_2}^{\epsilon_2}\ldots x_{i_k}^{\epsilon_k} \in R$, we have :*

$$f(x_{i_1})^{\epsilon_1}f(x_{i_2})^{\epsilon_2}\ldots f(x_{i_k})^{\epsilon_k} = 1,$$

there exists a unique extension $f : G \to H$ which is a homomorphism.

This property is very useful in practice, because it makes it easy to define homomorphisms : it suffices to assign values to the generators and to check that the relations are preserved.

Example 1.1.8. Consider the group $G = \langle a, b | aba^{-1}b^{-1} \rangle$. We claim that G is isomorphic to $\mathbb{Z}^2$. Indeed, the assignments $f(a) = (1,0)$ and $f(b) = (0,1)$ preserve the only relation in R, because :

$$f(a) + f(b) - f(a) - f(b) = (1,0) + (0,1) - (1,0) - (0,1) = (0,0),$$

and thus we can apply Proposition 1.1.7 to extend $f : G \to \mathbb{Z}^2$ as a surjective homomorphism. On the other hand, we saw in Example 1.1.6 that G is abelian. Therefore, by the universal property of $\mathbb{Z}^2$ as a free abelian group, the assignments $h(1,0) = a$ and $h(0,1) = b$ extend to a group homomorphism $h : \mathbb{Z}^2 \to G$. Moreover, notice that the assignments $a \mapsto a$ and $b \mapsto b$ preserve the relations in R and therefore they extend uniquely to a group homomorphism $G \to G$, which is the identity. But $h(f(a)) = a$ and $h(f(b)) = b$, which implies that $h \circ f = \mathrm{id}_G$. This shows that f is also injective, and therefore it is an isomorphism, which proves our claim.

We give a few more examples, for which similar arguments as in Example 1.1.8 may be used.

Example 1.1.9. The finite cyclic group C_n has the following presentation :

$$C_n = \langle a | a^n \rangle.$$

The product of two finite cyclic groups C_n and C_m has the following presentation :

$$C_n \times C_m = \langle a, b | a^n, b^m, aba^{-1}b^{-1} \rangle.$$

Example 1.1.10. The finite dihedral group D_n has the following presentation :

$$D_n = \langle a, b | a^2, b^2, (ab)^n \rangle.$$

The infinite dihedral group has the following presentation :

$$D_\infty = \langle a, b | a^2, b^2 \rangle.$$

1.2 Cayley graphs

Group actions can be thought of as realisations of abstract groups (such as those given by group presentations) in terms of symmetries of mathematical objects. For instance, Cayley's theorem allows us to see any group as a permutation group.

Theorem 1.2.1. *Let G be a group. Then G acts on itself by multiplication on the left.*

Proof. Let us write things explicitly. One must show that the map :

$$(g, x) \in G \times G \to gx \in G,$$

defines a group action. This follows immediately from the axioms in the definition of a group. $\square$

In a similar fashion, we have the following result.

Proposition 1.2.2. *Let G be a group and $H \leq G$ a subgroup. Then G acts on G/H by multiplication on the left.*

The following construction is fundamental in geometric group theory.

Definition 1.2.3. Let G be a group generated by a symmetric subset $S \subset G$ (this means that if $s \in S$, then $s^{-1} \in S$). The *Cayley graph* of G relatively to S, which we denote by $\mathcal{C}(G, S)$, is constructed as follows : the set of vertices is the group G, and $x, y \in G$ are connected by an edge whenever $y = xs$ for some $s \in S$.

Example 1.2.4. Consider the finite cyclic group $C_n = \langle a | a^n \rangle$ with generating subset $S = \{a, a^{-1}\}$. The Cayley graph $\mathcal{C}(C_n, S)$ is a cycle graph, which we have illustrated in Figure 1.1 when $n = 5$.

The natural action of a group on itself preserves the structure of the Cayley graph, and therefore one can see any group as an isometry group.

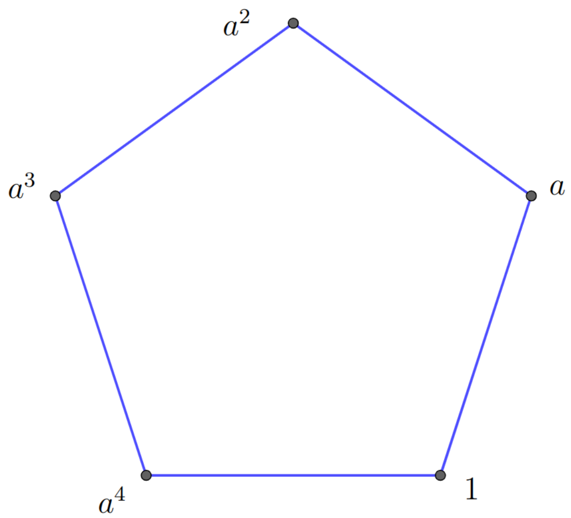

Figure 1.1: Cayley graph of C_5.

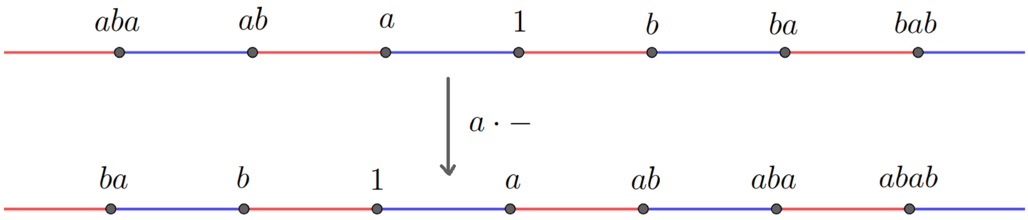

Figure 1.2: Cayley graph of D_∞ and the action of a by multiplication on the left.

Proposition 1.2.5. *Let G be a group generated by a symmetric subset $S \subset G$. Then G acts by graph automorphisms on $\mathcal{C}(G,S)$ (this means that the action of G permutes vertices and preserves edges of the Cayley graph).*

Proof. By Theorem 1.2.1, G acts on itself by multiplication on the left, which shows that G permutes the vertices of $\mathcal{C}(G,S)$. Moreover, we see that if $x, y \in G$ are connected by an edge, ie. if $y = xs$ for some $s \in S$, then $gy = gxs$ for all $g \in G$, which shows that the action of G preserves the edges of the Cayley graph. □

Example 1.2.6. Consider the infinite dihedral group $D_\infty = \langle a, b | a^2, b^2 \rangle$ with generating subset $S = \{a, b\}$. The Cayley graph $\mathcal{C}(D_\infty, S)$ is the simplical line, as illustrated in Figure 1.2. We can see that a acts as a reflexion of the simplicial line.

1.3 Unitary representations

In this section, we introduce the concept of unitary representations.

Definition 1.3.1. Let $\mathcal{H}$ be a Hilbert space (real or complex). We denote by $U(\mathcal{H})$ the *unitary group*, ie. the group of unitary operators on $\mathcal{H}$:

$$U(\mathcal{H}) = \left\{ L : \mathcal{H} \to \mathcal{H}, \text{ where } L \text{ is a bounded, invertible operator such that } L^* = L^{-1} \right\}.$$

We make the following elementary but important observation.

Lemma 1.3.2. *Let $\mathcal{H}$ be a Hilbert space. Then for all $L \in U(\mathcal{H})$ and for all $\xi \in \mathcal{H}$, we have : $||L\xi|| = ||\xi||$. Therefore :*

$$U(\mathcal{H}) = \{ L : \mathcal{H} \to \mathcal{H}, \text{ where } L \text{ is a linear isometry} \}.$$

Proof. It is a direct computation :

$$\langle L\xi, L\xi\rangle = \langle \xi, L^*L\xi\rangle = \langle \xi, L^{-1}L\xi\rangle = \langle \xi, \xi\rangle,$$

which shows that $||L\xi|| = ||\xi||$. $\square$

Remark. When $\mathcal{H}$ is a real Hilbert space, we will use the notation $O(\mathcal{H})$ instead of $U(\mathcal{H})$, and it is standard to speak of the *orthogonal group* instead of the unitary group.

With this in mind, a unitary representation is simply a group action by linear isometries on a Hilbert space.

Definition 1.3.3. Let G be a group. A *unitary representation* of G is a group homomorphism $\pi : G \to U(\mathcal{H})$.

Example 1.3.4. Consider the finite dihedral group $D_n = \langle a, b | a^2, b^2, (ab)^n\rangle$. The assignments :

$$\pi(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\pi(b) = \begin{pmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ \sin(2\pi/n) & -\cos(2\pi/n) \end{pmatrix},$$

extend to a group homomorphism and define a unitary representation $\pi : D_n \to O(\mathbb{R}^2)$. It allows one to realise D_n as the group of isometries of a regular n-gon embedded in $\mathbb{R}^2$. In this case, a and b act by orthogonal reflexions with respect to two lines through the origin forming an angle π/n.

We recall the construction of the direct sum of a family of Hilbert spaces.

Definition 1.3.5. Let $\{\mathcal{H}_i\}_{i\in I}$ be a family of Hilbert spaces. Their *direct sum* $\mathcal{H} = \oplus_{i\in I}\mathcal{H}_i$ is defined as follows : consider the product $\mathcal{K} = \Pi_{i\in I}\mathcal{H}_i$, equipped with the canonical vector space structure. Then :

$$\mathcal{H} = \left\{ \{\xi_i\}_{i\in I} \in \mathcal{K} : \sum_{i\in I} ||\xi_i||^2 < +\infty \right\}.$$

We will use the notation $\xi = \sum_{i\in I} \xi_i \in \mathcal{H}$. We equip $\mathcal{H}$ with the scalar product :

$$\langle \sum_{i\in I} \xi_i, \sum_{i\in I} \eta_i\rangle = \sum_{i\in I}\langle \xi_i, \eta_i\rangle.$$

In this way, one checks that $\mathcal{H}$ is a Hilbert space, and that it contains each of the $\mathcal{H}_i$ as a closed subspace.

Remark. Notice that if $i \neq j$, then the closed subspaces $\mathcal{H}_i$ and $\mathcal{H}_j$ are orthogonal. This implies that in the notation $\xi = \sum_{i\in I} \xi_i$, each component ξ_i represents the orthogonal projection of ξ on $\mathcal{H}_i$.

We can construct many unitary representations by taking direct sums.

Definition 1.3.6. Let G be a group and $\pi_i : G \to U(\mathcal{H}_i)$, $i \in I$ a family of unitary representations of G, and let $\mathcal{H} = \oplus_{i\in I}\mathcal{H}_i$. We define the *direct sum of representations* as the unitary representation $\pi : G \to U(\mathcal{H})$, defined for all $\xi = \sum_{i\in I} \xi_i$ by :

$$\pi(g)\xi = \sum_{i\in I} \pi_i(g)\xi_i.$$

We denote it by $\pi = \oplus_{i\in I}\pi_i$.

Example 1.3.7. Consider the finite dihedral group $D_n = \langle a, b | a^2, b^2, (ab)^n\rangle$. The assignments :

$$\rho(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho(b) = \begin{pmatrix} \cos(2\pi/n) & \sin(2\pi/n) & 0 \\ \sin(2\pi/n) & -\cos(2\pi/n) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

extend to a group homomorphism and define a unitary representation $\rho : D_n \to O(\mathbb{R}^3)$. Notice that ρ acts on the plane $\mathcal{H}_1 = \langle (1, 0, 0), (0, 1, 0)\rangle \subset \mathbb{R}^3$ in the same way as the unitary representation π which we described in Example 1.3.4, and it acts trivially on the line $\mathcal{H}_2 = \langle (0, 0, 1)\rangle \subset \mathbb{R}^3$. More precisely, consider the representation $\chi : D_n \to O(\mathbb{R})$ defined as $\chi(x) = 1$ for all $x \in D_n$. Then $\rho = \pi \oplus \chi$, where we have identified $\mathbb{R}^3 = \mathcal{H}_1 \oplus \mathcal{H}_2$.

We can also construct unitary representations whenever we have a group action.

Definition 1.3.8. Let G be a group acting on a set X. Consider the Hilbert space $\mathcal{H} = L^2(X)$ (X is endowed with the counting measure). For all $g \in G$ and $f \in L^2(X)$, the formula :

$$\pi(g)f(x) = f(g^{-1}x),$$

defines a function $\pi(g)f \in L^2(X)$. This defines a unitary representation $\pi : G \to U(\mathcal{H})$ which we call the *permutation representation* associated to the action of G on X.

Example 1.3.9. Let G be a group. Theorem 1.2.1 says that G acts on itself, and therefore we have a permutation representation acting on $L^2(G)$. We call it the *regular representation* of G. Similarly, if $H \leq G$ is a subgroup, Proposition 1.2.2 says that G acts on G/H, and therefore we have a permutation representation acting on $L^2(G/H)$. We call it the *quasi-regular representation* of G (relatively to H).

Remark. Since G also has an action on itself by multiplication on the right, given by :

$$(g, x) \in G \times G \mapsto xg^{-1} \in G,$$

we have another permutation representation on $L^2(G)$ which we call the *right-regular* representation of G. More explicitly, if we denote it by ρ, then for all $g \in G$ and $f \in L^2(G)$, we have :

$$\rho(g)f(x) = f(xg).$$

We will use the following terminology for fixed points under unitary representations.

Definition 1.3.10. Let G be a group and $\pi : G \to U(\mathcal{H})$ be a unitary representation. A vector $\xi \in \mathcal{H}$ is called *invariant* if $\pi(x)\xi = \xi$ for all $x \in G$.

The following results will be useful later, when we will specifically be interested in non-zero invariant vectors.

