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Market-Consistent Valuation and Funding of Cash Balance Pensions

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Abstract

Cash-balance (CB) pension plans are a mixture of Defined benefit (DB) and Defined contribution (DC) pension plans: they are legally recognized as DB plans but the plan sponsor makes an annual contribution to the participant's notional account hence the resemblance to DC plans. That way, many DB pension plans converted to CB plans in the aim of potentially reducing costs while staying in the DB category for regulatory purposes. This paper aims to analyze the CB pension and contributions using the methods and principles of financial economics to give a market consistent evaluation of the CB liability's costs. In this context, CB benefits, viewed as financial liabilities, are accumulated at guaranteed crediting interest rates which can be retrieved from government bonds yields for instance and valued using financial models.

To evaluate the CB liability's costs, we first determined the crediting rates using the 1-factor Hull & White model and the 2-factor Heath-Jarrow-Morton model. Then we calculated the market consistent value of the CB benefit at certain time t and due at a later date T, known as the valuation factor $V(t,T)$. These valuation factors helped us in the calculations of the actuarial liabilities (AL) and normal costs (NC) of one of three funding methods which takes account of past service and allows full interest credits up to a known future retirement/exit date. The second method covers past service but doesn't allow future interest credits. Having worked in a constant and deterministic environment till now, we introduced a constant salary growth rate, equal to the Euro area inflation rate for the month of June 2021[1] (= 1.9%), which was used in the third method to project all contributions to retirement. Moreover, this method assumes equal units of accrual of the projected benefit in each year of service. Last but not least, we modeled the inflation rate under the 1-factor H&W model which, after being simulated 1000 times, was used in the first two methods to study the 1-year evolution of the total costs.

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[1] https://ec.europa.eu/eurostat

The population used was made of 20 fictive individuals divided in 4 groups according to their past & future services. Furthermore, we used the Euro area data, and in some cases Belgian data, found on different websites[2] to do the different calculations hence the results are all expressed in the € currency. Those results showed that, even though the third method had the least costs, it is a flawed method and should be discontinued whereas the other two methods are more trustworthy regardless their positive and significant total costs. Also, the 1-year projection study put in evidence the fact that plan sponsors have to spare money so as to compensate the additional costs and always meet the liabilities.

Keywords: Cash-balance (CB) pension plans, Defined benefit (DB) pension plans, Defined contribution (DC) pension plans, Hull & White (H&W) model, Heath-Jarrow-Morton (HJM) model, Funding methods, Actuarial liabilities, Normal costs, Market consistent valuation, Valuation factors, Inflation rate.

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[2] Bloomberg; https://ec.europa.eu/eurostat; https://statbel.fgov.be

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List of abbreviations

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AL : Actuarial liability.

CB : Cash-balance.

CPI : Consumer Price Index.

DB : Defined benefit.

DC : Defined contribution.

DOL : Department of Labor (United States of America).

F : Fund.

H&W : Hull and White.

HJM : Heath-Jarrow-Morton.

IRS : Internal Revenue Service (United States of America).

NC : Normal cost.

ZC : Zero-coupon.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

A Cash-Balance (CB) pension plan can be referred to as a career-average-pay plan (Godwin N.H. and Key K.G., 2000 [4]). The employer of a certain organization sets up a notional account, for the employee's sake, to which an annual contribution calculated using a percentage on the employee's salary is made.
First introduced in 1984 by Bank of America, CB are hybrid plans regulated as Defined-Benefit (DB) pension plans since the benefits are based on a formula but resemble Defined-Contribution (DC) pension plans (Brown D.T., Dybvig P.H. and Marshall W.J., 2001 [2]). In other words, legally speaking these plans are DB pension plans since the employee's benefits are paid according to a formula determined in advance. However, the fact that the employer makes an annual contribution to the employee's notional account makes the plan look like DC pension plans. The employee's benefit is the balance of that account at any time and this balance can be transferred from one company to another in case the employee changes companies.
Pay-related credits (based on salary and wages) and interest-related credits (based on crediting rates linked to a market rate) nourish the employee's cash balance periodically (for instance, once a year).
The percentage and guaranteed growth rate, usually based on 30-year treasury bill interest rates, are defined beforehand and set to last till the employee's retirement.

In the late 1990's, some employers in the United States of America went from DB pension plans to CB ones mainly because they resembled much to the DC pension plans and were cheaper (Hardy M.R., Saunders D. and Zhu X., 2014 [5]). In addition, CB plans are more attractive to younger participants with shorter services and is more open to maximize shareholders' profit whereas DB plans seem to provide good benefits to older members.

In this work, we shall first describe in detail a Cash-Balance pension plan by giving out its advantages & drawbacks and compare it to other pension plans: the Defined-Benefit and the Defined-Contribution pension plans. Then we shall define two short-term interest rate models: the 1-factor Hull & White (H&W) and the 2-factor Heather-Jarrow-Morton (HJM) models which will be calculated/calibrated to European (and in some cases Belgian) data and will help us in the determination of different valuation factors $V(t,T)$, i.e the market consistent value of a CB benefit at t and due at T. Later on, these valuation factors will be used in one of the funding methods to find the actuarial liabilities (AL) and the normal costs (NC). Finally we shall study a 1-year evolution of the AL and the NC, along with the fund (F), so as to see the progress in the total costs and fund of a certain organisation using the inflation rate, which will be modeled under a 1-factor H&W model and simulated a 1000 times, as the growth rate.

1.2 Literature review

Cash-balance pension plans were considered to be an alternative to the Defined-Benefit pension plans' terminations because both employees and companies were better off converting than terminating (Maury M. and Shoaf V., 2001 [7]). Even though CB pension plans are mixture of both DB and DC pension plans, there is some important differences among those three pension plans. Authors like Kapinos K.A., Coronado J.L. and Copeland P.C. studied the differences and the reasons behind the CB pension plan conversions along with the advantages and inconveniences of CB pension plans.

Kapinos K.A. (2009 [6]) made an investigation to determine the main reasons (or determinants) behind the traditional DB plan's conversion to the CB plan in the 1990's using pension plan level 5500 (a form submitted to the United States of America's Internal Revenue Service [IRS], by companies providing pension plans, detailing information about the plan, the number of participants and other plan characteristics) data from the USA's Department of Labor [DOL] and the IRS.

Knowingly, DC plans offer advantages to both employers and employees: they have lower costs (company's view) and offer greater flexibility and portability (employee's view) compared to the traditional DB plans. However, in case of a plan change/conversion, the company must terminate its current plan and doing so, all the participants become vested. Moreover, when a company converted its DB pension plan into a DC pension plan and the plan happened to be over-funded (market value of plan assets greater than present discounted value of pension obligations), income taxes and a tax surcharge were applied on those assets.
CB plans, being a mixture of DB and DC plans and legally recognized as DB plans, the conversion from a DB pension plan to a CB pension plan didn't cost anything and no taxes were applied at all; only a plan modification needed to be filed to the DOL and IRS.
A comparison among the DB, DC and CB pension plans was made as seen on the figure 1 in Appendices .1 where we can observe the various features and differences among the three pension plans.

Coronado J.L. and Copeland P.C. (2004 [3]) made a study, similar to the one made by Kapinos K.A. (2009 [6]), whose goal was to estimate a hierarchy of the influences on the decision of a company to convert its DB pension plan to a CB pension plan. They start by explaining the reason why a company would opt for a DB pension plan in first place: benefits don't accrue evenly under this plan and the period where they accrue most promptly is the one right before retirement as seen on the figure 2 in Appendices .1. Companies tended to benefit from this back-loaded nature of benefits since it reduced turnover, due to the fact that employees leaving the company before retirement would lose some capital, and it also helped: improving productivity, encouraging work effort and regulating work behavior.
Nevertheless, DB pension plans weren't suitable for industries facing a change in production technology and happened to be characterized by younger and mobile employees. In this case, the companies could either terminate their DB plan and opt for a DC plan instead or convert to a CB plan. The first option (plan termination) was not optimal since a special tax, known as the reversion tax, was applied on excess assets in case of plan termination. Moreover, this tax was equal to 50% in 1990, compared to 15% before that year. The second option (CB plan conversion) was better as it helped the companies avoid the reversion tax and compete for more mobile workers: under the CB plan, other than the fact that employees could take a pre-retirement lump sum distribution from the plan if they left the firm, their benefits accrued earlier and evenly over their career.

Using data retrieved from companies in the S&P500 who had converted their DB pension plan to a CB pension plan by the year 1998, the authors concluded that the change in labor market conditions influenced the most those companies; i.e, conversions were mostly found in industries with younger, more mobile workers and tighter labor markets.