Proposition 1.3.11. *Let G be a group and $\pi_i : G \to U(\mathcal{H})$, $i \in I$ a family of unitary representations of G. If the direct sum $\pi = \oplus_{i\in I}\pi_i$ has non-zero invariant vectors, then at least one of the π_i has non-zero invariant vectors.*

Proof. If $\xi = \sum_{i\in I} \xi_i$ is invariant, then looking component by component, we see that for all $x \in G$:

$$\pi_i(x)\xi_i = \xi_i.$$

Therefore, if $\xi \neq 0$, then at least one component $\xi_i \neq 0$, and therefore π_i has non-zero invariant vectors. □

Proposition 1.3.12. *Let G be a group and denote by λ the regular representation of G. A function $f \in L^2(G)$ is invariant if and only if it is constant. Consequently, λ has non-zero invariant vectors if and only if G is finite.*

Proof. If f is invariant, then we see that for all $x \in G$:

$$f(x) = \lambda(x^{-1})f(1) = f(1),$$

which proves that f is constant. Similarly, we show that if f is constant, then it is invariant. Finally, observe that if G is infinite, the only constant function in $L^2(G)$ is zero. □

In a similar fashion, we prove the following result.

Proposition 1.3.13. *Let G be a group and $H \leq G$ a subgroup. Denote by λ the quasi-regular representation of G relatively to H. A function $f \in L^2(G/H)$ is invariant if and only if it is constant. Consequently, λ has non-zero invariant vectors if and only if G/H is finite.*

1.4 Finite groups

In this section, we review some classical results from the representation theory of finite groups. Unlike the previous section, we will only consider complex unitary representations. We refer to [14, Part 1] for more detail.

Definition 1.4.1. Let G be a group and $\pi : G \to U(\mathcal{H})$ a unitary representation. We say that π is an *irreducible representation* if the only closed subspaces $V \subset \mathcal{H}$ which are invariant, ie. such that $\pi(x)V \subset V$ for all $x \in G$, are either $V = 0$ or $V = \mathcal{H}$.

This definition is inspired by the construction of the direct sum of representations : in that case, each component is a closed invariant subspace. This suggests that irreducible representations are building blocks for arbitrary unitary representations. In general, it is not true that any unitary representation splits into a direct sum of irreducible representations, but it is a remarkable feature of finite groups !

Theorem 1.4.2. *Let G be a finite group and $\pi : G \to U(\mathcal{H})$ a unitary representation. Then π is a direct sum of irreducible representations.*

This implies that for finite groups, understanding all unitary representations reduces to understanding all irreducible representations. Naturally, there are still too many irreducible representation *a priori*, so we will need a way of comparing them.

Definition 1.4.3. Let G be a finite group and $\pi_1 : G \to U(\mathcal{H}_1)$ and $\pi_2 : G \to U(\mathcal{H}_2)$ two unitary representations. We say that π_1 and π_2 are *equivalent* if there exists an isomorphism $L : \mathcal{H}_1 \to \mathcal{H}_2$ such that $L\pi_1(g) = \pi_2(g)L$ for all $g \in G$.

Remark. In other words, unitary representations are equivalent if they are conjugate in the following sense : for all $g \in G$, $\pi_2(g) = L\pi_1(g)L^{-1}$. This can be understood as the fact that π_1 and π_2 act similarly. For finite dimensional representations, one can think of L as a change of basis, in which case we see that π_1 and π_2 act in the same way, only on a different basis.

Therefore, it is natural to try to establish a classification of all irreducible representations up to equivalence. We have the following important facts.

Proposition 1.4.4. *Let G be a finite group. Then :*

- *If $\pi : G \to U(\mathcal{H})$ is irreducible, then $dim(\pi) < +\infty$.*
- *There is a finite number of equivalence classes of irreducible representations of G.*

- *If $\{\pi_j\}_{j\in J}$ denotes a set of representatives for all equivalence classes of irreducible representations of G, then :*

$$|G| = \sum_{j\in J} dim(\pi_j)^2.$$

Later in this text, we will establish a list of irreducible representations for a family of finite groups, and the previous result will be useful in order to guarantee that our list is complete (up to equivalence). Moreover, the general theory gives a method for recognizing and distinguishing irreducible representations.

Definition 1.4.5. Let G be a finite group. Consider $\mathbb{C}^G$, the space of functions $f : G \to \mathbb{C}$, and equip it with the following scalar product :

$$\langle f_1, f_2\rangle = \frac{1}{|G|}\sum_{x\in G} f_1(x) f_2(x)^*.$$

This defines a Hilbert space structure on $\mathbb{C}^G$. If $\pi : G \to U(\mathcal{H})$ is a finite dimensional unitary representation, define its *character* $\chi_\pi : G \to \mathbb{C}$ as follows :

$$\chi_\pi(x) = \mathrm{tr}(\pi(x)).$$

Although the trace of an operator usually contains little information, we have the following important irreducibility criterion.

Proposition 1.4.6. *Let G be a finite group. Then :*

- *If $\pi : G \to U(\mathcal{H})$ is finite dimensional, then π is irreducible if and only if $||\chi_\pi|| = 1$.*
- *If π_1 and π_2 are irreducible representations, then they are non-equivalent if and only if $\langle \chi_{\pi_1}, \chi_{\pi_2}\rangle = 0$.*

Finally, we describe a method of construction of representations of finite groups using representations of subgroups.

Definition 1.4.7. Let G be a finite group and $H \leq G$ a subgroup. Let also $\rho : H \to U(\mathcal{K})$ be a finite dimensional unitary representation of H. Consider the space :

$$\mathcal{H} = \left\{ f : G \to \mathcal{K} \text{ such that } f(xh) = \rho(h)^{-1} f(x) \text{ for all } h \in H, x \in G \right\}.$$

It can be seen as a Hilbert space with the following scalar product :

$$\langle f_1, f_2\rangle = \frac{1}{|H|}\sum_{x\in G} \langle f_1(x), f_2(x)\rangle.$$

Then for all $g \in G$ and $f \in \mathcal{H}$, the formula :

$$\pi(g)f(x) = f(g^{-1}x),$$

defines a function $\pi(g)f \in \mathcal{H}$. This defines a unitary representation $\pi : G \to U(\mathcal{H})$ which we call the *induced representation*, and which we will sometimes denote by $\pi = \mathrm{Ind}(\rho)$.

The following result describes a canonical basis for the induced representation.

Proposition 1.4.8. *With the same assumptions as in Definition 1.4.7 : let R be a set of representatives for left-cosets of H in G and let $\{\xi_i\}_{i\in I}$ be an orthonormal basis of $\mathcal{K}$. We define a function :*

$$f_i^r(x) = \begin{cases} \rho(r^{-1}x)^{-1}\xi_i & \text{if } x \in rH \\ 0 & \text{else} \end{cases}$$

Then $\{f_i^r\}_{i\in I, r\in R}$ is an orthonormal basis of $\mathcal{H}$. In particular, $dim(\mathcal{H}) = dim(\mathcal{K})(G : H)$.

Proof. Let $f \in \mathcal{H}$. If $r \in R$, then for all $h \in H$ we see that :

$$f(rh) = \rho(h)^{-1} f(r).$$

Therefore, we see that the value of $f(r)$ determines all values of f on the left-coset rH. If $f(r) = \sum_{i\in I} \lambda_i^r \xi_i$ is the decomposition in the given basis of $\mathcal{K}$, we have that :

$$f(rh) = \sum_{i\in I} \lambda_i^r \rho(h)^{-1}\xi_i = \sum_{i\in I} \lambda_i^r f_i^r(rh).$$

If we do this for all $r \in R$, since G is a disjoint union of left-cosets, we finally obtain :

$$f = \sum_{r\in R}\sum_{i\in I} \lambda_i^r f_i^r.$$

It remains to check that the f_i^r are linearly independent, which can be seen by computing $\langle f_i^r, f_j^s\rangle$ for $i \neq j$ or $r \neq s$. $\square$

Example 1.4.9. Consider the finite dihedral group $D_n = \langle a, b | a^2, b^2, (ab)^n\rangle$. The subgroup $H = \langle ab\rangle$ is a cyclic group of order n so it has index 2 in D_n. Let $\chi : H \to \mathbb{C}^*$ be a representation. Since H is cyclic, it is uniquely determined by :

$$\chi(ab) = e^{i2k\pi/n},$$

for some $k = 0, \dots, n-1$. We will describe the induced representation $\pi : D_n \to U(\mathbb{C}^2)$ by looking at the action of a and b on an adapted basis of $\mathbb{C}^2$. The set $R = \{1, a\}$ is a set of representatives for left-cosets of H in D_n, and therefore we have a basis of $\mathbb{C}^2$ given by two vectors f^1 and f^a. Using the definitions, we find that $\pi(a)f^1 = f^a$, $\pi(a)f^a = f^1$,

$\pi(b)f^1 = \chi(ab)f^a$ and $\pi(b)f^a = \chi(ab)^{-1}f^1$. Since a and b generate D_n, this suffices to describe the whole representation. In matrix representation, we find :

$$\pi(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\pi(b) = \begin{pmatrix} 0 & e^{-i2k\pi/n} \\ e^{i2k\pi/n} & 0 \end{pmatrix}.$$

When $k = 1$, this unitary representation is equivalent to the one we gave in Example 1.3.4. Indeed, a change of basis which conjugates them is given by :

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in U(\mathbb{C}^2).$$

Chapter 2

Kazhdan's property (T)

Property (T) was initially introduced by Kazhdan in [9] as a tool to study lattices in real Lie groups. In fact, property (T) revealed to be an important concept in the study of locally compact groups, with striking applications in many different areas of mathematics, including graph theory and operator algebras to name a few. It is absolutely remarkable that this 3 page article (written at the age of 21 !) identified such a fundamental concept.

In this chapter, we will introduce Kazhdan's property (T) as a rigidity property of unitary representations of groups. Loosely speaking, a group has property (T) if one can guarantee the existence of non-trivial invariant vectors whenever there exist vectors which are *almost* invariant. We will see that all finite groups have property (T), but giving examples of infinite groups with Kazhdan's property is rather difficult.

Our main reference for this chapter is [1]. We will restrict ourselves to the study of discrete groups to avoid topological technicalities, but we will still be able to witness the essence of property (T) in this context.

2.1 Definition and examples

First, we need a definition for the loose idea of almost invariant vectors.

Definition 2.1.1. Let G be a group and $\pi : G \to U(\mathcal{H})$ a unitary representation. Given $\epsilon > 0$ and a finite subset $Q \subset G$, we say that a vector $\xi \in \mathcal{H}$ is (Q,ϵ)-*invariant* if :

$$\sup_{x\in Q} ||\pi(x)\xi - \xi|| < \epsilon ||\xi||.$$

We say that π *almost has invariant vectors* if it has (Q,ϵ)-invariant vectors for all $\epsilon > 0$ and finite $Q \subset G$.

We make the following elementary observation.

Lemma 2.1.2. *Let G be a group and $\pi : G \to U(\mathcal{H})$ a unitary representation. If $\xi \in \mathcal{H}$ is (Q, ϵ)-invariant, then $\xi \neq 0$. Therefore, $\eta = \xi/||\xi||$ is a unit (Q, ϵ)-invariant vector.*

For finite groups, we have the following remarkable rigidity property : the existence of almost invariant vectors suffices to deduce the existence of non-zero invariant vectors.

Proposition 2.1.3. *Let G be a finite group and $\pi : G \to U(\mathcal{H})$ a unitary representation. If π has $(G, \sqrt{2})$-invariant vectors, then it has non-zero invariant vectors.*

Proof. Let $\xi \in \mathcal{H}$ be a $(G, \sqrt{2})$-invariant vector. Without loss of generality (see Lemma 2.1.2), assume that $||\xi|| = 1$. Then by definition :

$$\sup_{x \in G} ||\pi(x)\xi - \xi|| < \sqrt{2}.$$

For all $x \in G$, we have :

$$||\pi(x)\xi - \xi||^2 = ||\pi(x)\xi||^2 + ||\xi||^2 - 2\mathrm{Re}\langle \pi(x)\xi, \xi\rangle = 2 - 2\mathrm{Re}\langle \pi(x)\xi, \xi\rangle.$$

Therefore, we see that :

$$\inf_{x \in G} \mathrm{Re}\langle \pi(x)\xi, \xi\rangle = 1 - \frac{1}{2} \sup_{x \in G} ||\pi(x)\xi - \xi||^2 > 0.$$

Set $\delta = \inf_{x \in G} \mathrm{Re}\langle \pi(x)\xi, \xi\rangle > 0$. Consider the average of the orbit of ξ :

$$\eta = \frac{1}{|G|} \sum_{x \in G} \pi(x)\xi.$$

We check that η is an invariant vector : indeed, for all $g \in G$, we have :

$$\pi(g)\eta = \frac{1}{|G|} \sum_{x \in G} \pi(g)\pi(x)\xi = \frac{1}{|G|} \sum_{x \in G} \pi(gx)\xi,$$

and by a change of variables $x \mapsto g^{-1}x$ which leaves the sum invariant, we obtain :

$$\pi(g)\eta = \frac{1}{|G|} \sum_{x \in G} \pi(x)\xi = \eta.$$

Moreover, we have :

$$\mathrm{Re}\langle \eta, \xi\rangle = \frac{1}{|G|} \sum_{x \in G} \mathrm{Re}\langle \pi(x)\xi, \xi\rangle \geq \delta > 0,$$

which shows that $\eta \neq 0$. ☐

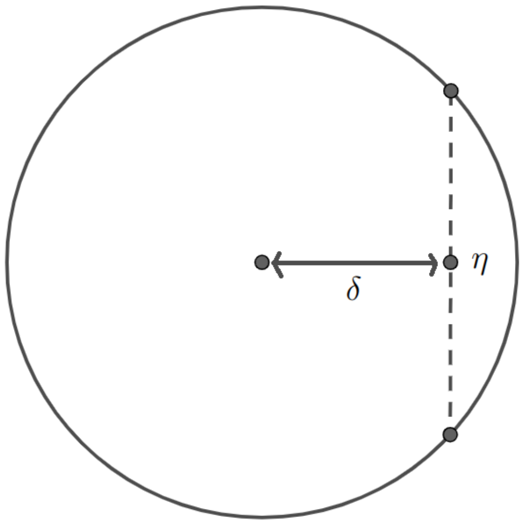

Figure 2.1: The average of the orbit is a non-zero vector.