Maury M. and Shoaf V. (2001 [7]) enumerated three main reasons why the CB pension plans kept expanding to many other companies (which were using DB pension plans) since their introduction in the mid-1980's:

- **Low cost**: companies could reduce their pension costs by 25-40% and since in the CB plan provisions weren't to be made anymore (no need to build up future pension liabilities and only current funding is required), the pension surplus could be used for contributions for the following few years hence reducing the company's liabilities thereby boosting earnings and gaining in competitiveness.
- **Flexibility**: workers are not obliged to stay longer in the enterprise so as to build their pensions.
  Under the DB pension plan, when an employee leaves the company, if the benefit hasn't become vested yet, the employee will get nothing at the moment and will wait his retirement date. Also, if the employee happens to go to another company, the benefits acquired till that moment will be frozen (till the retirement date) and the new company's model will be of application hence starting over the number of years spent in the company for instance.
  However, under the CB pension plan, when the employee leaves the company and happens to be vested, the balance on the assigned account is distributed as a lump sum that the employee can therefore invest or transfer the amount to the retirement account. Also, if the employee goes to a new company, that sum can be transferred to a new fictional account under the new company's regime so that the employee's pension could keep growing. Nevertheless, contrary to other pension plans, the benefits are not guaranteed under this plan.
- **Attractiveness to young people**: in a world where the workplace was changing, the fact that employees could easily calculate their pension balance at any time and the flexibility of the CB pension plan were great tools to attract young talents.

However, despite these great advantages for both the companies and the employees, there was the "**wearaway**" or "pension plateaus" feature which was being used by

companies who had recently adapted the CB pension plan and happened to be discriminating old workers.
This feature allowed employers not to contribute for employees whose earned benefit's present-value exceeded the amount that would have been accrued in the cash balance if it had been used from the beginning of the employee's occupation and start contributing when the inequality sign changes. Hence, the employers would spend a few years without contributing (which is at their advantage) but it also means that senior workers lose big since their pension is stagnant while they are closer than ever to their retirement.
A solution to avoid this discrimination was to allow senior employees who were at least 40 years old and had 10 years of service with the company to remain in the DB pension plan.

Brown D.T., Dybvig P.H. and Marshall W.J.(2001 [2]) made a study about the cost and the duration of CB pension plans depending on the choice of the rates.
In the figure 3 (see Appendices .1), an evaluation of the cost of funding a CB liability for a given particular benchmark yield and the duration were determined using diverse assumptions/criteria (maturity, presence or not of margins and yield curves/crediting rates) and the mean-reverting Vasicek Gaussian model of interest rates, with a mean reversion equal to 15% and a standard deviation equal to 1%, to describe uncertainty. The margins used were in link with the IRS (Internal Revenue Service) "safe-harbor", varying from 175 bps (for 3-month treasury bills discount rate) to 25 bps (for 5- or 7-year treasury yields). The values found in each "cost" column represent the cost per $1 dollar in cash balance, given a certain yield and maturity.
We can observe that costs under the panel B (taking into account the IRS margins) are higher than the corresponding ones in the panel A due to the additional basis points. We also notice that the longer the maturity and the the longer the yields, the less the cost when in panel A but the same can't be said when IRS margins are taken into account.

1.3 Framework, Assumptions, Notation

1.3.1 The CB Benefit

As in Hardy, M.R. and Saunders, D. and Zhu, X. (2014 [5]), we shall use a market consistent/arbitrage free model for interest rates to value payments by evaluating their expected discounted value. Let

- T : the retirement date and $T > t$;
- F_t : the participant's notional account value at t;
- F_0 : the participant's notional account value at the valuation date $t = 0$;
- F_T : the final lump-sum benefit for the participant at his/her retirement date T;
- $r^c(t)$: the continuously compounded *crediting* interest rate declared at t.

We have :

$$F_T = F_0 e^{\int_0^T r^c(t)dt} \tag{1.1}$$

hence the benefit is found using the continuous crediting and this crediting is done on an annual basis in most cases.

Hardy, M.R. and Saunders, D. and Zhu, X. (2014 [5]) found that only in 7% cases a fixed crediting rate was used and the other times the crediting rate would be market-dependent: for instance the yield on 30-year treasury bonds (in 41% cases), the 1-year treasury bond yield at t with a 1% margin (in 19% cases), treasury bond yields at other durations (in 20% cases) or the equity-based rate (remainder). However, the authors chose to use fixed treasury bond crediting rates in their paper. Later on in this paper we shall use k-year spot rates $r_k(t)$ with some fixed margins m as our continuously compounded crediting interest rates $r^c(t)$.

1.3.2 The Yield Curve

With $r(t)$ being the continuously compounded *risk-free* short rate of interest at t, then $r_k(t)$ is the k-year spot rate observed at t. Furthermore, $r(0) = lim_{k\to 0}\, r_k(0)$ where $r(0)$ is known at t=0.

Here, zero-coupon bond prices are of a big utility and can be defined here below:

- $P(t,T)$: the price at t of 1 unit (of currency) due at T (observable at t);
- $P(t,t+k) = e^{-k\ r_k(t)}$: the zero-coupon bond price, is the present value at t of 1 unit due at $t+k$;
- $P(0,T)$: the present value of 1 unit due at T.

Moreover, we can also define $f(t, t+k)$ as the forward rate contracted at t and applying at $t+k$.

Assuming an arbitrary free model for the future term term structure, the market value of a payment at $t=0$ and due at $t=T$ must be equal to the expected discounted value of that payment. In other words :

$$P(0,T) = E_0^Q \left[e^{-\int_0^T r(s)ds} \right] \tag{1.2}$$

where 0 indicates the time t and Q the risk-neutral valuation.

Finally some relations among the notations above can be deducted, where :

$$P(t, t+k) = e^{-\int_t^{t+k} f(t,s)ds} \tag{1.3}$$

which leads to

$$f(t, t+k) = -\frac{d}{dk} \log P(t, t+k) = -\frac{1}{P(t, t+k)} \frac{d}{dk} P(t, t+k)$$

and

$$P(t, t+k) = e^{-k\ r_k(t)} = e^{-\int_0^k f(t,t+u)du} = E_t^Q \left[e^{-\int_t^{t+k} r(s)ds} \right] \tag{1.4}$$

with $r(t) = f(t, t)$.

1.3.3 The Valuation Formula

Knowing that F_T is the (random) payoff at T, its value at t=0 is :

$$_0V = E^Q \left[F_T\ e^{-\int_0^T r(t)dt} \right] = F_0 E^Q \left[e^{\int_0^T (r^c(t) - r(t))dt} \right] \tag{1.5}$$

hence reducing the cost of CB benefit to just the difference between the crediting interest rate $r^c(t)$ and the risk-free discount (instantaneous market) rate $r(t)$.

Let $\boldsymbol{V(t,T)}$ be **the valuation factor and the market consistent value of a CB benefit** at t and due at T, we have $V(0,T)$ that can be written as :

$$V(0,T) = E^Q \left[e^{\int_0^T (r^c(t) - r(t))dt} \right] \tag{1.6}$$

As mentioned above, we shall assume that $r^c(t) = r_k(t) + m$ with $r_k(t)$ being the k-year spot rate that can be retrieved from the zero-coupon bond price $[p(t, t+k) = e^{-kr_k(t)}]$ and m being a constant margin. Hence the calculation of the ZC bond prices will be our main priority in the determination of valuation factors.

CHAPTER 2

SHORT-TERM INTEREST RATE MODELS

2.1 One-factor Hull & White model

2.1.1 General model description

The one-factor H&W model for interest rates is an extension of the Vasicek model.

The Vasicek model helps define instantaneous-spot-rate dynamics which is, under the risk-neutral measure:

$$dr(t) = k[\theta - r(t)]dt + \sigma dW(t),\ r(0) = r_0$$

where r_0, k (the speed of the mean reversion), θ (the mean reversion level) and σ (the rate volatility) are positive constants.

This model was found useful for providing analytical pricing of bonds and bond options. It is however time-homogeneous and produces endogenous term structure of interest rates (the current term of rates is an output of the model instead of being an input) hence leading to a poor reproduction of the yield curve and so on.
The 1-factor H&W model extending the Vasicek model resolves these two problems mentioned above by introducing a time-varying parameter: in this case, θ becomes $\theta(t)$ which allows the model to reproduce accurately the curve at $t = 0$. Other than that, the model has the same characteristics as the Vasicek model: possibility of having negative rates with a positive probability, the instantaneous-short-rate $r(t)$ follows a Gaussian distribution and the analytical pricing of bonds and bond

options.

The short rate dynamics becomes then:

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma dW(t) \tag{2.1}$$

where $\theta(t)$ is to be chosen wisely in order to fit the term structure of interest rates being observed in the market at the moment and a, σ are positive constants.
Let $f^M(0,T)$ be the market instantaneous forward rate at $t = 0$ for the maturity T:

$$f^M(0,T) = -\frac{\partial lnP^M(0,T)}{\partial T} \tag{2.2}$$

with $P^M(0,T)$ being the market discount factor, then:

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial T} + af^M(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{2.3}$$

where $\frac{\partial f^M}{\partial T}$ represents the partial derivative of f^M according to its second argument.
After integration, $r(t)$ becomes:

$$\begin{aligned} r(t) &= r(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \sigma \int_s^t e^{-a(t-u)}dW(u) \qquad (2.4) \\ &= r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)}dW(u) \qquad (2.5) \end{aligned}$$

with

$$\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2$$

The mean and variance of $r(t)$ conditionally to $\mathcal{F}_s$ are as follow:

$$\begin{aligned} E\{r(t)|\mathcal{F}_s\} &= r(s)e^{-a(t-s)} + \alpha(t) - \alpha(s)e^{-a(t-s)} \\ V\{r(t)|\mathcal{F}_s\} &= \frac{\sigma^2}{2a}(1 - e^{-2a(t-s)}) \end{aligned}$$

As for the zero-coupon bond price at t and with a $t + k$ mature date, it is defined as follow :

$$P(t, t+k) = E_t^Q\left[e^{-\int_t^{t+k} r(s)ds}\right] = \exp\left(A(t, t+k) - B(t, t+k)r(t)\right) \tag{2.6}$$

with

$$B(t, t+k) = \frac{1 - e^{-ak}}{a}$$

$$A(t,t+k) = \log\frac{P(0,t+k)}{P(0,t)} + f(0,t)B(t,t+k) - \frac{\sigma^2}{4a}B(t,t+k)^2(1-e^{-2at})$$

where $P(0,t),\ P(0,t+k)\ \&\ f(0,t)$ are retrieved from the yield curve information at $t = 0$.