Remark. The previous result has a geometric interpretation : if ξ is a unit $(G, \sqrt{2})$-invariant vector, then its orbit is strictly contained in one hemisphere of the unit sphere of $\mathcal{H}$. Therefore, the average of the orbit is not equal to the center of the unit sphere. Figure 2.1 illustrates the situation.

It is clear from the definition that if a representation has non-zero invariant vectors, then it almost has invariant vectors. However, the converse does not hold in general, as highlighted by the following example.

Example 2.1.4. Consider the group $\mathbb{Z}$ and denote by λ its regular representation (recall Example 1.3.9). We claim that λ almost has invariant vectors. Indeed, if $Q \subset G$ is finite, let $M = \sup_{x\in Q}|x|$ and consider the characteristic function $\chi_{[0,L]} \in L^2(\mathbb{Z})$, with $L \geq 0$. Then we may compute for all $x \in Q$:

$$||\lambda(x)\chi_{[0,L]} - \chi_{[0,L]}||^2 = \sum_{n\in\mathbb{Z}}(\chi_{[0,L]}(-x+n) - \chi_{[0,L]}(n))^2$$

$$= \sum_{n\in\mathbb{Z}}(\chi_{[x,x+L]}(n) - \chi_{[0,L]}(n))^2 = \sum_{n\in\mathbb{Z}}(\chi_{[x,x+L]\Delta[0,L]}(n))^2 = 2|x|,$$

where we have used the notation Δ for the symmetric difference of sets. We have illustrated the computation in Figure 2.2. Thus, we find that :

$$\sup_{x\in Q}||\lambda(x)\chi_{[0,L]} - \chi_{[0,L]}||^2 = 2\sup_{x\in Q}|x| = 2M.$$

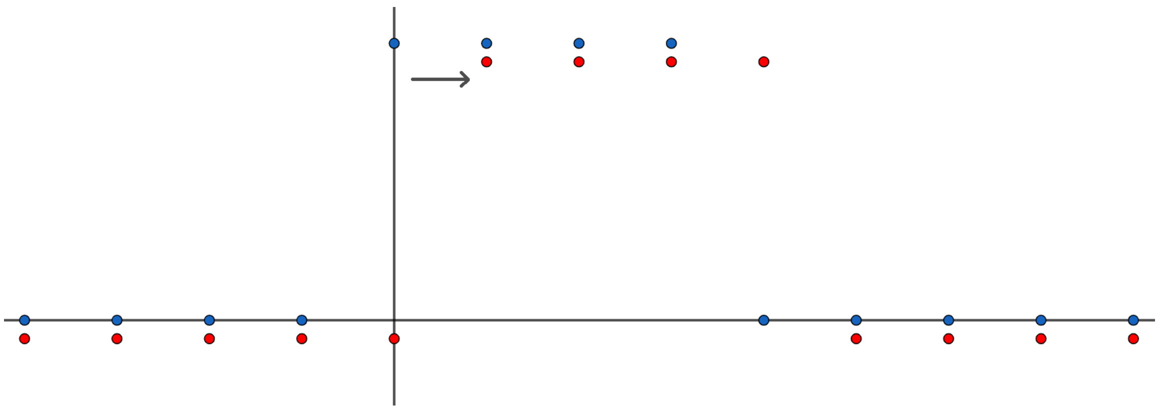

Figure 2.2: In blue, $\chi_{[0,3]}$ and in red, $\lambda(1)\chi_{[0,3]} = \chi_{[1,4]}$.

Let $\epsilon > 0$. Since $||\chi_{[0,L]}|| = \sqrt{L+1}$, the unit vector $\xi = \chi_{[0,L]}/\sqrt{L+1}$ satisfies :

$$\sup_{x\in Q} ||\lambda(x)\xi - \xi|| = \sqrt{\frac{2M}{L+1}} < \epsilon,$$

as long as we take L sufficiently large. This shows that ξ is (Q,ϵ)-invariant, and proves our claim. However, λ does not have non-zero invariant vectors as a result of Proposition 1.3.12.

We would like to insist on the fact that this phenomenon cannot be witnessed for finite dimensional representations, and therefore that it is not surprising that Example 2.1.4 involves an infinite dimensional representation.

Proposition 2.1.5. *Let G be a finitely generated group and $\pi : G \to U(\mathcal{H})$ a unitary representation almost having invariant vectors. If $\mathcal{H}$ is finite dimensional, then π has non-zero invariant vectors.*

Proof. Let $S \subset G$ be a finite generating subset of G. By assumption, π almost has invariant vectors and thus in particular, for all $\epsilon > 0$, π has (S,ϵ)-invariant vectors, and therefore :

$$\inf_{||\xi||=1} \sup_{s\in S} ||\pi(s)\xi - \xi|| < \epsilon.$$

This implies that :

$$\inf_{||\xi||=1} \sup_{s\in S} ||\pi(s)\xi - \xi|| = 0.$$

Consider a sequence of unit vectors $\{\xi_n\}_{n\in\mathbb{N}}$ such that :

$$\lim_{n\to+\infty} \sup_{s\in S} ||\pi(s)\xi_n - \xi_n|| = 0.$$

If $\mathcal{H}$ is finite dimensional, then the unit sphere is compact, so that we may extract a converging subsequence $\{\xi_{n_k}\}_{k\in\mathbb{N}}$. Denote by $\xi = \lim_{k\to+\infty} \xi_{n_k}$ its limit. Then we see that for all $s \in S$, we must have $||\pi(s)\xi - \xi|| = 0$, which means that $\pi(s)\xi = \xi$ for all $s \in S$. Since S is a generating subset, we conclude that $\pi(x)\xi = \xi$ for all $x \in G$, which shows that ξ is a non-zero invariant vector. ☐

This brings us to the definition of Kazhdan's property (T).

Definition 2.1.6. Let G be a group. We say that G has **Kazhdan's property (T)** if whenever a unitary representation almost has invariant vectors, then it has a non-zero invariant vector.

Example 2.1.4 shows that $\mathbb{Z}$ does not have property (T). A consequence of Proposition 2.1.3 is the following.

Corollary 2.1.7. *Let G be a finite group. Then G has property (T).*

Proof. If $\pi : G \to U(\mathcal{H})$ almost has invariant vectors, then in particular it has $(G, \sqrt{2})$-invariant vectors (because G is finite), and by Proposition 2.1.3, we may deduce the existence of non-trivial invariant vectors. ☐

2.2 Kazhdan constants

In this section, we will see that Kazhdan's property can be formulated in terms of Kazhdan constants, which give a quantitative measure of property (T).

Definition 2.2.1. Let G be a group generated by a finite symmetric subset $S \subset G$. We define the *Kazhdan constant* of G relative to S as follows :

$$\kappa(G, S) = \inf \Big\{ \inf_{||\xi||=1} \sup_{s \in S} ||\pi(s)\xi - \xi||, \text{ where } \pi : G \to U(\mathcal{H}) \text{ has no non-zero invariant vectors} \Big\}.$$

Example 2.2.2. If G is a finite group, $S = G$ is a finite, symmetric generating subset. Proposition 2.1.3 shows that $\kappa(G, S) \geq \sqrt{2}$ (by contraposition).

We have the following immediate observation, which really captures the same rigidity phenomenon that we encountered for finite groups in Proposition 2.1.3 : if a representation has a unit vector which is not displaced *too much* under the action of a generating subset, then it has non-zero invariant vectors.

Proposition 2.2.3. *Let G be a group generated by a finite symmetric subset $S \subset G$. If $\kappa(G,S) > 0$, then whenever a unitary representation $\pi : G \to U(\mathcal{H})$ has (S,ϵ)-invariant vectors with $\epsilon < \kappa(G,S)$, then π has non-zero invariant vectors.*

Proof. Saying that π has (S,ϵ)-invariant vectors with $\epsilon < \kappa(G,S)$ amounts to saying that :

$$\inf_{||\xi||=1} \sup_{s\in S} ||\pi(s)\xi - \xi|| < \epsilon < \kappa(G,S).$$

By the definition of the Kazhdan constant, this implies that π has non-zero invariant vectors. □

We will need the following lemma.

Lemma 2.2.4. *Let G be a group and $\pi : G \to U(\mathcal{H})$ a unitary representation. If π has (Q,ϵ)-invariant vectors, then it has $(Q^k, k\epsilon)$-invariant vectors for all $k \geq 1$.*

Proof. Let $\xi \in \mathcal{H}$ be a unit (Q,ϵ)-invariant vector. If $x, y \in Q$, we have by triangle inequality :

$$\begin{aligned} ||\pi(xy)\xi - \xi|| &\leq ||\pi(x)\pi(y)\xi - \pi(x)\xi|| + ||\pi(x)\xi - \xi|| \\ &\leq ||\pi(y)\xi - \xi|| + ||\pi(x)\xi - \xi|| < 2\epsilon, \end{aligned}$$

which proves that ξ is $(Q^2, 2\epsilon)$-invariant. By induction, and using the same argument repeatedly, we prove that ξ is $(Q^k, k\epsilon)$-invariant for all $k \geq 1$. □

The next theorem gives an equivalent definition for property (T) in terms of Kazhdan constants.

Theorem 2.2.5. *Let G be a group generated by a finite symmetric subset $S \subset G$. Then G has property (T) if and only if $\kappa(G,S) > 0$.*

Remark. We will see in Theorem 2.3.1 that it is not restrictive to consider finitely generated groups.

Proof. If $\kappa(G,S) > 0$, we show that G has property (T). Indeed, if $\pi : G \to U(\mathcal{H})$ almost has invariant vectors, then in particular it has (S,ϵ)-invariant vectors with $\epsilon < \kappa(G,S)$. Proposition 2.2.3 implies that π has non-zero invariant vectors.

If G has property (T), we show that $\kappa(G,S) > 0$. By contradiction, assume that $\kappa(G,S) = 0$. Then by definition, there exists a sequence of unitary representations without non-zero invariant vectors $\{\pi_n\}_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$:

$$\inf_{||\xi||=1} \sup_{s\in S} ||\pi_n(s)\xi - \xi|| < \frac{1}{n}.$$

Consider the direct sum (recall Definition 1.3.6) :

$$\pi = \bigoplus_{n \in \mathbb{N}} \pi_n.$$

We claim that π almost has invariant vectors. Indeed, if $Q \subset G$ is finite, since S is a symmetric generating subset of G, we have $Q \subset S^k$ for some $k \geq 1$. By construction, π_n has $(S, 1/n)$-invariant vectors, and thus by Lemma 2.2.4 it has $(S^k, k/n)$-invariant vectors as well and therefore it has $(Q, k/n)$-invariant vectors. As a result, for all $\epsilon > 0$, for sufficiently large n, we have $k/n \leq \epsilon$ and thus π_n has (Q, ϵ)-invariant vectors, from which implies that π has (Q, ϵ)-invariant vectors and proves our claim. Since G has property (T), we deduce that π has non-zero invariant vectors. By Proposition 1.3.11, we conclude that one of the π_n has non-zero invariant vectors, a contradiction. □

2.3 Necessary conditions

For now, our only source of examples of groups with property (T) are finite groups. In this section, we give three important necessary conditions which are satisfied by groups with Kazhdan's property, thereby yielding more examples of infinite groups without property (T). First, we show that property (T) implies finite generation.

Theorem 2.3.1. *Let G be a group. If G has property (T), then G is finitely generated.*

Proof. Denote by $\mathcal{C}$ the collection of all finitely generated subgroups of G. For all $H \in \mathcal{C}$, let $\lambda_{G/H}$ be the associated quasi-regular representation of G (recall Example 1.3.9). Consider the direct sum :

$$\pi = \bigoplus_{H \in \mathcal{C}} \lambda_{G/H}.$$

We claim that π almost has invariant vectors. Indeed, if $Q \subset G$ is finite, then it generates a subgroup $H = \langle Q \rangle \in \mathcal{C}$. Consider the Dirac vector $\delta_H \in L^2(G/H)$. It is a non-zero H-invariant vector, and in particular it is Q-invariant, which shows that π has (Q, ϵ)-invariant vectors for all $\epsilon > 0$ and proves our claim. Since G has property (T), we deduce that π has non-zero invariant vectors. By Proposition 1.3.11, we conclude that one of the $\lambda_{G/H}$ has non-zero invariant vectors, and therefore by Proposition 1.3.13, G/H is finite. Let Q be a finite generating subset of H, and let R be a finite set of representatives of G/H. Then $S = Q \cup R$ is a finite generating subset of G. □

If a group has property (T), then so do all of its quotients.

Proposition 2.3.2. *Let G_1 and G_2 be two groups and $f : G_1 \to G_2$ a surjective group homomorphism. If G_1 has property (T), then G_2 has property (T).*

Proof. Let $\pi : G_2 \to U(\mathcal{H})$ be a unitary representations almost having invariant vectors. We claim that $\rho = \pi \circ f : G_1 \to U(\mathcal{H})$ almost has invariant vectors. Indeed, if $\epsilon > 0$ and $Q \subset G_1$ is finite, then $f(Q) \subset G_2$ is finite and by assumption, π has $(f(Q), \epsilon)$-invariant vectors, which amounts to say that ρ has (Q, ϵ)-invariant vectors, which proves our claim. If G_1 has property (T), then ρ has a non-zero invariant vector $\xi \in \mathcal{H}$. Then for all $y \in G_2$, since f is surjective, we have $y = f(x)$ for some $x \in G_1$, and therefore :

$$\pi(y)\xi = \pi(f(x))\xi = \rho(x)\xi = \xi,$$

which shows that π has non-zero invariant vectors. $\square$

Example 2.3.3. The group $\mathbb{Z}^n$ does not have property (T), because it has $\mathbb{Z}$ as a quotient (which does not have property (T), as a result of Example 2.1.4). Similarly, the free group F_n on n generators does not have property (T) because its abelianisation is isomorphic to $\mathbb{Z}^n$.