2.1.2 H&W model for inflation rate $i(t)$

We have just defined and shown the Hull and White 1-factor short rate model's dynamics. Here we shall define and give the dynamics of the inflation rate under the same model, which would be correlated to the model described in the section above. However, even though the model's parameters will be estimated along with the interest rate's parameters (section 2.1.5.1) for practical reasons, it will only be used later on in the funding methods' part (**section 3.3.2**).

Let a reference index, in this case the Consumer Price Index (CPI), be a representative basket of goods and services. Inflation can then be defined as the increase in percentage of the CPI over time :

$$i(t) = \frac{I(t)}{I(t-1)} - 1$$

where $i(t)$ is the inflation rate and $I(t)$ is the price index, both at time t.

The inflation rate is related to the nominal and real interest rates, according to the Fisher equation, as follow :

$$1 + R = \frac{1+r}{1+i}$$

where R, r and i are respectively the real interest rate, the nominal interest rate and the inflation rate. This equation can also be approximated as : $R \approx r - i$.

The aim of this section is to model the inflation rate under a mean-reverting short term rate. We chose the H&W 1-factor inflation model $i(t)$ correlated to the H&W 1-factor interest rate model $r(t)$ (section 2.1.1) since there is a direct link between those two rates. Let the inflation rate dynamics be defined as follow :

$$di(t) = [\theta_I(t) - a_I\ i(t)]dt + \sigma dW_I(t) \tag{2.7}$$

where a_I, σ_I are positive constants. As for $\theta_I(t)$, it must be chosen wisely in order to fit the term structure of interest rates being observed in the market at the moment and in a similar way as in equation [2.3] we have that, under the risk neutral measure:

$$\theta_I(t) = \frac{\partial f_I(0,t)}{\partial T} + a_I f_I(0,t) - \frac{\sigma_I^2}{2a_I}(1-e^{-2a_I t}) + \rho_{i,r}\sigma\sigma_I\left[\frac{1}{a}(1-e^{-at}) + \frac{1}{a_I}(1-e^{-a_I t})e^{-at}\right] \tag{2.8}$$

with a, σ being the same positive constants used in the short interest rate $r(t)$ dynamics, $\rho_{i,r}$ being the correlation between the two rates and $f_I(0,T)$ being the forward inflation rate

$$f_I(0,T) = \frac{\partial}{\partial T} ln \hat{I}(0,T) = \frac{\partial}{\partial T} ln\left(\frac{P_I(0,T)}{P(0,T)}\right) \tag{2.9}$$

where $P_I(0,T)$, $I(T)$ and $\hat{I}(0,T)$ are, respectively, the price of the zero-coupon index-linked bond, the price index (CPI) at T and the projected price index

$$P_I(t,t+k) = I(t)\ E_t^Q\left[e^{-\int_t^{t+k}[r(s)-i(s)]ds}\right] \tag{2.10}$$

$$I(T) = I(0)e^{\int_0^T i(s)ds} \tag{2.11}$$

After integration, $i(t)$ becomes:

$$i(t) = i(s)e^{-a_I(t-s)} + \int_s^t e^{-a_I(t-u)}\theta_I(u)du + \sigma_I\int_s^t e^{-a_I(t-u)}dW_I(u) \tag{2.12}$$

with $d\langle W, W_I\rangle = \rho_{i,r}dt$.

2.1.3 H&W's calibration and parameters' estimation

In this part, we shall give all the necessary tools to estimate and calibrate the 1-factor H&W short rate model's parameters. We shall start with the constant parameters a and σ then end with the calibration of $\theta(t)$. The same thing can also be done for the 1-factor H&W inflation rate model's parameters: a_I and σ_I and $\theta_I(t)$.

The constant parameters a and σ can be estimated using the Vasicek model, under the real measure $\mathbb{P}$. The model was defined above as:

$$dr_t = k[\theta - r_t]dt + \sigma dW_t \tag{2.13}$$

with k, θ, σ being positive constants. Considering the use of short-term rates, for instance on a monthly basis, the variation between two successive rates can be written as :

$$r_{t+1} - r_t = \Delta r_t = k[\theta - r_t]\Delta_t + \sigma\sqrt{\Delta_t}\ \mathbb{N}(0,1) \tag{2.14}$$

where we would have $\Delta_t = 1/12$ in this case and the three constant parameters (k, θ, σ) remaining unknown. However, knowing that the monthly variation Δr_t follows the Gaussian distribution as follow :

$$\Delta r_t \sim \mathbb{N}\Big(k(\theta - r_t)\Delta_t, \sigma^2\Delta_t\Big),$$

we can estimate the parameters k and θ by log-likelihood maximisation, which is equivalent to minimizing the sum of squared differences between the variation Δr_t and its mean :

$$\hat{k}, \hat{\theta} = \arg\min_{k,\theta} \sum_{i=1}^{n} \Bigg(\Delta r_t - k(\theta - r_t)\Delta_t\Bigg)^2 \tag{2.15}$$

After finding k and θ, the variance σ^2 can then be estimated as follow :

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n} \Bigg(\Delta r_t - k(\theta - r_t)\Delta_t\Bigg)^2 \tag{2.16}$$

With all that done, we can now estimate the constant parameters a and σ of the 1-factor H&W model. The short rate dynamics in equation [2.1] can be rewritten as :

$$dr(t) = a\Big[\frac{\theta(t)}{a} - r(t)\Big]dt + \sigma dW(t) \tag{2.17}$$

where we can observe a great resemblance between this equation [2.17] and the equation [2.13]. However, in the H&W case, the mean reversion level is time-varying $\theta(t)$ and the short term rate, at time t tending to infinity, reverts to $\frac{\theta(t)}{a}$ at rate a. All that being said, a and σ can be estimated as follow:

- $\hat{a} = \hat{k}$,
- $\hat{\sigma}_{H\&W} = \hat{\sigma}_{Vasicek}$.

As for the parameter $\theta(t)$, it can be calibrated using the Nelson Siegel whose rate is equal to:

$$r^{NS}(t) = \beta_0 + \beta_1 \frac{1 - exp(-t/\tau)}{t/\tau} + \beta_2\Big(\frac{1 - exp(-t/\tau)}{t/\tau} - exp(-t/\tau)\Big) \tag{2.18}$$

with $\beta_0, \beta_1, \beta_2, \tau$ being constant parameters to be estimated by minimization of the sum of square of the residual:

$$\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, \hat{\tau} = \arg \min_{\beta_0,\beta_1,\beta_2,\tau} \sum_t \left(r^{NS}(t) - r(t) \right)^2 \tag{2.19}$$

where $r(t)$ usually refer to spot rates. These parameters intervene in the determination and calibration of $\theta(t)$ [2.3] as they help in the calculation of the market instantaneous forward rate $f(0,T)$ [2.2] and its derivative $\partial f(0,T)$ as follow:

$$\begin{aligned} f(0,t) &= R_t + t\frac{\partial R_t}{\partial t} \\ \frac{\partial}{\partial t} f(0,t) &= 2\frac{\partial R_t}{\partial t} + t\frac{\partial^2 R_t}{\partial t^2} \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} R_t &= \beta_0 + (\beta_1 + \beta_2)\Big(1 - e^{-t/\tau}\Big)\frac{\tau}{t} - \beta_2 e^{-t/\tau} \\ \frac{\partial R_t}{\partial t} &= (\beta_1 + \beta_2)\Big(\frac{e^{-t/\tau}}{t} - \tau\frac{1 - e^{-t/\tau}}{t^2}\Big) + \beta_2 \frac{1}{\tau} e^{-t/\tau} \\ \frac{\partial^2 R_t}{\partial t^2} &= (\beta_1 + \beta_2)\Bigg(- \frac{e^{-t/\tau}}{t\tau} - \frac{e^{-t/\tau}}{t^2} + \frac{2\tau}{t^3}\Big(1 - e^{-t/\tau}\Big) - \frac{e^{-t/\tau}}{t^2}\Bigg) - \beta_2 \frac{1}{\tau^2} e^{-t/\tau} \end{aligned}$$

2.1.4 Valuation Formula : 1-factor H&W model

The valuation of the benefit can be done using a market consistent model of the term structure of interest rates. Here we shall use a 1-factor H&W model:

$$dr(t) = (\theta(t) - ar(t))dt + \sigma dW_t$$

with $a > 0,\ \sigma > 0$ being constant parameters and $\theta(t)$ being a deterministic function of t as seen above in equation [2.1].
Combining the 1-factor H&W model's zero-coupon bond price in the equation [2.6] with the fact that $p(t, t+k) = e^{-k\ r_k(t)}$, we can retrieve the k-year spot rate:

$$r_k(t) = \frac{B(t,t+k)r(t) - A(t,t+k)}{k} \tag{2.21}$$

then, assuming the crediting rate to be:

$$r^c(t) = r_k(t) + m \tag{2.22}$$

we can finally deduce the valuation formula, which hands out the valuation factor per 1 unit of participant's fund at $t = 0$, in this model using the equation [1.6]:

$$V(0,T) = E_0^Q \left[e^{\int_0^T (r_k(t)+m-r(t))dt} \right] \tag{2.23}$$

$$= E_0^Q \left[e^{\int_0^T (\frac{B(t,t+k)r(t)-A(t,t+k)}{k}+m-r(t))dt} \right] \tag{2.24}$$

with $r(t)$ being the random variable and the valuation formula can be rewritten as:

$$V(0,T) = \exp(mT)\ \exp\left(\int_0^T -\frac{A(t,t+k)}{k} dt \right)\ E_0^Q \left[e^{-\int_0^T \gamma\ r(t)dt} \right] \tag{2.25}$$

where

$$\gamma = \left(1 - \frac{B(t,t+k)}{k} \right) = 1 - \left(\frac{1-e^{-ak}}{ak} \right)$$

$$\begin{aligned} \int_0^T \frac{A(t,t+k)}{k} dt &= \int_0^T f(0,T)B(t,t+k)dt + \int_0^T \log \frac{p(0,t+k)}{p(0,t)} dt \\ &- \frac{\sigma^2}{4a} \int_0^T B(t,t+k)^2 (1-e^{-2at}) \\ &= -\log p(0,T) - \int_0^T ((t+k)r_{t+k}(0) - tr_t(0))dt \\ &+ \frac{\sigma^2}{4a} B(t,t+k)^2 \left(T - \left(\frac{1-e^{-2aT}}{2a} \right) \right) \end{aligned}$$

$$\begin{aligned} E_0^Q \left[e^{-\int_0^T \gamma\ r(t)dt} \right] = p(0,T)^\gamma \exp \Bigg(\frac{\sigma^2 \gamma}{2a^2} \Bigg(\left(\frac{1-e^{-aT}}{a} \right)(1-2\gamma) + \frac{(1-e^{-aT})^2}{2a} \\ + \frac{\gamma(1-e^{-2aT})}{2a} - T(1-\gamma) \Bigg) \Bigg) \end{aligned}$$

2.1.5 Practical case

In this section, we first determined the parameters a, a_I, σ and σ_I using the 1-factor H&W models stated above. Then we used the interest rate and the inflation rate models fitted to the Zero-Coupon yields bootstrapped from, respectively, the Euro EURIBOR Interest Rate Swaps (based on 6-month EURIBOR) and the Zero-Coupon inflation swaps, both retrieved from Bloomberg as at **March 23, 2021**. Afterwards, we calculated the two mean reversion levels ($\theta(t)$ & $\theta_I(t)$) and from their respective yields we managed to get the different k-year spot rates, for $k = \{1, 5, 10, 20, 30\}$. The interest rate model's spot rates will be used later on as cred-

iting rates (section 2.1.5.2 & 2.2.6.2). Exits before maturity T are ignored and the time horizon (years to retirement) goes from 5 to 30 years, with an interval of 5 years: $T = \{5, 10, 15, 20, 25, 30\}$.
Finally, we were able to determine the valuation factors $V(0, T)$ and do a parameters' sensitivity analysis.

2.1.5.1 Estimation of the 1-factor H&W interest rate and inflation rate models parameters

From equations [2.15 - 2.16], we used the Euro Interbank Offered Rate (EURIBOR) and the Euro area historical inflation rates from January 2015 to January 2020 (see figures 4 and 5 in Appendix .2) to estimate the Vasicek model's parameters for the interest rate model ($r(t)$) and the inflation rate model ($i(t)$) respectively. Since the constant parameters are the same in the 1-factor H&W models, we get the following results:

Parameters	$r(t)$ estimates	$i(t)$ estimates
Δ_t	1/12	1/12
Mean reverting speed ($\hat{a}$ - $\hat{a_I}$)	3.3187	0.8548
Volatility ($\hat{\sigma}$ - $\hat{\sigma_I}$)	0.001367	0.003016

Table 2.1 – Parameters' estimates for the H&W models

For the mean reversion level $\theta(t)$, we first estimated the Nelson Siegel parameters using equation [2.19] with $r(t)$ referring to spot rates retrieved from the Euro EURIBOR Interest Rate Swaps (based on 6-month EURIBOR) mentioned above, as seen in the figure 2.1 which shows us the comparison between the market spot rates and the calculated Nelson Siegel rates for $r(t)$. We did the same thing for $\theta_I(t)$ but this time we used the spot rates retrieved from the Euro swap breakeven rates, which happen to be the difference between the Euro EURIBOR Interest Rate Swaps, based on 6-month EURIBOR, and the Zero-Coupon inflation swaps (in other words: $r(t) - i(t)$) and we can see the comparison between those spot rates and the calculated Nelson Siegel rates for $(r(t) - i(t))$ in the figure 2.2.

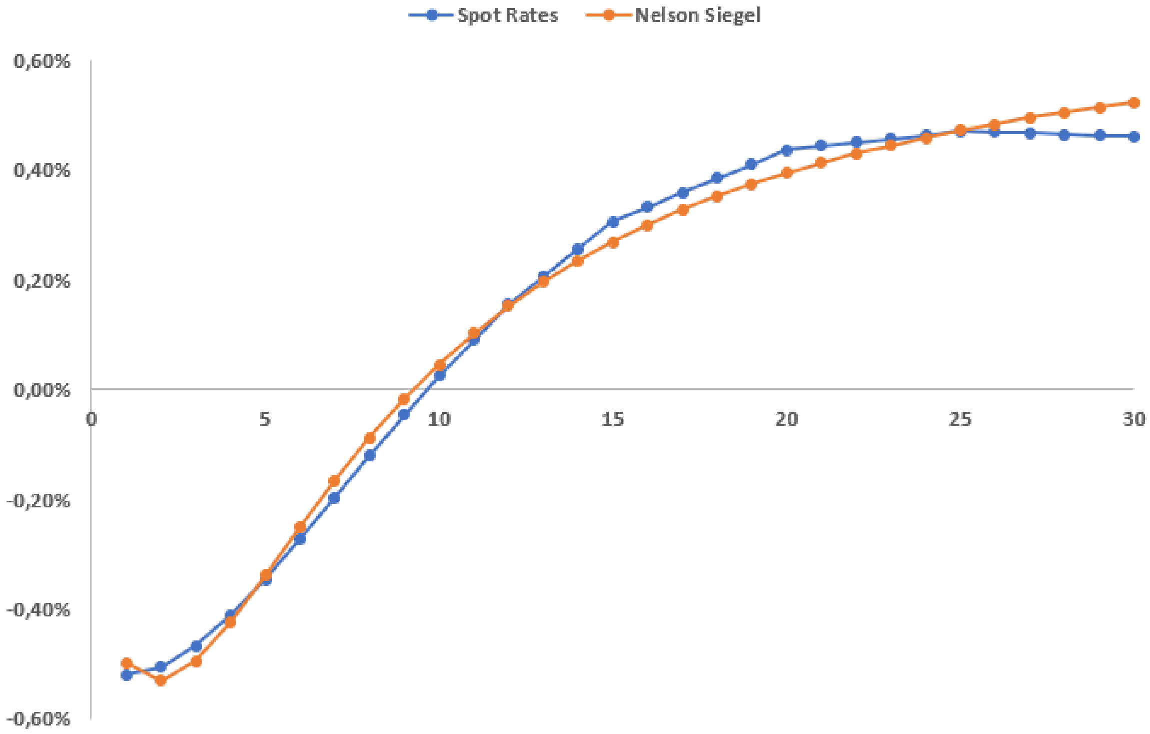

Figure 2.1 – Comparison between the market spot rates (retrieved from Euro EURIBOR Interest Rate Swaps) and the Nelson Siegel rates for $r(t)$ and $t = 1, ..., 30$