Finally, property (T) is inherited by all subgroups of finite index. Informally, this is because subgroups of finite index are sufficiently large (see [1, Theorem 1.7.1]).

Proposition 2.3.4. *Let G be a group and $H \leq G$ a subgroup of finite index. If G has property (T), then H has property (T).*

Example 2.3.5. The infinite dihedral group $D_\infty = \langle a, b | a^2, b^2 \rangle$ does not have property (T). Indeed, consider the subgroup $H = \langle ab \rangle$. It is isomorphic to $\mathbb{Z}$ (which does not have property (T)) and has index 2 in D_∞.

Example 2.3.6. The group $SL_2(\mathbb{Z})$ does not have property (T). To see this, we show that it contains a subgroup of finite index isomorphic to the free group F_2 on 2 generators (which does not have property (T), because of Example 2.3.3). Indeed, consider the homomorphism $SL_2(\mathbb{Z}) \to SL_2(\mathbb{F}_2)$ which consists in taking entries of matrices in $SL_2(\mathbb{Z})$ modulo 2. $SL_2(\mathbb{F}_2)$ is isomorphic to the symmetric group S_3. This shows that the kernel $\Gamma \leq SL_2(\mathbb{Z})$ has index 6. It is well-known that Γ is isomorphic to F_2 (see [4, Example 7.63]).

There exist infinite groups with property (T). For instance, $SL_n(\mathbb{Z})$ has property (T) for all $n \geq 3$. A proof of this statement is far beyond the scope of this text : a proof such as in [1] makes use of tools from abstract harmonic analysis. On the other hand, one may wonder if it is possible to compute the Kazhdan constant relative to the finite generating subset consisting of all elementary matrices. To this day, this remains an open problem. The best known estimates rely on computer calculations (see [7]).

Chapter 3

Estimating Kazhdan constants

In this chapter, we state and prove two results which yield lower bounds on Kazhdan constants, and therefore give interesting criteria for Kazhdan's property (T). We have seen in the previous chapter that property (T) for a group G generated by a finite symmetric subset $S \subset G$ can be identified by computing the Kazhdan constant, which we defined as :

$$\kappa(G,S) = \inf \left\{ \inf_{||\xi||=1} \sup_{s \in S} ||\pi(s)\xi - \xi||, \text{ where } \pi \text{ has no non-zero invariant vectors} \right\}.$$

Heuristically, finding the exact value of $\kappa(G,S)$ requires an explicit understanding of all unitary representations of G, which is typically difficult, even for finite groups. We will describe two methods which considerably reduce the complexity of the problem, at the cost of only giving lower bounds on Kazhdan constants. Still, recall from Theorem 2.2.5 that $\kappa(G,S) > 0$ is sufficient to prove property (T), and some remarkable applications of these ideas provide several examples of infinite groups with property (T).

We will focus on the most simple and essential ideas, but one may find more information in [1] for the spectral approach and [8] for the geometric approach.

3.1 Spectral method

In this section, we introduce a spectral criterion for property (T). It involves the spectrum of the Laplace operator, an object of particular interest in graph theory.

Definition 3.1.1. Let G be a group generated by a finite symmetric subset $S \subset G$. The *Laplace operator* $\Delta : L^2(G) \to L^2(G)$ is defined as follows :

$$\Delta f(x) = f(x) - \frac{1}{|S|}\sum_{s \in S} f(xs) = \frac{1}{|S|}\sum_{s \in S} f(x) - f(xs).$$

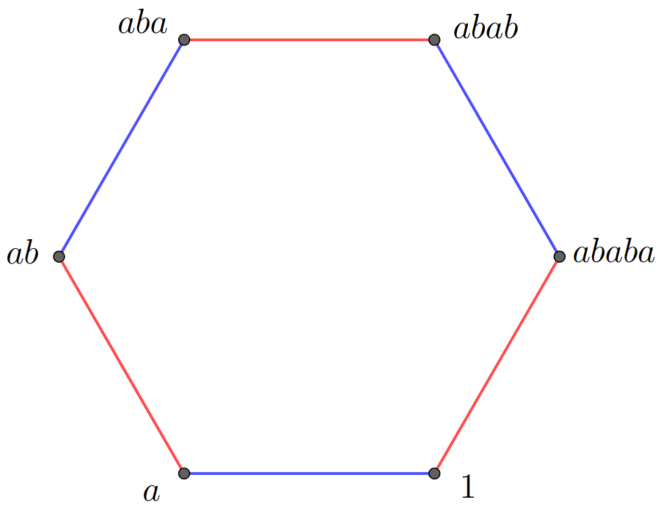

Figure 3.1: Cayley graph of D_3 with respect to $S = \{a, b\}$.

Remark. This definition comes from graph theory, where the Laplace operator is defined as an averaging operator over all neighbors of a given vertex. In this case, recall the construction of the Cayley graph from Definition 1.2.3 : the vertices of the graph are elements of the group, and $x, y \in G$ are adjacent if $y = xs$ for some $s \in S$.

Example 3.1.2. Consider the dihedral group $D_3 = \langle a, b | a^2, b^2, (ab)^3 \rangle$ with $S = \{a, b\}$ (see Figure 3.1 for an illustration of the Cayley graph). Using Dirac functions as a basis of $L^2(D_3)$, we see that the Laplace operator has the following matrix representation :

$$\Delta = \begin{pmatrix} 1 & -1/2 & 0 & 0 & 0 & -1/2 \\ -1/2 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & -1/2 & 1 & -1/2 \\ -1/2 & 0 & 0 & 0 & -1/2 & 1 \end{pmatrix}.$$

Conveniently, the Laplace operator on the Cayley graph can be expressed in terms of unitary representations, which facilitates the proof of several properties.

Proposition 3.1.3. *Let G be a group generated by a finite symmetric subset $S \subset G$. Denote by ρ the right-regular representation of G. Then for all $f \in L^2(G)$, we have :*

$$\Delta f = \frac{1}{|S|} \sum_{s \in S} f - \rho(s) f.$$

Proof. This is immediate if we recall the definition of the right-regular representation (see Example 1.3.9). Then for all $x \in G$, we have :

$$\rho(s)f(x) = f(xs),$$

and we recognize the definition of the Laplace operator. □

Proposition 3.1.4. *Let G be a group generated by a finite symmetric subset $S \subset G$. Then the Laplace operator is a self-adjoint operator.*

Proof. For all $f \in L^2(G)$, we compute :

$$\langle \Delta f, f\rangle = \frac{1}{|S|}\sum_{s\in S}\langle f - \rho(s)f, f\rangle = \frac{1}{|S|}\sum_{s\in S}\langle f, f - \rho(s)^* f\rangle.$$

Since ρ is a unitary representation, we have $\rho(s)^* = \rho(s)^{-1} = \rho(s^{-1})$. Then, using the fact that S is symmetric, we see that the change of variables $s \mapsto s^{-1}$ leaves the sum invariant, so that :

$$\langle \Delta f, f\rangle = \frac{1}{|S|}\sum_{s\in S}\langle f, f - \rho(s^{-1})f\rangle = \frac{1}{|S|}\sum_{s\in S}\langle f, f - \rho(s)f\rangle = \langle f, \Delta f\rangle,$$

which proves that Δ is self-adjoint. □

Proposition 3.1.5. *Let G be a group generated by a finite symmetric subset $S \subset G$. Then for all $f \in L^2(G)$, we have :*

$$\langle \Delta f, f\rangle = \frac{1}{2|S|}\sum_{s\in S}||\rho(s)f - f||^2.$$

In particular, the Laplace operator is positive, ie. $\langle \Delta f, f\rangle \geq 0$ for all $f \in L^2(G)$. Moreover, $\Delta f = 0$ if and only if f is constant.

Proof. For all $f \in L^2(G)$, we compute :

$$\langle \Delta f, f\rangle = \frac{1}{|S|}\sum_{s\in S}\langle f - \rho(s)f, f\rangle = \frac{1}{|S|}\sum_{s\in S}||f||^2 - \langle \rho(s)f, f\rangle.$$

For all $s \in S$, since ρ is unitary, we observe that :

$$\langle \rho(s)f, f\rangle + \langle \rho(s^{-1})f, f\rangle = \langle \rho(s)f, f\rangle + \langle f, \rho(s)f\rangle = 2\mathrm{Re}\langle \rho(s)f, f\rangle.$$

Therefore, using the fact that S is symmetric, which implies that the change of variables $s \mapsto s^{-1}$ leaves any sum over S invariant, we see that :

$$2\langle \Delta f, f\rangle = \frac{1}{|S|}\sum_{s\in S}||f||^2 - \langle \rho(s)f, f\rangle + \frac{1}{|S|}\sum_{s\in S}||f||^2 - \langle \rho(s^{-1})f, f\rangle$$

$$= \frac{1}{|S|} \sum_{s \in S} 2||f||^2 - 2\mathrm{Re}\langle \rho(s)f, f\rangle = \frac{1}{|S|} \sum_{s \in S} ||\rho(s)f - f||^2,$$

which proves the formula. Moreover, we now see that $\Delta f = 0$ if and only if $||\rho(s)f - f|| = 0$ for all $s \in S$, ie. $\rho(s)f = f$ for all $s \in S$. Since S is a generating subset, we see that this is equivalent to saying that f is an invariant vector under ρ, but by Proposition 1.3.12, this amounts to saying that f is constant. $\square$

The Laplace operator is a self-adjoint, positive operator, and therefore it has a positive spectrum. We will be interested in the following quantity.

Definition 3.1.6. Let G be a group generated by a finite symmetric subset $S \subset G$. The *spectral gap* of the associated Laplace operator is defined as the following quantity :

$$\Lambda = \inf \left\{ \langle \Delta f, f\rangle, \text{ where } f \in L^2(G), ||f|| = 1, \text{ and non-constant} \right\}.$$

Remark. If G is finite, then $L^2(G)$ is finite dimensional, and thus ruling out constant functions guarantees that $\Lambda > 0$. In that case, Λ is the smallest non-zero eigenvalue of Δ. If G is infinite, the only constant function in $L^2(G)$ is zero, and therefore the condition is redundant. However, although $\Delta f \neq 0$ for all $||f|| = 1$, it is possible that $\Lambda = 0$.

A generalisation of the Laplace operator applies to vector valued functions, but it can be expressed explicitly in terms of the usual Laplace operator.

Proposition 3.1.7. *Let G be a group generated by a finite symmetric subset $S \subset G$. Given a Hilbert space $\mathcal{H}$, consider the space :*

$$\Omega(\mathcal{H}) = \left\{ F : G \to \mathcal{H} \text{ such that } \frac{1}{|G|} \sum_{x \in G} ||F(x)||^2 < +\infty \right\}.$$

It can be seen as a Hilbert space with the following scalar product :

$$\langle F_1, F_2 \rangle = \frac{1}{|G|} \sum_{x \in G} \langle F_1(x), F_2(x) \rangle.$$

Then, we have an isomorphism $\Omega(\mathcal{H}) = L^2(G) \otimes \mathcal{H}$. Moreover, if we define the action of the Laplace operator on $F \in \Omega(\mathcal{H})$ by :

$$\Delta F(x) = \frac{1}{|S|} \sum_{s \in S} F(x) - F(xs),$$

then it can be seen as the tensor product $\Delta \otimes 1$ of the usual Laplace operator on $L^2(G)$ with the identity operator on $\mathcal{H}$. In particular, it has the same spectral gap $\Lambda > 0$.

Proof. If $f \otimes \xi \in L^2(G) \otimes \mathcal{H}$, then $F(x) = f(x)\xi$ defines a function $F \in \Omega(\mathcal{H})$. This assignment extends to an injective linear mapping $L^2(G) \otimes \mathcal{H} \to \Omega(\mathcal{H})$. To show that it is surjective, notice that if $F \in \Omega(\mathcal{H})$, we have :

$$F = \sum_{x \in G} \delta_x F(x),$$

where $\delta_x \in L^2(G)$ denotes the Dirac function concentrated at x. The claim on the Laplace operator follows immediately, considering how it acts on functions of the form $F = f\xi$. Indeed, we compute :

$$\Delta F(x) = \frac{1}{|S|} \sum_{s \in S} F(x) - F(xs) = \frac{1}{|S|} \sum_{s \in S} f(x)\xi - f(xs)\xi = \Delta f(x)\xi,$$

which proves that $\Delta F = \Delta f \xi$ in this case, and suffices to conclude. □

We can finally state the spectral criterion. The trick consists in considering the action of a unitary representation on vector valued functions, for which we may use the spectral gap to obtain interesting estimates.