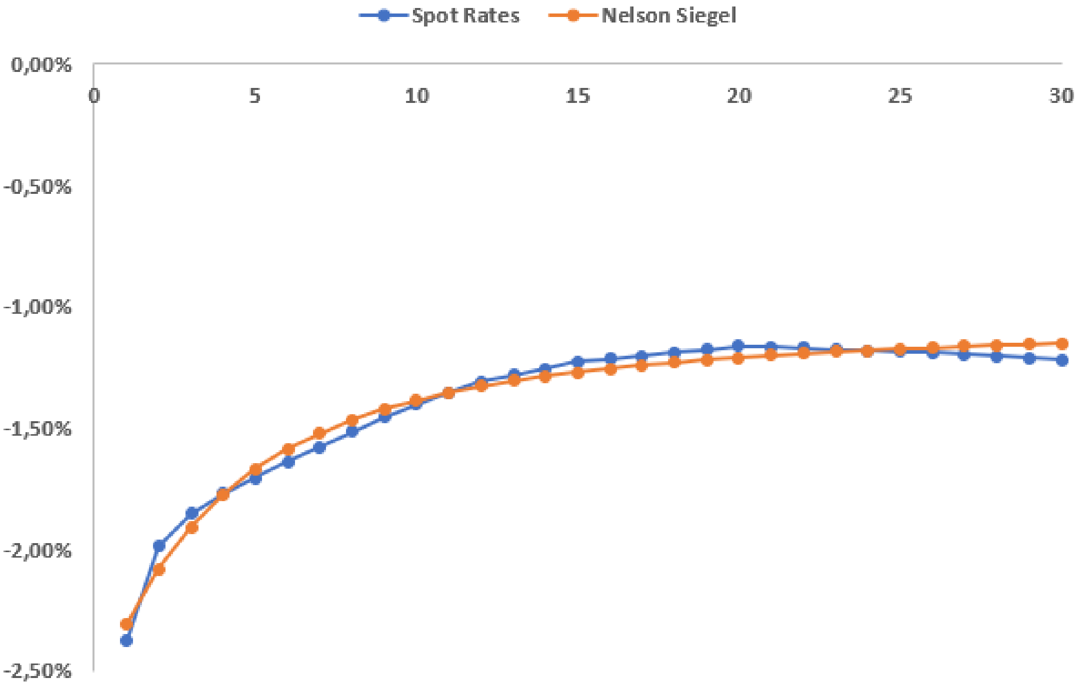

Figure 2.2 – Comparison between the market spot rates (retrieved from breakeven inflation swap rates) and the Nelson Siegel rates for $r(t)$ and $t = 1, ..., 30$

With $\rho_{i,r} = 56.53\%$, the correlation between the 2 rates, we get:

Parameters	$r(t)$ estimates	$i(t)$ estimates
$\hat{\beta}_0$	0.007816	-0.010289
$\hat{\beta}_1$	-0.01118	-0.015747
$\hat{\beta}_2$	-0.02271	-0.000097
$\hat{\tau}$	2.28134	2.280085

Table 2.2 – Parameters' estimates for the Nelson Siegel model [2.18]

Then we calculated the market instantaneous forward rates $\{f(0,t), f_I(0,t)\}$ and their respective derivatives $\{\partial f(0,t), \partial f_I(0,t)\}$ [2.20] which helped us determine the different mean reversion levels ($\theta(t)$ & $\theta_I(t)$), with $t = 1, ..., 30$ as seen on the figures 2.3 and 2.4 where we can observe a similar shape between them but with different values.

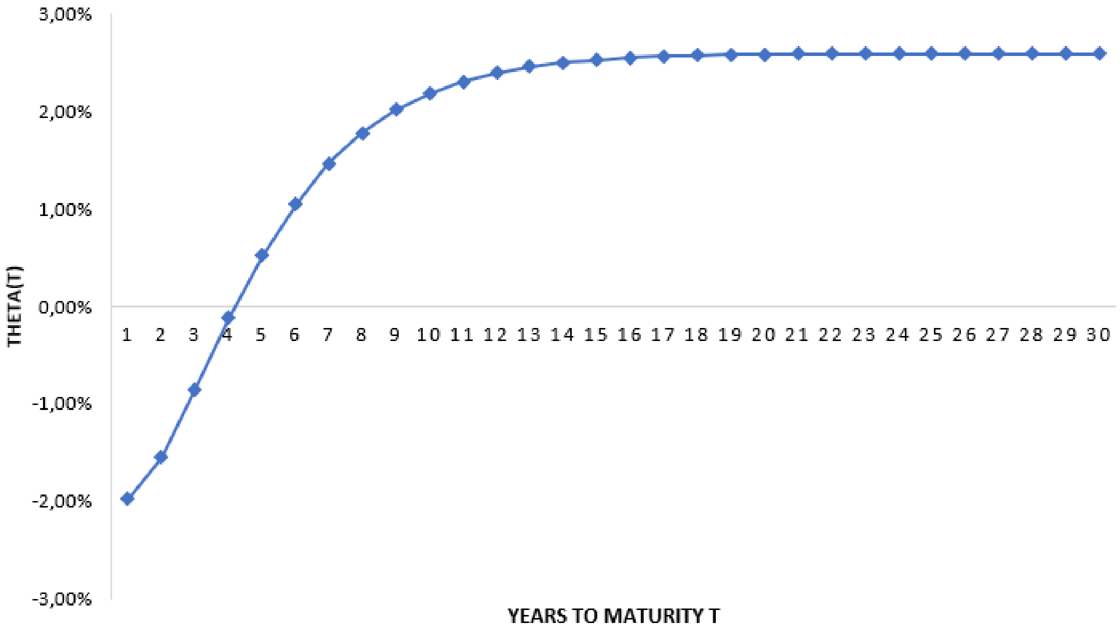

Figure 2.3 – H&W interest rate model's $\theta(t)$ parameter, for $t = 1, ..., 30$

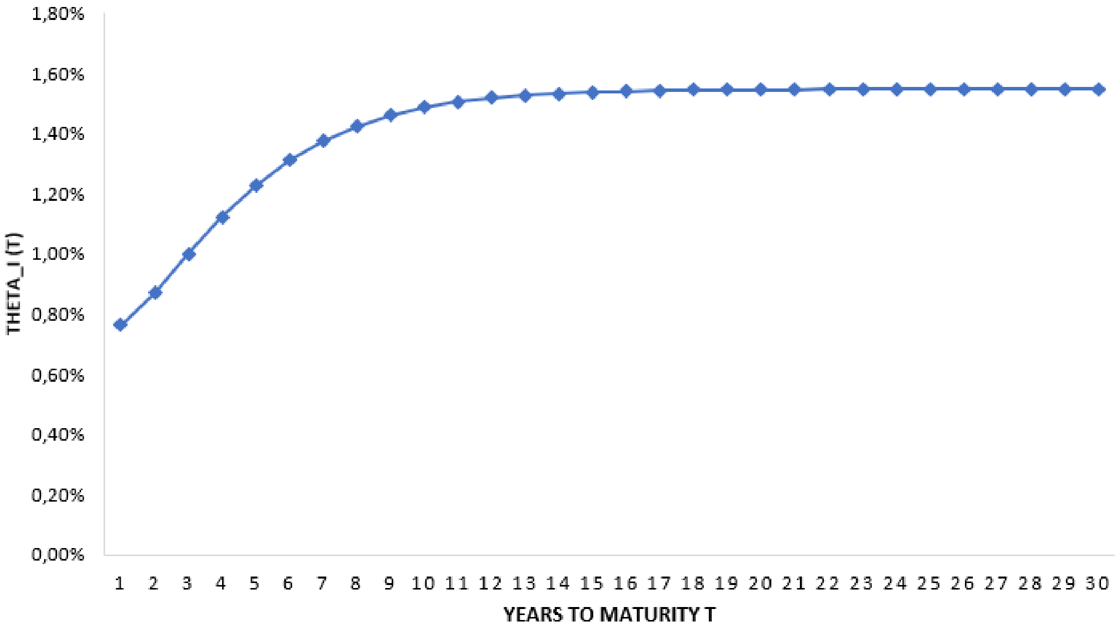

Figure 2.4 – H&W inflation rate model's $\theta_I(t)$ parameter, for $t = 1, ..., 30$

Finally, we were able to simulate a one sample path of the 1-factor H&W interest rate $r(t)$ and inflation rate $i(t)$ for $t = 1, ..., 30$, as seen on the figures 6 and 7 (in Appendix .2). The interest rate model's estimated parameters will be used in the next sections (2.1.5.2 & 2.1.5.3) whereas the ones of the inflation rate model will be used later on in the section 2.2.6.2.

2.1.5.2 Valuation factors V(0,T)

Now that we have determined the parameters $\hat{a} = 3.3187$ and $\hat{\sigma} = 0.1367\%$, we can calculate the valuation factors $V(0, T)$, for $T = \{5, 10, 15, 20, 25, 30\}$, using the equation [2.25].

As seen in the table 2.3, the crediting rates [2.22] were equal to the k-year spot rates $r_k(t)$ in the first three cases, meaning that $m = 0$, but we added two cases with non-null margins : $m = 100$ basis points for the $r_5(t)$ and $m = 150$ basis points for the $r_1(t)$.