Theorem 3.1.8. *Let G be a group generated by a finite symmetric subset $S \subset G$. Denote by Λ the spectral gap of the associated Laplace operator. Then $\kappa(G, S) \geq \sqrt{2\Lambda}$. In particular, if $\Lambda > 0$, then G has property (T).*

Proof. We show that if $\pi : G \to U(\mathcal{H})$ has no non-zero invariant vectors, then :

$$\inf_{||\xi||=1} \sup_{s \in S} ||\pi(s)\xi - \xi||^2 \geq 2\Lambda.$$

Indeed, if $\xi \in \mathcal{H}$ is a unit vector, define a function $F \in \Omega(\mathcal{H})$ as follows :

$$F(x) = \pi(x)\xi.$$

We see that :

$$||F||^2 = \frac{1}{|G|} \sum_{x \in G} ||F(x)||^2 = \frac{1}{|G|} \sum_{x \in G} ||\pi(x)\xi||^2 = \frac{1}{|G|} \sum_{x \in G} ||\xi||^2 = 1.$$

Moreover, since π has no non-zero invariant vectors, F is non-constant. Therefore, we have : $\langle \Delta F, F \rangle \geq \Lambda$. We compute :

$$\langle \Delta F, F \rangle = \frac{1}{|G|} \sum_{x \in G} \langle \Delta F(x), F(x) \rangle = \frac{1}{|G|} \sum_{x \in G} \frac{1}{|S|} \sum_{s \in S} \langle F(x) - F(xs), F(x) \rangle.$$

Similarly as in Proposition 3.1.5, we find that :

$$\frac{1}{|S|} \sum_{s \in S} \langle F(x) - F(xs), F(x) \rangle = \frac{1}{|S|} \sum_{s \in S} \langle \pi(x)\xi - \pi(xs)\xi, \pi(x)\xi \rangle$$

$$= \frac{1}{|S|} \sum_{s\in S} \langle \xi - \pi(s)\xi, \xi \rangle = \frac{1}{2|S|} \sum_{s\in S} ||\xi - \pi(s)\xi||^2,$$

where we use in a crucial way that S is symmetric and π is unitary. If we put everything together, we find that :

$$\langle \Delta F, F \rangle = \frac{1}{|G|} \sum_{x\in G} \frac{1}{2|S|} \sum_{s\in S} ||\pi(s)\xi - \xi||^2 = \frac{1}{2|S|} \sum_{s\in S} ||\pi(s)\xi - \xi||^2.$$

Finally, we obtain :

$$\Lambda \leq \langle \Delta F, F \rangle = \frac{1}{2|S|} \sum_{s\in S} ||\pi(s)\xi - \xi^2|| \leq \frac{1}{2} \sup_{s\in S} ||\pi(s)\xi - \xi||^2,$$

which proves the theorem. □

Remark. As we have noted previously, finite groups satisfy $\Lambda > 0$, and therefore Theorem 3.1.8 gives an alternative proof of the fact that finite groups have property (T).

Another spectral criterion, known as Zuk's criterion (see [1, Theorem 5.6.1]), has some spectacular applications. Notably, let us mention the following informal result : an infinite group "picked at random" has a great chance of having property (T). We refer to [10] for further reading on property (T) for random groups using Zuk's criterion, which makes our statement more precise.

3.2 Geometric method

In this section, we develop a geometric criterion for property (T). We will state and prove the main theorem only in the most simple case, but a general approach can be found in [8], from which we have extracted the essential ideas.

We start with the following elementary, but crucial observation.

Proposition 3.2.1. *Consider the real Hilbert space $\mathcal{H} = \mathbb{R}^2$ and let $\mathcal{H}^1, \mathcal{H}^2 \subset \mathcal{H}$ be two lines through the origin forming an angle $0 < \beta \leq \pi/2$. Then for all $\xi \in \mathcal{H}$, if we denote by ξ_1 and ξ_2 the orthogonal projections of ξ on $\mathcal{H}^1$ and $\mathcal{H}^2$ respectively, we have :*

$$||\xi||^2 = d^t M^{-1} d,$$

where $d = (||\xi - \xi_1||, ||\xi - \xi_2||)$ and M denotes the following positive definite matrix :

$$M = \begin{pmatrix} 1 & -\cos(\beta) \\ -\cos(\beta) & 1 \end{pmatrix}.$$

Remark. By assumption, $\beta > 0$ and therefore the matrix M has the following inverse :

$$M^{-1} = \frac{1}{\sin^2(\beta)} \begin{pmatrix} 1 & \cos(\beta) \\ \cos(\beta) & 1 \end{pmatrix}.$$

The formula can then be written down explicitly as :

$$||\xi||^2 = \frac{1}{\sin^2(\beta)}(||\xi - \xi_1||^2 + ||\xi - \xi_2||^2 + 2\cos(\beta)||\xi - \xi_1||||\xi - \xi_2||),$$

which is what we will prove. However, introducing the matrix M is useful for generalisations of this result.

Proof. Denote by η the projection of ξ on $\mathcal{H}^1$ parallel to $\mathcal{H}^2$, as suggested on Figure 3.2. We see that :

$$||\xi_1 - \eta|| = \frac{\cos(\beta)}{\sin(\beta)}||\xi - \xi_1||,$$

$$||\eta|| = \frac{1}{\sin(\beta)}||\xi - \xi_2||.$$

Since both η and ξ_1 belong to the line $\mathcal{H}^1$, we find :

$$||\xi_1|| = ||\xi_1 - \eta|| + ||\eta|| = \frac{1}{\sin(\beta)}(\cos(\beta)||\xi - \xi_1|| + ||\xi - \xi_2||).$$

We may finally compute :

$$||\xi||^2 = ||\xi - \xi_1||^2 + ||\xi_1||^2$$

$$= ||\xi - \xi_1||^2 + \frac{1}{\sin^2(\beta)}(\cos^2(\beta)||\xi - \xi_1||^2 + ||\xi - \xi_2||^2 + 2\cos(\beta)||\xi - \xi_1||||\xi - \xi_2||)$$

$$= \frac{1}{\sin^2(\beta)}(||\xi - \xi_1||^2 + ||\xi - \xi_2||^2 + 2\cos(\beta)||\xi - \xi_1||||\xi - \xi_2||),$$

which proves the proposition. □

We introduce the following standard definition for the angle between closed subspaces of a Hilbert space.

Definition 3.2.2. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}^1, \mathcal{H}^2 \subset \mathcal{H}$ be two non-trivial closed subspaces with $\mathcal{H}^1 \cap \mathcal{H}^2 = \{0\}$. We define the *angle* $\alpha(\mathcal{H}^1, \mathcal{H}^2)$ between both subspaces as follows : it is the unique number $0 \leq \alpha \leq \pi/2$ such that :

$$\cos(\alpha) = \sup\left\{|\langle \xi_1, \xi_2 \rangle| \text{ where } \xi_1 \in \mathcal{H}^1, \xi_2 \in \mathcal{H}^2 \text{ with } ||\xi_1|| = ||\xi_2|| = 1\right\}.$$

Remark. If $\mathcal{H}^1$ and $\mathcal{H}^2$ are finite dimensional, we may show that $\alpha(\mathcal{H}^1, \mathcal{H}^2) > 0$ (because the unit sphere is compact in finite dimensional spaces). By convention, if $\mathcal{H}^1 \cap \mathcal{H}^2 \neq 0$, we set $\alpha(\mathcal{H}^1, \mathcal{H}^2) = 0$. Similarly, if $\mathcal{H}^1$ or $\mathcal{H}^2$ is trivial, we set $\alpha(\mathcal{H}^1, \mathcal{H}^2) = \pi/2$.

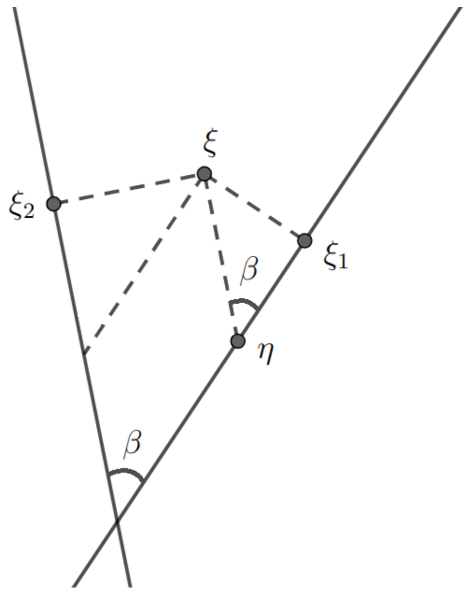

Figure 3.2

In other words, the angle between two closed subspaces is the smallest angle which can be formed by pairs of vectors in each of them. We will need the next lemma to relate Proposition 3.2.1 with the general case of complex Hilbert spaces.

Lemma 3.2.3. *Let $\mathcal{H}$ be a complex Hilbert space. Then it can be seen as a real Hilbert space with the scalar product :*

$$\langle \xi_1, \xi_2 \rangle_{\mathbb{R}} = Re\langle \xi_1, \xi_2 \rangle_{\mathbb{C}}.$$

Moreover, if $\mathcal{H}^1, \mathcal{H}^2 \subset \mathcal{H}$ are two non-trivial closed subspaces with $\mathcal{H}^1 \cap \mathcal{H}^2 = \{0\}$, we have :

$$\cos(\alpha(\mathcal{H}^1, \mathcal{H}^2)) = \sup \left\{ Re\langle \xi_1, \xi_2 \rangle \text{ where } \xi_1 \in \mathcal{H}^1, \xi_2 \in \mathcal{H}^2 \text{ with } ||\xi_1|| = ||\xi_2|| = 1 \right\},$$

implying that the angle between $\mathcal{H}^1$ and $\mathcal{H}^2$ is the same, regardless of whether we consider the real or complex structure on $\mathcal{H}$.

Proof. The first claim is easy to check, so we focus on the second claim. Observe that for any $\xi_1, \xi_2 \in \mathcal{H}$ with $||\xi_1|| = ||\xi_2|| = 1$, we have that $\langle \xi_1, \xi_2 \rangle = re^{i\theta}$ for some $r \in \mathbb{R}$ and $\theta \in [0, 2\pi[$. Therefore, we see that $\langle e^{-i\theta}\xi_1, \xi_2 \rangle = r$, and thus $|\langle \xi_1, \xi_2 \rangle| = |\mathrm{Re}\langle e^{-i\theta}\xi_1, \xi_2 \rangle| = |r|$. In other words, any value attained by the complex modulus of the complex scalar product can also be reached by the real scalar product (possibly with a different pair of vectors), which proves our claim. □

Proposition 3.2.4. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}^1, \mathcal{H}^2 \subset \mathcal{H}$ be two closed subspaces with $\mathcal{H}^1 \cap \mathcal{H}^2 = 0$, such that $\alpha(\mathcal{H}^1, \mathcal{H}^2) \geq \alpha > 0$. Then for all $\xi \in \mathcal{H}$, if we denote by ξ_1*

and ξ_2 the orthogonal projections of ξ on $\mathcal{H}^1$ and $\mathcal{H}^2$ respectively, we have :

$$||\xi||^2 \leq d^t M^{-1} d,$$

where $d = (||\xi - \xi_1||, ||\xi - \xi_2||)$ and M denotes the following positive definite matrix :

$$M = \begin{pmatrix} 1 & -\cos(\alpha) \\ -\cos(\alpha) & 1 \end{pmatrix}.$$

Proof. Denote by ξ_{12} the orthogonal projection of ξ on the (real) plane $\langle \xi_1, \xi_2 \rangle \subset \mathcal{H}$ and denote by β the angle formed by ξ_1 and ξ_2. We may apply Proposition 3.2.1 to obtain the following :

$$||\xi_{12}||^2 = \frac{1}{\sin^2(\beta)}(||\xi_{12} - \xi_1||^2 + ||\xi_{12} - \xi_2||^2 + 2\cos(\beta)||\xi_{12} - \xi_1||||\xi_{12} - \xi_2||).$$

Since $||\xi_{12} - \xi_1||^2 = ||\xi - \xi_1||^2 - ||\xi - \xi_{12}||^2$ and $||\xi_{12} - \xi_2||^2 = ||\xi - \xi_2||^2 - ||\xi - \xi_{12}||^2$, we have :

$$\begin{aligned} ||\xi_{12}||^2 = \frac{1}{\sin^2(\beta)}(&||\xi - \xi_1||^2 + ||\xi - \xi_2||^2 - 2||\xi - \xi_{12}||^2 \\ &+ 2\cos(\beta)\sqrt{||\xi - \xi_1||^2 - ||\xi - \xi_{12}||^2}\sqrt{||\xi - \xi_2||^2 - ||\xi - \xi_{12}||^2}). \end{aligned}$$

Therefore, we have :

$$\begin{aligned} ||\xi||^2 &= ||\xi_{12}||^2 + ||\xi - \xi_{12}||^2 \\ &= \frac{1}{\sin^2(\beta)}(||\xi - \xi_1||^2 + ||\xi - \xi_2||^2 - 2||\xi - \xi_{12}||^2 + \sin^2(\beta)||\xi - \xi_{12}||^2 \\ &\quad + 2\cos(\beta)\sqrt{||\xi - \xi_1||^2 - ||\xi - \xi_{12}||^2}\sqrt{||\xi - \xi_2||^2 - ||\xi - \xi_{12}||^2}), \end{aligned}$$

from which we may deduce that :

$$||\xi||^2 \leq \frac{1}{\sin^2(\beta)}(||\xi - \xi_1||^2 + ||\xi - \xi_2||^2 + 2\cos(\beta)||\xi - \xi_1||||\xi - \xi_2||).$$

Since $\beta \geq \alpha(\mathcal{H}^1, \mathcal{H}^2) \geq \alpha$, we obtain the estimate :

$$||\xi||^2 \leq \frac{1}{\sin^2(\alpha)}(||\xi - \xi_1||^2 + ||\xi - \xi_2||^2 + 2\cos(\alpha)||\xi - \xi_1||||\xi - \xi_2||),$$

which proves the proposition. □

Corollary 3.2.5. *With the exact same assumptions as Proposition 3.2.4, if we denote by $0 < \Lambda \leq 1$ the smallest eigenvalue of M, then :*

$$||\xi||^2 \leq \frac{1}{\Lambda}(||\xi - \xi_1||^2 + ||\xi - \xi_2||^2).$$

Proof. By Proposition 3.2.4, we have :

$$||\xi||^2 \leq d^t M^{-1} d \leq C d^t d,$$

where $C > 0$ denotes the largest eigenvalue of M^{-1}. Therefore, $C = 1/\Lambda$, where Λ is the smallest eigenvalue of M, and the corollary follows. □

Finally, we introduce the concept of representation angle between subgroups.

Definition 3.2.6. Let G be a group generated by two finite subgroups $A, B \leq G$. We define the *representation angle* $\alpha(A, B)$ between both subgroups as follows :

$$\alpha(A,B)) = \inf \left\{ \alpha(\mathcal{H}^A, \mathcal{H}^B), \text{ where } \pi : G \to U(\mathcal{H}) \text{ has no non-zero invariant vectors} \right\}.$$

Remark. If G is finite, the representation angle is reached by a non-trivial irreducible representation of G. This is due to the fact that unitary representations of finite groups are direct sums of irreducible representations (recall Theorem 1.4.2). Moreover, Proposition 1.4.4 says that irreducible representations are finite dimensional, and therefore $\mathcal{H}^A$ and $\mathcal{H}^B$ are finite dimensional, so that the angle $\alpha(\mathcal{H}^A, \mathcal{H}^B) > 0$ for each of them. But up to equivalence, there are only finitely many irreducible representations, and therefore $\alpha(A, B) > 0$.

In other words, the representation angle is the smallest angle which can be formed by invariant subspaces which intersect trivially, ie. for representations without non-zero invariant vectors.

Example 3.2.7. Consider the dihedral group $D_n = \langle a, b | a^2, b^2, (ab)^n \rangle$. It is generated by the pair of subgroups $A = \langle a \rangle$ and $B = \langle b \rangle$. A standard computation shows that $\alpha(A, B) = \pi/n$. The infimum is reached by the following representation of dimension 2, given by :

$$\pi(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\pi(b) = \begin{pmatrix} \cos(2\pi/n) & \sin(2\pi/n) \\ \sin(2\pi/n) & -\cos(2\pi/n) \end{pmatrix}.$$

It corresponds to an action by isometries of a regular n-gon : a and b act by orthogonal reflexions with respect to two lines through the origin. We have illustrated the situation in Figure 3.3 when $n = 6$.

For infinite groups, the representation angle could be zero.

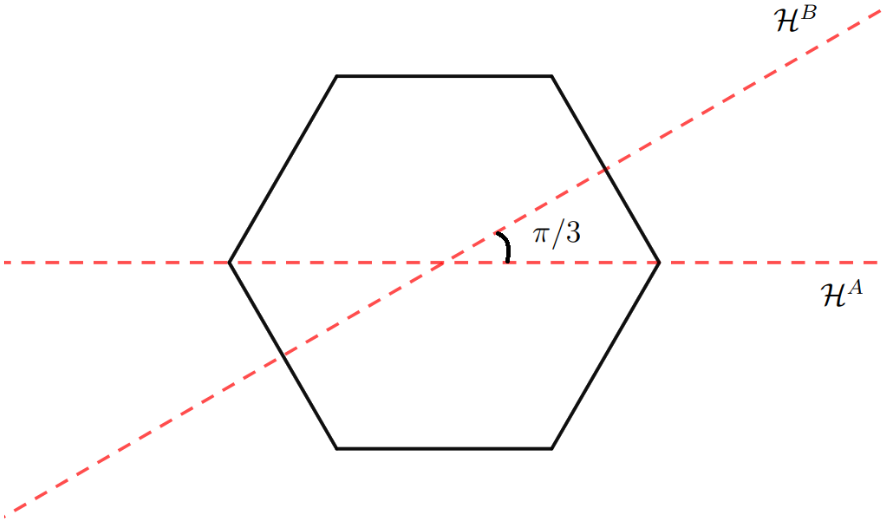

Figure 3.3: Invariant subspaces under the action of $\pi : D_6 \to O(\mathbb{R}^2)$

Example 3.2.8. Consider the infinite dihedral group $D_\infty = \langle a, b | a^2, b^2 \rangle$. We claim that in this case, the representation angle between $A = \langle a \rangle$ and $B = \langle b \rangle$ is zero. Indeed, for any $\theta \in \mathbb{R}$, define a unitary representation as follows :

$$\pi(a) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\pi(b) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}.$$

In other words, a and b act by orthogonal reflexions with respect to two lines through the origin which form an angle θ. Taking the limit $\theta \to 0$, we see that $\alpha(A, B) = 0$.

The previous geometric estimates can be used to obtain a lower bound on Kazhdan constants.

Theorem 3.2.9. *Let G be a group generated by two finite subgroups $A, B \leq G$. If $\alpha(A, B) > 0$, then we denote by $0 < \Lambda \leq 1$ the smallest eigenvalue of the positive definite matrix :*

$$M = \begin{pmatrix} 1 & -\cos(\alpha(A, B)) \\ -\cos(\alpha(A, B)) & 1 \end{pmatrix},$$

and $\kappa(G, A \cup B) \geq \sqrt{2\Lambda} > 0$. In particular, G has property (T).

Proof. We show that if $\pi : G \to U(\mathcal{H})$ has $(A \cup B, \epsilon)$-invariant vectors with $\epsilon < \sqrt{2\Lambda}$, then π has non-zero invariant vectors. Indeed, if $\xi \in \mathcal{H}$ is a unit$(A \cup B, \epsilon)$-invariant vector, then it is (A, ϵ)-invariant with $\epsilon < \sqrt{2\Lambda} \leq \sqrt{2}$. Therefore, in virtue of Proposition

2.1.3 applied to the restriction $\pi : A \to U(\mathcal{H})$, we know that the average of the orbit of ξ is a non-zero A-invariant vector, and that if we denote by ξ_A the orthogonal projection of ξ on $\mathcal{H}^A \neq 0$, then :

$$||\xi - \xi_A|| \leq \frac{\epsilon}{2}.$$

In a similar fashion, we show that $\mathcal{H}^B \neq 0$ and that if ξ_B denotes the orthogonal projection of ξ on $\mathcal{H}^B$, then :

$$||\xi - \xi_B|| \leq \frac{\epsilon}{2}.$$

Assume by contradiction that π has no non-zero invariant vectors. In other words, $\mathcal{H}^A \cap \mathcal{H}^B = 0$, and in particular $\alpha(\mathcal{H}^A, \mathcal{H}^B) \geq \alpha(A, B)$. Then we may apply Corollary 3.2.5 to obtain the following estimate :

$$||\xi||^2 \leq \frac{1}{\Lambda}(||\xi - \xi_A||^2 + ||\xi - \xi_B||^2) \leq \frac{2\epsilon^2}{4\Lambda} < 1.$$

But ξ is a unit vector, and thus we obtain a contradiction. □

Let us illustrate the criterion with a simple example.

Example 3.2.10. Consider the finite dihedral group $D_n = \langle a, b | a^2, b^2, (ab)^n \rangle$ with the pair of subgroups $A = \langle a \rangle$ and $B = \langle b \rangle$. We have seen in Example 3.2.7 that the representation angle is given by $\alpha(A, B) = \pi/n$. A quick computation shows that in this case, $\Lambda = 1 - \cos(\pi/n)$. Therefore, Theorem 3.2.9 states that : $\kappa(D_n, A \cup B) \geq \sqrt{2}\sqrt{1 - \cos(\pi/n)}$.

This criterion can be generalized to groups generated by any finite number of finite subgroups. Let us state (without proof) the generalization in the case of groups generated by three finite subgroups (see [8, Theorem 1.2] for a general version).

Theorem 3.2.11. *Let G be a group generated by three finite subgroups $A, B, C \leq G$. Consider the subgroup $H_1 = \langle A \cup B \rangle$, and denote by α_1 the representation angle between subgroups $A, B \leq H$. Similarly, we define $H_2 = \langle A \cup C \rangle$ and $H_3 = \langle B \cup C \rangle$, and the appropriate representation angles α_2 and α_3. If $\alpha_1 + \alpha_2 + \alpha_3 > \pi$, then $\kappa(G, A \cup B \cup C) > 0$ and therefore G has property (T).*

Remark. The condition on the sum of angles comes from the study of the matrix :

$$M = \begin{pmatrix} 1 & -\cos(\alpha_1) & -\cos(\alpha_2) \\ -\cos(\alpha_1) & 1 & -\cos(\alpha_3) \\ -\cos(\alpha_2) & -\cos(\alpha_3) & 1 \end{pmatrix},$$

which plays the same role as before in yielding geometric estimates of the form :

$$||\xi||^2 \leq d^t M^{-1} d,$$

where $d = (||\xi - \xi_A||, ||\xi - \xi_B||, ||\xi - \xi_C||)$ is a vector containing the distance of ξ to the subspaces left invariant under the action of A, B and C. It is a classical computation to show that M is positive definite if and only if $\alpha_1 + \alpha_2 + \alpha_3 > \pi$.

While Theorem 3.2.9 requires only a strictly positive angle between subgroups to deduce property (T), Theorem 3.2.11 motivates the search for good estimates, which would guarantee that the condition for the sum of angles is satisfied. Let us illustrate the theorem with a few examples.

Example 3.2.12. Let m_1, m_2 and m_3 be positive integers, and consider the group :

$$G = \langle a, b, c | a^2, b^2, c^2, (ab)^{m_1}, (ac)^{m_2}, (bc)^{m_3} \rangle.$$

It is generated by the three finite subgroups $A = \langle a \rangle$, $B = \langle b \rangle$ and $C = \langle c \rangle$. Moreover, if we denote by $H_1 = \langle A \cup B \rangle$, $H_2 = \langle A \cup C \rangle$ and $H_3 = \langle B \cup C \rangle$, then we see that all three of them are isomorphic to the finite dihedral groups D_{m_1}, D_{m_2} and D_{m_3}, respectively. In view of Example 3.2.7, we see that the representation angles are given by $\alpha_1 = \pi/m_1$, $\alpha_2 = \pi/m_2$ and $\alpha_3 = \pi/m_3$. Therefore, if :

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} > 1,$$

the condition on the sum of angles is satisfied and by Theorem 3.2.11, G has property (T). However, in this case, it can be shown that G is finite, so that the theorem doesn't really tell us anything new.

Example 3.2.13. Consider the group :

$$G = \langle a, b, c | a^2, b^2, c^2, (ab)^3, (ac)^3, (bc)^3 \rangle.$$

It is generated by the three finite subgroups $A = \langle a \rangle$, $B = \langle b \rangle$ and $C = \langle c \rangle$. If we consider the associated subgroups H_1, H_2 and H_3, we see that they are all isomorphic to the finite dihedral group D_3, and so the representation angles are given by $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$. In particular, the condition on the sum of angles is not satisfied, because $\alpha_1 + \alpha_2 + \alpha_3 = \pi$. It can be shown that G does not have property (T). Indeed, property (FH) is the property that every action of a group by affine isometries on a Hilbert space has a global fixed point. It is well-known that property (T) implies property (FH) (see [1, Theorem 2.12.4]). It then suffices to observe that in this case, G has a natural action by affine isometries of $\mathbb{R}^2$ without fixed points (see Figure 3.4). Therefore, the condition on the sum of angles cannot be relaxed.

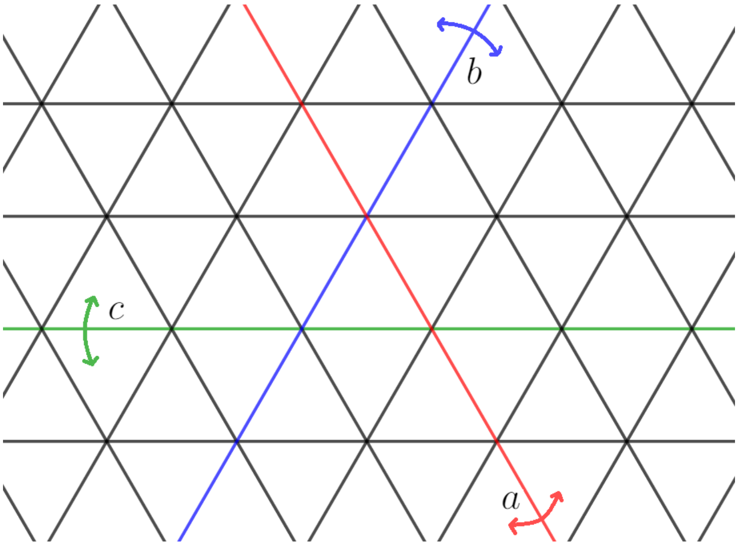

Figure 3.4: An affine action of G is given by reflexions along the three colored lines, and preserves the tiling of the plane by equilateral triangles.

Example 3.2.14. Let p be an odd prime number. Consider the group :

$$G = \langle a, b, c | a^p, b^p, c^p, [a, b, a], [a, b, b], [b, c, b], [b, c, c], [a, c, a], [a, c, c, a], [a, c, c, c] \rangle.$$

It is generated by three finite subgroups $A = \langle a \rangle$, $B = \langle b \rangle$, $C = \langle c \rangle$, which are cyclic of order p. The subgroups $H_1 = \langle A \cup B \rangle$, $H_2 = \langle A \cup C \rangle$ and $H_3 = \langle B \cup C \rangle$ appear in families of finite p-groups, similar to those which we will study in Chapter 4. It is possible to compute the representation angles α_1, α_2 and α_3 explicitly, and to check that the condition on the sum of angles is satisfied and thus Theorem 3.2.11 applies, showing that G has property (T). On the other hand, other criteria can be applied to show that the group presentation for G defines an infinite hyperbolic group (see [2, Theorem 1.3]).