		Valuation factor					
k	$r_k(t)$	5	10	15	20	25	30
10	0.03%	1.000633	1.001266	1.001899	1.002533	1.003167	1.003802
20	0.44%	1.011002	1.022126	1.033372	1.044742	1.056236	1.067858
30	0.46%	1.011624	1.023384	1.035280	1.047315	1.059489	1.071805
5+100BP	-0.35%	1.042223	1.086228	1.132091	1.179891	1.229710	1.281631
1+150BP	-0.52%	1.063962	1.132016	1.204422	1.281460	1.363425	1.450633

Table 2.3 – Valuation factors $V(0,T)$ per 1 € of account balance at March 23 2021, with H&W parameters $\hat{a} = 3.3187$ and $\hat{\sigma} = 0.1367\%$

The results are a representation of a market consistent valuation of the interest rate guarantee, per 1 € of each participant's account balance as at **March 23, 2021**. In general, having a valuation factor which is greater than 1 means that, under the CB plan, the interest rate guarantee has a positive market value and the surplus over 1 will be the cost of that interest rate guarantee or, in other words, the additional amount to be added to the fund coming from the employer's pockets. On the contrary, a valuation factor whose value is less or equal to 1 is beneficial to the employer since no additional cost/premium has to be paid or accounted for.
In our case, the valuation factors are greater than 1, for any k-year spot rate and/or time horizon T meaning that the interest rate guarantee has a positive but not very significant market value. Moreover, the more years the participant has before retiring/exiting (T) and the longer the spot rates' time horizon (k), the more the participant earns; with an exception for fixed crediting rates. For instance, if we consider the case with the 30-year spot rate and 30 years to exit ($T = 30$) as crediting rate $r_{30}(30)$, the valuation factor is equal to $V(0,30)$=1.072 € per 1 € of fund meaning that if a 100 € contribution is made into the notional account at the valuation date, the additional liability will be equal to 107.2 € and the premium, 7.2 €, represents the cost of the interest rate guarantee.

2.1.5.3 Parameter Sensitivity

Since the model in use is arbitrary free, it is fitted to the initial yield curve at valuation date hence a parameter sensitivity is of great importance. Parameters a and σ control the variability and the speed of the mean regression and their fluctuation tend to give close results whereas k and T when high give sensitive factors.
Here we did two valuation factors' sensitivity studies, for a time to retirement equal to 20 years ($T = 20$) as seen in the tables 2.4 and 2.5.

In the table 2.4, we studied the sensitivity of the valuation factors $V(0,T)$ due to variations in the 1-factor H&W estimated parameters $\hat{a}$ and $\hat{\sigma}$. We had 5 scenarios, from A to E, with scenario A being the original scenario.

In scenarios B and C, we respectively increased and decreased the speed of the mean regression by 3, without changing any other parameters, resulting in a (almost) null change of the different valuation factors.

In scenarios D and E, we treated the valuation factors' sensitivity due to changes in the variability $\hat{\sigma}$, without changing other parameters :

- Scenario D : an increase of the variability by 1% led to a very small negative and non significant variation (-0.0003%) of the different valuation factors ;
- Scenario E : a decrease of the variability by 0.1% led to no change (0%) of the different valuation factors.

Overall the variations of the different valuation factors are quite non significant/null and these factors are all greater than 1 meaning that the interest rate guarantee still has a positive market value.

$\hat{a}$	3.318677761	6.318677761	0.318677761	3.318677761	3.318677761
$\hat{\sigma}$	0.001366525	0.001366525	0.001366525	0.021366525	0.000366525
		$\hat{a}+3$	$\boldsymbol{\hat{a}-3}$	$\boldsymbol{\hat{\sigma}+1\%}$	$\hat{\sigma}-0.1\%$
k	A	B	C	D	E
10	1.003	1.003	1.003	1.003	1.003
20	1.045	1.045	1.045	1.045	1.045
30	1.047	1.047	1.047	1.047	1.047
5+100BP	1.180	1.180	1.180	1.180	1.180
1+150BP	1.281	1.281	1.281	1.281	1.281
		0.0000%	**-0.0005%**	**-0.0003%**	0.0000%

Table 2.4 – Sensitivity of the valuation factors $V(0,T)$ due to variations in the 1-factor H&W interest rate model parameters $\hat{a}$ and $\hat{\sigma}$, for $T = 20$

In the table 2.5, we studied the impact that variations in the spot rates $r_k(t)$ might have on the valuation factors. We also had 5 scenarios {A,F,G,H,I}, with scenario A being the original scenario.

In scenarios F and G, we studied the impact of a 0.5% increase and 0.5% decrease of the spot rates respectively. We notice that a 0.5% increase leads to 5.13% increase of the valuation factors and a 0.5% decrease leads to a nearly 5% decrease; the valuation factors' impact tends to be 10 times bigger than the spot rates' variations.

In scenarios H and I, we did the same study as in the 2 preceding scenarios but with the spot rates varying by 1% instead of 0.5%. The observations remain the same: the valuation factors' impact is around 10 times bigger than the spot rates' variations, where we have a positive impact of 10.5% in the valuation factors for a

1% increase in spot rates and a negative impact of 9.5% for a 1% decrease.
In general, the interest rate guarantee has a positive and significant market value: the valuation factors are greater than 1 in most cases hence being beneficial for the participant. However, we observe 6 cases where those factors are less than 1 and all the 6 cases take place in scenarios with negative variations of the spot rates; meaning that, in cases with no margins ($m = 0$) and a negative variation of the spot rates greater than their initial values (in absolute value), the interest rate guarantee will most certainly have negative market value: the valuation factors will be less than 1 and the employer will get the most out of the situation.

		$r_k(t)$	$r_k(t)$+0.5%	**$r_k(t) - 0.5\%$**	$r_k(t) + 1\%$	**$r_k(t) - 1\%$**
k	$r_k(t)$	A	F	G	H	I
10	0.03%	1.003	1.054	0.954	1.108	0.907
20	0.44%	1.045	1.098	0.994	1.155	0.945
30	0.46%	1.047	1.101	0.996	1.157	0.948
5+100BP	-0.35%	1.180	1.240	1.122	1.304	1.068
1+150BP	-0.52%	1.281	1.347	1.219	1.416	1.160
			5.13%	**-4.88%**	10.52%	**-9.52%**

Table 2.5 – Sensitivity of the valuation factors $V(0, T)$ due to variations in the k-year spot rates $r_k(t)$, for $T = 20$

The two cases with non-zero margins ($m > 0$) always have positive and significant market values, regardless the variations/sensitivities studied in these different scenarios: the participant benefits the most from these situations.

2.2 Heath-Jarrow-Morton (HJM) model

2.2.1 Model description

The Heath-Jarrow-Morton (HJM) model is a dynamic interest rate model and happens to be compliant with the Solvency II regulation. Its framework is a very general approach that encloses the Vasicek and the H&W models; the difference being that the HJM model proposes dynamics for instantaneous forward rates instead of modeling short term rates.

This model states that, under Q, the instantaneous forward rate at time u of maturity T is defined as follow:

$$df(u,T) = \alpha(u,T)du + \sigma(u,T)^{\top}dW(u) \tag{2.26}$$

where the drift $\alpha(u,T)$ and the p-vector $\sigma(u,T) = (\sigma_1(u,T), ..., \sigma_p(u,T))^{\top}$ are $\mathcal{F}_u$-adapted processes. Moreover, to ensure that forward rate processes are well defined, $\alpha(u,T)$ and $\sigma(u,T)$ must also satisfy these assumptions:

- $\int_0^T |\alpha(u,T)|du$ and $\int_0^T |\sigma_i^2(u,T)|du$, for $i = 1, ..., p$, are almost surely finite;
- $\int_0^T \int_0^s |\sigma_i^2(u,s)|duds$, for $i = 1, ..., p$, is finite;
- $\mathbb{E}\left(\int_0^s |\sigma(u,s)^{\top}dW_s^Q|ds\right) \leq \infty$.

which leads us to:

$$\begin{aligned} f(t,T) &= f(0,T) + \int_0^t \alpha(u,T)du + \int_0^t \sigma(u,T)^{\top}dW(u) \\ &= f(0,T) + \sum_{i=1}^{p}\int_0^t \alpha_i(u,T)du + \sum_{i=1}^{p}\int_0^t \sigma_i(u,T)^{\top}dW_i(u) \end{aligned} \tag{2.27}$$

whereas the risk-free rate is defined as:

$$\begin{aligned} r_t = f(t,t) &= f(0,t) + \int_0^t \alpha(u,t)du + \int_0^t \sigma(u,t)^{\top}dW(u) \\ &= f(0,t) + \sum_{i=1}^{p}\int_0^t \alpha_i(u,t)du + \sum_{i=1}^{p}\int_0^t \sigma_i(u,t)^{\top}dW_i(u) \end{aligned} \tag{2.28}$$

The zero-coupon bond can also be deducted, under the assumption that $\int_0^T \int_0^s |\alpha(u,s)|duds$

is finite, as follow:

$$\begin{aligned} P(t,T) &= \exp\left(-\int_t^T f(t,s)ds\right) \\ &= \exp\left(-\int_t^T f(0,s)ds - \sum_{i=1}^{p}\int_0^t\int_t^T \alpha_i(u,s)dsdu - \sum_{i=1}^{p}\int_0^t\int_t^T \sigma_i(u,s)^\top dsdW_i(u)\right) \end{aligned} \tag{2.29}$$

Let $(B_t)_{t\geq 0}$, known as the cash account, be the value of a deposit of one monetary unit capitalized at risk-free rate till time t and is equal to: $B_t = \exp\left(\int_0^t r_s ds\right)$. Knowing that $r_s = f(s,s)$ (from equations [2.27] and [2.28]), the cash account is also equal to:

$$B_t = \exp\left(\int_0^t f(0,s)ds + \sum_{i=1}^{p}\int_0^t\int_u^t \alpha_i(u,s)dsdu + \sum_{i=1}^{p}\int_0^t\int_u^t \sigma_i(u,s)^\top dsdW_i(u)\right) \tag{2.30}$$

Proposition 2.2.1. *If the market is arbitrage free then the drift of forward rates* $\alpha(t,T)$ *is related to* $\sigma(t,T)$ *by the next relation:*

$$\alpha(t,T) = \sigma(t,T)^\top \int_t^T \sigma(t,u)du = \sum_{i=1}^{p}\sigma_i(t,T)^\top \int_t^T \sigma_i(t,u)du \tag{2.31}$$

Proof. See Appendices .3. □

A direct consequence of this proposition is that zero-coupon bonds earn on average the risk-free rate under Q.