Chapter 4

A family of p-groups

In this chapter, we will be interested in a specific family of finite p-groups (where $p \geq 5$ is a prime number). More specifically, they are obtained as the p-Sylow subgroups (ie. maximal p-subgroups) of the Chevalley groups of type G_2 on the finite field $\mathbb{F}_p$, whose construction originates back in the pioneering work of Chevalley in [3]. In [5], they are given by means of a presentation : they have six generators x_γ, where $\gamma \in \{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}$, all of which have order p and satisfy the following commutation relations (those which do not appear here are trivial commutation relations) :

$$\begin{aligned}
&[x_\alpha, x_\beta] = x_{\alpha+\beta} x_{\alpha+2\beta} x_{\alpha+3\beta} x_{2\alpha+3\beta},\\
&[x_{\alpha+\beta}, x_\beta] = x_{\alpha+2\beta}^2 x_{\alpha+3\beta}^3 x_{2\alpha+3\beta}^3,\\
&[x_{\alpha+2\beta}, x_\beta] = x_{\alpha+3\beta}^3,\\
&[x_{\alpha+2\beta}, x_{\alpha+\beta}] = x_{2\alpha+3\beta}^3,\\
&[x_{\alpha+3\beta}, x_\alpha] = x_{2\alpha+3\beta}^{-1}.
\end{aligned}$$

This defines a finite group of order p^6, which we will denote by $U_6(p)$. We will be interested in the representation angle between subgroups $A = \langle x_\alpha \rangle$ and $B = \langle x_\beta \rangle$, which together generate the group $U_6(p)$. In [2, Section 7], a similar computation for groups of the same kind (but of orders p^3 and p^4, corresponding to p-Sylow subgroups in Chevalley groups on $\mathbb{F}_p$ of type A_2 and C_2, respectively) is used to give examples of infinite hyperbolic groups with property (T), such as those in Example 3.2.14. Our motivation in this problem is that it could yield new examples of such groups.

We will start by giving a shorter presentation of the group, which will make some computations easier. Then, we establish the complete list of all irreducible representations of $U_6(p)$, and finally we will attempt a computation of the representation angle. We will give an upper bound, and using a result from [5], we will give a lower bound. Our estimates improve those which can be found in the current published literature, but we were not able to determine the exact value of the representation angle. Still, those estimates are sufficient to apply Theorem 3.2.11 and obtain new examples of groups with property (T), for which the tools from [2] guarantee that they are infinite hyperbolic.

4.1 A shorter group presentation

Let $p \geq 5$ be a prime number and consider the following presentation :

$$U = \langle a, b | a^p, b^p, [a,b,a,a], [a,b,a,b], [a,b,b,b,b], [a,b,b,a] = [a,b,b,b,a] = [a,b,a]^6 \rangle,$$

where we have used the notation :

$$[x_1, \dots, x_n] = [\dots[[[x_1, x_2], x_3], \dots], x_n],$$

for iterated commutators. The aim of this section is to prove that $U_6(p) = U$, with the natural identification $x_\alpha = a$ and $x_\beta = b$. A first observation is the following.

Proposition 4.1.1. *The assignments $a \mapsto x_\alpha$ and $b \mapsto x_\beta$ extend to a surjective homomorphism $U \to U_6(p)$.*

Proof. It suffices to check that the relations in the group presentation of U are satisfied in $U_6(p)$. These computations are straightforward and not very instructive, so we will only detail one of them. First, we compute :

$$[x_\alpha, x_\beta, x_\alpha] = [[x_\alpha, x_\beta], x_\alpha] = x_{\alpha+\beta} x_{\alpha+2\beta} x_{\alpha+3\beta} x_{2\alpha+3\beta} x_\alpha x_{2\alpha+3\beta}^{-1} x_{\alpha+3\beta}^{-1} x_{\alpha+2\beta}^{-1} x_{\alpha+\beta}^{-1} x_\alpha^{-1}.$$

Notice that x_α commutes with every other generator except $x_{\alpha+3\beta}$. Therefore, we see that :

$$[x_\alpha, x_\beta, x_\alpha] = [x_{\alpha+3\beta}, x_\alpha] = x_{2\alpha+3\beta}^{-1}.$$

Also notice that $x_{2\alpha+3\beta}$ is central. In particular, the following relations hold :

$$[x_\alpha, x_\beta, x_\alpha, x_\alpha] = 1,$$

$$[x_\alpha, x_\beta, x_\alpha, x_\beta] = 1.$$

The remaining computations are similar. To see that the extension is surjective, it suffices to show that x_α and x_β generate $U_6(p)$. A computation shows that :

$$[x_\alpha, x_\beta, x_\beta] = x_{\alpha+2\beta}^2 x_{\alpha+3\beta}^6 x_{2\alpha+3\beta}^3,$$

$$[x_\alpha, x_\beta, x_\beta, x_\beta] = x_{\alpha+3\beta}^6.$$

These suffice to express every generator of $U_6(p)$ in terms of x_α and x_β. □

Remark. It is important that we chose $p \geq 5$, so that 2, 3 and 6 are prime with p, and so that we avoid any potential degeneracy of some of the relations computed above.

We identify some additional relations which are satisfied in U, which can be computed directly.

Proposition 4.1.2. *In U, we have the following commutation relations :*

$$\begin{aligned}
&[a,b,b,b][a,b,b] = [a,b,b][a,b,b,b],\\
&[a,b,b,b][a,b] = [a,b][a,b,b,b],\\
&[a,b,b][a,b] = [a,b,a]^{-6}[a,b][a,b,b].
\end{aligned}$$

Proof. Let us prove the second relation. We have :

$$[a,b,b,b][a,b] = [a,b,b,b]aba^{-1}b^{-1} = [a,b,b,b,a]a[a,b,b,b]ba^{-1}b^{-1}.$$

Since $[a,b,b,b,b] = 1$, b and $[a,b,b,b]$ commute and therefore :

$$\begin{aligned}
[a,b,b,b][a,b] &= [a,b,b,b,a]ab[a,b,b,b]a^{-1}b^{-1} = [a,b,b,b,a]ab[a,b,b,b,a]^{-1}a^{-1}[a,b,b,b]b^{-1}\\
&= [a,b,b,b,a]ab[a,b,b,b,a]^{-1}a^{-1}b^{-1}[a,b,b,b].
\end{aligned}$$

But $[a,b,b,b,a] = [a,b,a]^6$ is central in U, and therefore we may conclude that

$$[a,b,b,b][a,b] = [a,b][a,b,b,b].$$

The other two relations require similar arguments. □

Proposition 4.1.3. *In U, we have the following commutation relations for all $n \in \mathbb{N}$:*

$$\begin{aligned}
&[a,b,b]b^n = b^n[a,b,b,b]^n[a,b,b],\\
&[a,b]b^n = b^n[a,b,b,b]^{n(n+1)/2}[a,b,b]^n[a,b],\\
&[a,b]a^n = [a,b,a]^n a^n[a,b],\\
&ab^n = b^n[a,b,b,b]^{n(n+1)(n+2)/6}[a,b,b]^{n(n+1)/2}[a,b]^n[a,b,a]^{6n}a,\\
&b[a,b]^n = [a,b]^n[a,b,b]^{-n}b[a,b,a]^{6n(n+1)/2},\\
&ba^n = a^n[a,b]^{-n}[a,b,a]^{-n(n+1)/2}b.
\end{aligned}$$

Proof. Each of these relations is shown to hold by induction on $n \in \mathbb{N}$. The commutation relations in Proposition 4.1.2 are particularly useful in these computations. □

Proposition 4.1.4. *In U, we have the following additional relations :*

$$[a,b,b,b]^p = [a,b,b]^p = [a,b]^p = [a,b,a]^p = 1.$$

Proof. Taking $n = p$ in Proposition 4.1.3 yields the desired relations, after observing that if $p \geq 5$ is prime, then p divides $p(p+1)/2$ and $p(p+1)(p+2)/6$. □

With all of this in mind, we can prove the following proposition.

Proposition 4.1.5. *U is a finite group of order p^6. In particular, $U = U_6(p)$.*

Proof. We claim that every element $w \in U$ has a representation of the form :

$$w = b^{e_1}[a,b,b,b]^{e_2}[a,b,b]^{e_3}[a,b]^{e_4}a^{e_5}[a,b,a]^{e_6}.$$

Indeed, we see that $w = 1$ has such a representation with $e_1 = \cdots = e_6 = 0$. Then, we observe that if some w can be written in this way, then also wa, wa^{-1}, wb and wb^{-1}. Since a and b generate U, this proves our claim. Moreover, we claim that such a representation is unique (with the e_i chosen modulo p). Indeed, assume that :

$$b^{e_1}[a,b,b,b]^{e_2}[a,b,b]^{e_3}[a,b]^{e_4}a^{e_5}[a,b,a]^{e_6} = b^{f_1}[a,b,b,b]^{f_2}[a,b,b]^{f_3}[a,b]^{f_4}a^{f_5}[a,b,a]^{f_6}.$$

Then in the abelianisation of U, we find that :

$$b^{e_1}a^{e_5} = b^{f_1}a^{f_5},$$

from which we deduce that $e_1 = f_1$ and $e_5 = f_5$. After simplification, we see that :

$$[a,b,b,b]^{e_2}[a,b,b]^{e_3}[a,b]^{e_4}[a,b,a]^{e_6} = [a,b,b,b]^{f_2}[a,b,b]^{f_3}[a,b]^{f_4}[a,b,a]^{f_6}.$$

In the central quotient $U/\langle[a,b,a]\rangle$, we see that $[a,b,b,b]$, $[a,b,b]$ and $[a,b]$ generate an abelian group, and therefore :

$$[a,b,b,b]^{e_2}[a,b,b]^{e_3}[a,b]^{e_4} = [a,b,b,b]^{f_2}[a,b,b]^{f_3}[a,b]^{f_4},$$

implies that $e_2 = f_2$, $e_3 = f_3$ and $e_4 = f_4$. Finally, we see that $e_6 = f_6$, and this proves our claim. Therefore, we see that $|U| \leq p^6$. However, in Proposition 4.1.1 we gave a surjective homomorphism $U \to U_6(p)$, which proves that $|U| \geq |U_6(p)| = p^6$, and thus $|U| = p^6$ and both groups are isomorphic. □

From now on, we will freely identify the group $U_6(p)$ with this new presentation.

4.2 Irreducible representations

We will now establish the list of irreducible representations of $U_6(p)$, up to equivalence. We will induction of representations in a crucial way to construct them as explicitly as possible.

Proposition 4.2.1. *$U_6(p)$ has p^2 inequivalent irreducible representations of degree 1.*

Proof. The abelianisation of $U_6(p)$ has order p^2, and therefore there are p^2 inequivalent irreducible representations of degree 1. □

We will need the following lemma, which can be found in [11, Lemma 6.7].

Lemma 4.2.2. *Let G be a finite p-group and $H \leq G$ a subgroup. If $(G : H) = p$, then H is normal in G.*

Proposition 4.2.3. *$U_6(p)$ has $p^3 - 1$ inequivalent irreducible representations of degree p.*

Proof. Consider the subgroup $S = \langle a, [a,b], [a,b,b], [a,b,b,b] \rangle \leq U_6(p)$. Notice that it contains $[a,b,a]$ as a commutator, and so $|S| = p^5$, which shows that S has index p and therefore it is normal in $U_6(p)$ by Lemma 4.2.2. Let $\chi : S \to \mathbb{C}^*$ be a representation of degree 1. We compute the induced representation $\pi = \mathrm{Ind}(\chi) : U_6(p) \to U(\mathbb{C}^p)$ (recall the construction from Definition 1.4.7). Since $U_6(p)/S$ is isomorphic to a cyclic group of order p (generated by the image of b), we can describe the action of π on an adapted basis $\{\xi_n\}_{n=0,\dots,p-1}$ of $\mathbb{C}^p$:

$$\begin{aligned}
&\pi(b)\xi_n = \xi_{n+1}, \\
&\pi(a)\xi_n = \chi([a,b,b,b])^{n(n+1)(n+2)/6}\chi([a,b,b])^{n(n+1)/2}\chi([a,b])^n\chi(a)\xi_n.
\end{aligned}$$

Let us compute the character χ_π explicitly. For $w = b^{e_1}[a,b,b,b]^{e_2}[a,b,b]^{e_3}[a,b]^{e_4}a^{e_5}[a,b,a]^{e_6}$, we can check that :

$$\begin{aligned}
\pi(w)\xi_n = &\chi([a,b,b,b])^{e_2+e_3n+e_4n(n+1)/2+e_5n(n+1)(n+2)/6}\chi([a,b,b])^{e_3+e_4n+e_5n(n+1)/2} \\
&\chi([a,b])^{e_4+ne_5}\chi(a)^{e_5}\xi_{n+e_1}.
\end{aligned}$$

Therefore, we see that $\chi_\pi(w) = 0$ if $e_1 \neq 0$, and if $e_1 = 0$ we have :

$$\begin{aligned}
\chi_\pi(w) = \sum_{n=0}^{p-1} &\chi([a,b,b,b])^{e_2+e_3n+e_4n(n+1)/2+e_5n(n+1)(n+2)/6}\chi([a,b,b])^{e_3+e_4n+e_5n(n+1)/2} \\
&\chi([a,b])^{e_4+ne_5}\chi(a)^{e_5}.
\end{aligned}$$

If χ_1, χ_2 are two characters of S, denote by π_1 and π_2 the associated induced representations. We compute $\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle$ directly :

$$= \frac{1}{p^6} \sum_{n,m,e_6=0}^{p-1} \sum_{e_2=0}^{p-1} (\chi_1([a,b,b,b])\chi_2([a,b,b,b])^{-1})^{e_2}$$

$$\sum_{e_3=0}^{p-1} (\chi_1([a,b,b,b])^n \chi_2([a,b,b,b])^{-m} \chi_1([a,b,b])\chi_2([a,b,b])^{-1})^{e_3}$$

$$\sum_{e_4=0}^{p-1} (\chi_1([a,b,b,b])^{n(n+1)/2} \chi_2([a,b,b,b])^{-m(m+1)/2} \chi_1([a,b,b])^n \chi_2([a,b,b])^{-m}$$

$$\chi_1([a,b])\chi_2([a,b])^{-1})^{e_4}$$

$$\sum_{e_5=0}^{p-1} (\chi_1([a,b,b,b])^{n(n+1)(n+2)/6} \chi_2([a,b,b,b])^{-m(m+1)(m+2)/6} \chi_1([a,b,b])^{n(n+1)/2}$$

$$\chi_2([a,b,b])^{-m(m+1)/2} \chi_1([a,b])^n \chi_2([a,b])^{-m} \chi_1(a)\chi_2(a)^{-1})^{e_5}.$$

Here, we see that if $\chi_1 = \chi_2$, then $\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = 1$ if and only if $\chi([a,b,b,b])$, $\chi([a,b,b])$ and $\chi([a,b])$ are not simultaneously equal to 1. Moreover, we see that if $\chi_1([a,b,b,b]) \neq \chi_2([a,b,b,b])$, then $\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = 0$. If $\chi_1([a,b,b,b]) = \chi_2([a,b,b,b]) \neq 1$, then taking $\chi_1([a,b,b]) = \chi_2([a,b,b]) = 1$, we have p^2 different choices for $\chi([a,b])$ and $\chi(a)$ which yield $\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = 0$. If $\chi_1([a,b,b,b]) = \chi_2([a,b,b,b]) = 1$, and if $\chi_1([a,b,b]) \neq \chi_2([a,b,b])$, we see that $\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = 0$. If $\chi_1([a,b,b]) = \chi_2([a,b,b]) \neq 1$, then taking $\chi_1([a,b]) = \chi_2([a,b]) = 1$, we have p different choices for $\chi(a)$ which yield $\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = 0$. And finally, if $\chi_1([a,b,b]) = \chi_2([a,b,b]) = 1$, we see that if $\chi_1([a,b]) \neq \chi_2([a,b])$, then $\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = 0$. If $\chi_1([a,b]) = \chi_2([a,b]) \neq 1$, we must take $\chi_1(a) = \chi_2(a) = 1$ to obtain $\langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = 0$. In total, this proves that this construction yields $(p-1)p^2 + (p-1)p + (p-1) = p^3 - 1$ inequivalent, irreducible representations of degree p. $\square$

Remark. Notice that all irreducible representations of $U_6(p)$ which we have described so far are trivial on the central element $[a,b,a]$, and therefore they factor through the quotient $U_6(p)/\langle [a,b,a] \rangle$, which has order p^5. Since $p^5 = p^2 + (p^3-1)p^2$, by Proposition 1.4.4, we see that we have described the full list of irreducible representations of said quotient (up to equivalence). This remark will be useful later when we apply a result to obtain an upper bound on $\cos(\alpha(A,B))$.

Proposition 4.2.4. *$U_6(p)$ has $p^2 - p$ inequivalent irreducible representations of degree p^2.*

Proof. Consider the subgroup $R = \langle b, [a,b] \rangle \leq U_6(p)$. It has order p^5 and therefore has index p, which implies that it is normal in $U_6(p)$ by Lemma 4.2.2. We will induce irreducible representations of degree p of R to obtain irreducible representations of degree p^2 of $U_6(p)$. Consider the subgroup $T = \langle b, [a,b,b], [a,b,a] \rangle \leq R$. It has order p^4 so it has index p (in R), and therefore it is normal in R, by Lemma 4.2.2. Let $\chi : T \to \mathbb{C}^*$ be a representation of degree 1 such that $\chi([a,b,a]) \neq 1$. We compute the induced representation $\rho = \mathrm{Ind}(\chi) : R \to U(\mathbb{C}^p)$. Since R/T is isomorphic to a cyclic group of order p (generated by the image of $[a,b]$), we can describe the action of ρ on an adapted basis $\{\xi_n\}_{n=0,\ldots,p-1}$ of $\mathbb{C}^p$:

$$\begin{aligned}
&\rho([a,b])\xi_n = \xi_{n+1},\\
&\rho(b)\xi_n = \chi(b)\chi([a,b,b])^{-n}\chi([a,b,a])^{6n(n+1)/2}\xi_n.
\end{aligned}$$

A similar computation as in the proof of Proposition 4.2.3 shows that ρ is irreducible, and that we obtain $p^2 - p$ inequivalent representations by fixing $\chi([a,b,b]) = 1$. We finally compute the induced representation $\pi = \mathrm{Ind}(\rho) : U_6(p) \to U(\mathbb{C}^{p^2})$. Since $U_6(p)/R$ is isomorphic to a cyclic group of order p (generated by the image of a), we can describe the action of π on an adapted basis $\{\xi_n^m\}_{n,m=0,\ldots,p-1}$ of $\mathbb{C}^{p^2}$:

$$\begin{aligned}
&\pi(a)\xi_n^m = \xi_n^{m+1},\\
&\pi(b)\xi_n^m = \chi(b)\chi([a,b,a])^{6n(n+1)/2-m(m+1)/2}\xi_{n-m}^m.
\end{aligned}$$

Once again, a direct computation of characters shows that π is irreducible and that different choices of ρ yield different induced representations π, so that we have indeed constructed $p^2 - p$ irreducible representations of degree p^2. $\square$

Since $p^6 = p^2 + (p^3 - 1)p^2 + (p^2 - p)p^4$, by Proposition 1.4.4, the list of irreducible representations of $U_6(p)$ (up to equivalence) is complete.

4.3 Representation angle

We will now attempt a computation of the representation angle between the subgroups $A = \langle a\rangle$ and $B = \langle b\rangle$.

Proposition 4.3.1. *The representation angle satisfies :*

$$\cos(\alpha(A,B)) \geq \sqrt{\frac{3}{p}}.$$

Proof. Let $\pi : U_6(p) \to U(\mathbb{C}^p)$ be an irreducible representation of degree p. We start by identifying the invariant subspaces. It is easy to see that the B-invariant subspace has dimension 1 and is spanned by $\xi_B = \sum_{n=0}^{p-1} \xi_n$. On the other hand, the A-invariant subspace is spanned by basis vectors ξ_i, and the number of basis elements which are fixed by A can be determined by looking at the number of solution of a cubic equation in the finite field $\mathbb{F}_p$. Indeed, we are looking at values of $n \in \mathbb{N}$ (modulo p) which satisfy the equation :

$$\chi([a,b,b,b])^{n(n+1)(n+2)/6}\chi([a,b,b])^{n(n+1)/2}\chi([a,b])^n\chi(a) = 1.$$

Since $\mathbb{F}_p$ is a field, there can be at most 3 distinct solutions. Notice that when $\chi([a,b,b,b]) \neq 1$ and $\chi([a,b,b]) = \chi([a,b]) = \chi(a) = 1$ (which appears in our list of irreducible representations of degree p), there are exactly three solutions for $n = 0, -1, -2$. Therefore, in this case, the A-invariant subspace has dimension 3 and is spanned by ξ_0, ξ_{-1} and ξ_{-2}. For all λ_0, λ_1, λ_2 such that $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 = 1$, we compute :

$$|\langle \lambda_0 \xi_0 + \lambda_1 \xi_{-1} + \lambda_2 \xi_{-2}, \xi_B \rangle| = |\lambda_0 + \lambda_1 + \lambda_2|,$$

which is maximal when $\lambda_0 = \lambda_{-1} = \lambda_{-2} = \sqrt{3}/3$. Finally, since $||\xi_B|| = \sqrt{p}$, we have that :

$$\cos(\alpha(\mathcal{H}^A, \mathcal{H}^B)) = \sqrt{\frac{3}{p}},$$

and therefore :

$$\cos(\alpha(A, B)) \geq \sqrt{\frac{3}{p}},$$

which proves the proposition. □

We will need the following result, which can be found in [5, Theorem 4.1].

Proposition 4.3.2. *Let G be a group generated by two finite subgroups $A, B \leq G$. Let $H \leq Z(G)$ be a central subgroup, and denote by A' and B' the image of A and B in the quotient G/H, respectively. Moreover, let m be the minimal dimension of an irreducible representation of G which is non-trivial on H and such that there exist non-zero A-invariant vectors and non-zero B-invariant vectors. Then we have the following estimate :*

$$\cos(\alpha(A, B)) \leq \sqrt{\cos(\alpha(A', B')) + \frac{1}{m}}.$$

A direct application yields the following bound.

Proposition 4.3.3. *The representation angle satisfies :*

$$\cos(\alpha(A, B)) \leq \sqrt{\sqrt{\frac{3}{p}} + \frac{1}{p^2}}.$$

Proof. We have to compute m, the minimal dimension of an irreducible representation of $U_6(p)$ which is non-trivial on the central element $[a, b, a]$. We see from our description of the irreducible representations of $U_6(p)$ that $m = p^2$. Moreover, a previous remark shows that we have in fact given the complete list of irreducible representations of the quotient $U_6(p)/\langle [a, b, a] \rangle$, and in fact Proposition 4.3.1 proves that if we denote by A' and B' the images of the subgroups A and B in the quotient, then the representation angle satisfies :

$$\cos(\alpha(A', B')) = \sqrt{\frac{3}{p}}.$$

Therefore, we may apply Proposition 4.3.2, which tells us that :

$$\cos(\alpha(A,B)) \leq \sqrt{\cos(\alpha(A',B')) + \frac{1}{m}} = \sqrt{\sqrt{\frac{3}{p}} + \frac{1}{p^2}},$$

which is what we wanted to prove. $\square$

We were unable to determine a precise value of $\cos(\alpha(A,B))$. Still, although the following result is not optimal, it yields a number of new examples of infinite hyperbolic groups with property (T), in the spirit of Example 3.2.14.

Theorem 4.3.4. *For any prime $p \geq 53$, consider the group :*

$$\begin{aligned} G_p = \langle a,b,c | a^p, b^p, c^p, [a,b,a,a], [a,b,a,b], [a,b,b,b,b], [a,b,b,a] = [a,b,b,b,a] = [a,b,a]^6, \\ [a,c,a,a], [a,c,a,c], [a,c,c,c,c], [a,c,c,a] = [a,c,c,c,a] = [a,c,a]^6, \\ [b,c,b,b], [b,c,b,c], [b,c,c,c,c], [b,c,c,b] = [b,c,c,c,b] = [b,c,b]^6] \rangle. \end{aligned}$$

Then G_p has property (T).

Proof. Consider the finite subgroups $A = \langle a \rangle$, $B = \langle b \rangle$ and $C = \langle c \rangle$. Together, they generate G_p. Moreover, we see that the subgroups $H_1 = \langle A \cup B \rangle$, $H_2 = \langle A \cup C \rangle$ and $H_3 = \langle B \cup C \rangle$ are all isomorphic to $U_6(p)$, and therefore by Proposition 4.3.3, the associated representation angles $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$ satisfy :

$$\alpha \geq \arccos(\sqrt{\sqrt{\frac{3}{p}} + \frac{1}{p^2}}).$$

The right hand-side of the lower bound is an increasing function of p, and a numerical computation ensures that for $p \geq 53$, we have $\alpha > \pi/3$. In particular, the sum of angles satisfies the condition $\alpha_1 + \alpha_2 + \alpha_3 = 3\alpha > \pi$, and thus we may apply Theorem 3.2.11 to conclude that G_p has property (T). $\square$

Remark. With the tools developed in [2], it is possible to check that G_p is indeed infinite hyperbolic.

Conclusion

My thesis has been an exciting journey and has introduced me to a strikingly rich subject, the study of locally compact groups, with Kazhdan's property (T) as a motivating and unifying theme. I would like to thank my advisor again for guiding me and introducing me, little by little, to various aspects of this theory. I would like to end this text with a number of openings for further study, and that I look forward to discover more in depth.

We have introduced unitary representations of groups without consideration of potentially relevant topological data. More specifically, unitary representations of topological groups can be required to satisfy some kind of continuity (which we have ignored by considering discrete groups only). In that case, some ideas become irrelevant, and must be adapted to the new setting. For instance, our construction of the regular representation used the counting measure, but in general it makes more sense to equip groups with what we call a *Haar measure*, which can be thought of as the generalisation of the Lebesgue measure on locally compact groups. Abstract harmonic analysis is a subfield of the representation theory of groups which studies analytical features of this nature. We refer to [6] for an introduction to this subject.

Another aspect of group theory which we have not mentioned in this text, although it is relevant in the study of locally compact groups : operator algebras. To each group corresponds an algebra, and therefore we have access to a rich theory of operator algebras at disposal. Regarding property (T) in particular, a recent discovery due to Ozawa (see [12]) unveiled the following profound result : whether a group has property (T) or not can be witnessed inside the group algebra. With the help of computers, it has been shown that $\mathrm{Aut}(F_5)$ (the automorphism group of the free group F_5) has property (T). The work of [7] implies that in fact, $\mathrm{Aut}(F_n)$ has property (T) for all $n \geq 5$. In a similar fashion, their method provides a proof of the fact that $SL_n(\mathbb{Z})$ has property (T) for all $n \geq 3$.

Finally, an important theme in geometric group theory is the study of rigidity properties of geometric representations of groups. For instance, property (FH) is the property that every action by affine isometries on a Hilbert space has a global fixed point. In the case of countable discrete groups, property (T) is equivalent to property (FH). This is a particular case of the Delorme-Guichardet Theorem (see [1, Theorem 2.12.4]). Understanding these geometric aspects of property (T) remains an interesting source of problems to look at.

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