Corollary 2.2.1.1. *Under the risk neutral measure, the zero-coupon bond is solution of the SDE, for* $t \leq T$*:*

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \left(\int_t^T \sigma(t,s)ds\right)^\top dW_t. \tag{2.32}$$

Proof. See Appendices .3. □

As for the ZC bond price, from the general formula $P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$ and using the Itô's lemma with the variable $-\int_t^T f(t,s)ds$ on the equation [5] (in Appendix .3), we can rewrite that equation as:

$$d\ln P(t,T) = \left(r_t - \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^\top \left(\int_t^T \sigma(t,s)ds\right)\right) - \left(\int_t^T \sigma(t,s)ds\right)^\top dW_t.$$

which leads to this ZC bond price formula:

$$\begin{aligned} P(t,T) &= P(0,T)\exp\left(\int_0^t r_s + \frac{1}{2}S(s,T)^\top S(s,T)ds + \int_0^t S(s,T)dW(s)\right) \\ &= \frac{P(0,T)}{P(0,t)}\exp\left(\frac{1}{2}\int_0^t S(s,T)^\top S(s,T)ds + \int_0^t S(s,T)dW(s)\right) \\ &= \frac{P(0,T)}{P(0,t)}\exp\left(\frac{1}{2}\sum_{i=1}^p\int_0^t S_i(s,T)^\top S_i(s,T)ds + \sum_{i=1}^p\int_0^t S_i(s,T)dW_i(s)\right) \end{aligned} \tag{2.33}$$

where $S(t,T) = -\int_t^T \sigma(t,s)ds$.

Theorem 2.2.1. *Let* $\sigma_i(s,t) = \varsigma_i(r(s))e^{-\int_s^t \kappa_i(x)dx}$*, with* ς_i *and* κ_i *being deterministic functions for* $1 \leq i \leq p$*, be a separable and exponentially decaying volatility, then the bond price is given by the following formula:*

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left(-\Phi(t,T) - \Psi(t,T) - \gamma_1(t,T)[r(t) - f(0,t)]\right) \tag{2.34}$$

where $\gamma_i(t,T) = \int_t^T e^{-\int_t^u \kappa_i(x)dx}du$*, for* $1 \leq i \leq p$*, and*

$$\begin{aligned} \Phi(t,T) &= \frac{1}{2}\sum_{i=1}^p \gamma_i^2(t,T)\epsilon_i(t); \\ \Psi(t,T) &= \sum_{i=2}^p \zeta_i(t)[\gamma_i(t,T) - \gamma_1(t,T)]; \\ \zeta_i(t) &= \int_0^t \alpha_i(s,t)ds + \int_0^t \sigma_i(s,t)dW_i(s); \\ \alpha_i(s,u) &= \sigma_i(s,u)\int_s^u \sigma_i^2(s,v)dv; \\ \epsilon_i(t) &= \int_0^t \sigma_i^2(s,t)ds. \end{aligned}$$

Proof. See Appendix .4. ☐

2.2.2 HJM 1-factor model

The HJM 1-factor model is the one where $p = 1$ in the equations [2.27-2.29] and in the proposition 2.2.1.

2.2.2.1 Special case : $\boldsymbol{\sigma(t,T) = \sigma e^{-a(T-t)}}$

Let's consider the case where we have a separable and exponentially decaying volatility equal to

$$\sigma(t,T) = \sigma e^{-a(T-t)} \tag{2.35}$$

with a and σ being real constants. That way,

$$S_1(t,T) = S(t,T) = \int_t^T \sigma(t,s)ds = \sigma\left(\frac{1-e^{-a(T-t)}}{a}\right) = \sigma B(t,T)$$

The short rate dynamics can be written as follow

$$\begin{aligned} dr(t) &= \Bigg(\frac{\partial f(0,t)}{\partial T} + \frac{\sigma^2}{a}(e^{-at} - e^{-2at}) - a\Big(r(t) - f(0,t) - \frac{\sigma^2}{2a^2}(1-e^{-at})^2\Big)\Bigg)dt + \sigma dW_1(t) \\ &= \Big(\underbrace{\frac{\partial f(0,t)}{\partial T} + af(0,t) + \frac{\sigma^2}{2a}(1-e^{-2at})}_{\theta(t)} - ar(t)\Big)dt + \sigma dW_1(t) \end{aligned} \tag{2.36}$$

which is identical to the short-rate dynamics of the 1-factor H&W interest rate model in equation [2.1]. In a similar way, we can show that the ZC bond price of the 1-factor HJM model with the volatility defined as in equation [2.35] is the same as the one from the 1-factor H&W interest rate model in equation [2.6]. (See Appendices .5)

2.2.3 HJM 2-factor model

We shall consider here the case where $p = 2$ in the equations [2.27-2.29] and in the proposition 2.2.1, which leads to the HJM 2-factor model.

2.2.3.1 Special case : $\boldsymbol{\sigma(t,T) = (\sigma_1 e^{-a_1(T-t)})^\top(\sigma_2 e^{-a_2(T-t)})}$

As seen above, we consider the case where we have separable and exponentially decaying volatilities $\sigma_1(t,T)$ and $\sigma_2(t,T)$ which are equal to, for $i = 1,2$

$$\sigma_i(t,T) = \sigma_i e^{-a_i(T-t)} \tag{2.37}$$

with a_i and σ_i being real constants. We also have that

$$S_i(t,T) = \int_t^T \sigma_i(t,s)ds = \sigma_i \left(\frac{1-e^{-a_i(T-t)}}{a_i}\right) = \sigma_i B_i(t,T)$$

for $i = 1,2$.

Under Q, the instantaneous forward rate at time t of maturity T is defined as follow:

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma_1(t,T)dW_1(t) + \sigma_2(t,T)dW_2(t) \\ &= \alpha(t,T)dt + \sigma_1 e^{-a_1(T-t)}dW_1(t) + \sigma_2 e^{-a_2(T-t)}dW_2(t) \end{aligned} \tag{2.38}$$

The short rate process $r(t) = f(t,t)$ defined in equation [2.28] can be rewritten as

$$\begin{aligned} r_t &= f(0,t) + \sum_{i=1}^{2}\int_0^t \alpha_i(u,t)du + \sum_{i=1}^{2}\int_0^t \sigma_i(u,t)^\top dW_i(u) \\ &= f(0,t) + X_1(t) + X_2(t) \end{aligned} \tag{2.39}$$

where under an Equivalent Martingale Measure Q, $X_1(t)$ and $X_2(t)$ are defined as follow

$$dX_1(t) = \left(-a_1X_1(t) + \sum_{k=1}^{2} Z_{1,k}(0,t)\right)dt + \sigma_1 d\tilde{W}_1^Q(t)\ ,\ X_1(0) = 0 \tag{2.40}$$

$$dX_2(t) = \left(-a_2X_2(t) + \sum_{k=1}^{2} Z_{2,k}(0,t)\right)dt + \sigma_2 d\tilde{W}_2^Q(t)\ ,\ X_2(0) = 0 \tag{2.41}$$

with $d\langle\tilde{W}_1,\tilde{W}_2\rangle = \rho dt$ being the correlation and $Z_{ik}(u,t) = \int_u^t \mathrm{cov}(X_i(s),X_k(s))ds$, for $i,k = 1,2$, giving

$$
\begin{aligned}
Z_{11}(u,t) &= \int_u^t \operatorname{cov}(X_1(s),X_1(s))ds = \int_u^t \sigma_1^2 e^{-2a_1(t-s)}ds = \frac{\sigma_1^2(1-e^{-2a_1(t-u)})}{2a_1} \\
Z_{12}(u,t) &= Z_{21}(u,t) = \int_u^t \operatorname{cov}(X_1(s),X_2(s))ds \\
&= \int_u^t \rho_{1,2}\sigma_1\sigma_2 e^{-(a_1+a_2)(t-s))}ds = \frac{\rho_{1,2}\sigma_1\sigma_2}{a_1+a_2}\left(1-e^{-(a_1+a_2)(t-u)}\right) \\
Z_{22}(u,t) &= \int_u^t \operatorname{cov}(X_2(s),X_2(s))ds = \int_u^t \sigma_2^2 e^{-2a_2(t-s)}ds = \frac{\sigma_2^2(1-e^{-2a_2(t-u)})}{2a_2}
\end{aligned}
$$

with $\rho_{1,2}$ being the correlation between $X_1(t)$ and $X_2(t)$.
From the formulas in equations [2.40] and [2.41], we can generate $X_i(t)$ using the analytical strong solution form of a generalized linear stochastic differential equation, for $s \leq t$ and $i = 1,2$

$$
\begin{aligned}
X_i(t) &= e^{-a_i(t-s)}\Bigg(X_i(s) + \int_s^t Z_{i,1}(s,u)e^{-a_i(u-s)}du + \int_s^t Z_{i,2}(s,u)e^{-a_i(u-s)}du \\
&\quad + \int_s^t \sigma_i e^{-a_i(u-s)}d\tilde{W}_i^Q(u)\Bigg) \\
&= X_i(s)e^{-a_i(t-s)} + \int_s^t Z_{i,1}(s,u)e^{-a_i(t-u)}du + \int_s^t Z_{i,2}(s,u)e^{-a_i(t-u)}du \\
&\quad + \int_s^t \sigma_i e^{-a_i(t-u)}d\tilde{W}_i^Q(u)
\end{aligned} \tag{2.42}
$$

where the corresponding $X_1(t)$ and $X_2(t)$ equations are found in Appendix .6.
Therefore, the forward rate from the equation [2.27] becomes

$$
\begin{aligned}
f(t,T) &= f(0,T) + \sum_{i=1}^{2}\int_0^t \alpha_i(u,T)du + \sum_{i=1}^{2}\int_0^t \sigma_i(u,T)^\top dW_i(u) \\
&= f(0,T) + \sum_{i=1}^{2} e^{-a_i(T-t)}\left[X_i(t) + \sum_{k=1}^{2} B_k(t,T)Z_{ik}(0,t)\right]
\end{aligned} \tag{2.43}
$$

where $B_k(t,T) = \frac{1-e^{-a_i(T-t)}}{a_i}$ and $r(t)$ in equation [2.39] becomes

$$
\begin{aligned}
r(t) &= f(s,t) + X_1(t) + X_2(t) \\
&= f(s,t) + X_1(s)e^{-a_1(t-s)} + \frac{\sigma_1^2(1-e^{-a_1(t-s)})^2}{2a_1^2} + X_2(s)e^{-a_2(t-s)} \\
&\quad + \frac{\sigma_2^2(1-e^{-a_2(t-s)})^2}{2a_2^2} + \frac{\rho_{1,2}\sigma_1\sigma_2}{a_1a_2}(1-e^{-a_1(t-s)})(1-e^{-a_2(t-s)}) \\
&\quad + \int_s^t \sigma_1 e^{-a_1(t-u)}d\tilde{W}_1^Q(u) + \int_s^t \sigma_2 e^{-a_2(t-u)}d\tilde{W}_2^Q(u)
\end{aligned} \tag{2.44}
$$

with

$$\begin{aligned}E^Q\left[r(t)|\mathcal{F}_s\right] = f(s,t) + X_1(s)e^{-a_1(t-s)} + \frac{\sigma_1^2(1-e^{-a_1(t-s)})^2}{2a_1^2} + X_2(s)e^{-a_i(t-s)} \\ + \frac{\sigma_2^2(1-e^{-a_2(t-s)})^2}{2a_2^2} + \frac{\rho_{1,2}\sigma_1\sigma_2}{a_1a_2}(1-e^{-a_1(t-s)})(1-e^{-a_2(t-s)})\end{aligned} \tag{2.45}$$

$$\mathrm{Var}^Q\left[r(t)|\mathcal{F}_s\right] = \frac{\sigma_1^2(1-e^{-2a_1(t-s)})}{2a_1} + \frac{\sigma_2^2(1-e^{-2a_2(t-s)})}{2a_2} + 2\frac{\rho_{1,2}\sigma_1\sigma_2}{(a_1+a_2)}(1-e^{(a_1+a_2)t}) \tag{2.46}$$

where it is noticed that $r(t)$ is conditionally Gaussian. Furthermore, by defining μ_r, σ_r and the risk-neutral probability for the short rate to be negative $Q(r(t) < 0)$ as follow

$$\begin{aligned}\mu_r &:= E^Q\left[r(t)|\mathcal{F}_0\right] = f(0,t) + \frac{\sigma_1^2(1-e^{-a_1t})^2}{2a_1^2} + \frac{\sigma_2^2(1-e^{-a_2t})^2}{2a_2^2} \\ &+ \frac{\rho_{1,2}\sigma_1\sigma_2}{a_1a_2}(1-e^{-a_1t})(1-e^{-a_2t}) \\ \sigma_r &:= \mathrm{Var}^Q\left[r(t)|\mathcal{F}_0\right] = \frac{\sigma_1^2(1-e^{-2a_1t})}{2a_1} + \frac{\sigma_2^2(1-e^{-2a_2t})}{2a_2} + 2\frac{\rho_{1,2}\sigma_1\sigma_2}{(a_1+a_2)}(1-e^{(a_1+a_2)t}) \\ &Q(r(t)<0) = E_Q[\mathbb{1}_{\{r(t)<0\}}] = E_Q\left[\mathbb{1}_{\left\{\frac{r(t)-\mu_r}{\sigma_r}<-\frac{\mu_r}{\sigma_r}\right\}}\right] = \Phi\left(-\frac{\mu_r}{\sigma_r}\right) > 0\end{aligned}$$

We observe a perfect resemblance in this 2-factor HJM model special case's mean and variance of the short rate and in the G2++ model ones defined in Damiano Brigo et Fabio Mercurio (2006 [1]) on page 147: meaning that the two models have the same dynamics of the short rate.
The ZC bond price can be expressed as follow

$$\begin{aligned}P(t,T) &= E\left[e^{-\int_t^T r(s)ds}|\mathcal{F}_t\right] = e^{-\int_t^T f(t,u)du} \\ &= \frac{p(0,T)}{p(0,t)}\exp\left(-B_1(t,T)X_1(t) - B_2(t,T)X_2(t) - \frac{1}{2}\sum_{i,j=1}^{2} B_i(t,T)B_j(t,T)Z_{ij}(0,t)\right) \\ &= A(t,T)\exp\Big(-B_1(t,T)X_1(t) - B_2(t,T)X_2(t)\Big)\end{aligned} \tag{2.47}$$

where

$$A(t,T) = \frac{p(0,T)}{p(0,t)} \exp\left(\frac{1}{2}[V(t,T) - V(0,T) + V(0,t)]\right)$$

$$\begin{aligned} V(t,T) = &\frac{\sigma_1^2}{a_1^2}\left(T - t + \frac{2}{a_1}e^{-a_1(T-t)} - \frac{1}{2a_1}e^{-2a_1(T-t)} - \frac{3}{2a_1}\right) \\ &+ \frac{\sigma_2^2}{a_2^2}\left(T - t + \frac{2}{a_2}e^{-a_2(T-t)} - \frac{1}{2a_2}e^{-2a_2(T-t)} - \frac{3}{2a_2}\right) \\ &+ 2\frac{\rho_{1,2}\sigma_1\sigma_2}{a_1a_2}\left(T - t + \frac{e^{-a_1(T-t)} - 1}{a_1} + \frac{e^{-a_2(T-t)} - 1}{a_2} - \frac{e^{-(a_1+a_2)(T-t)} - 1}{a_1 + a_2}\right) \end{aligned}$$

2.2.4 HJM's econometric estimation

In order to use the HJM model for risk management purposes, its scenarios must be compliant with the history of financial markets. The parameters are then estimated, in this case, from time-series samples under the real measure $\mathbb{P}$.
Furthermore, under $\mathbb{P}$, the instantaneous forward rate at time u of maturity T found above in the equation [2.26] becomes:

$$df(u,T) = (\alpha(u,T) + \sigma(u,T)^\top\theta_u)du + \sigma(u,T)^\top d\tilde{W}_u \tag{2.48}$$

where $\theta_u = (\theta_u^1, ..., \theta_u^p)^\top$ is a p-vector (of $\mathcal{F}_t$-adapted processes) that defines the market risk premiums of each risk factors and $\tilde{W}_u = W_u - \theta_u$ is a p-vector of Brownian motions under $\mathbb{P}$.
The following Radon-Nykodym derivatives show the change of measure from $\mathbb{P}$ to $\mathbb{Q}$:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_t = \exp\left(-\int_0^t \theta_s^\top d\tilde{W}_s - \frac{1}{2}\int_0^t ||\theta_s||^2 ds\right)$$

Let $\gamma(u,T) = \alpha(u,T) + \sigma(u,T)^\top\theta_u$ be the drift of forward rates under $\mathbb{P}$. Then the forward rates of maturity $t+\tau$, under the real measure $\mathbb{P}$, is expressed as:

$$f(t,t+\tau) = f(0,t+\tau) + \int_0^t \gamma(u,t+\tau)du + \int_0^t \sigma(u,t+\tau)^\top d\tilde{W}_u \tag{2.49}$$

There exists a link between the forward rates and the bond price at time t of maturity $t+\tau$ where $P(t,t+\tau) = \exp\left(-\int_t^{t+\tau} f(t,s)ds\right)$. The dynamics of the ZC bond prices will help us estimate our model under $\mathbb{P}$ where the differential of the integral of forward rates is:

$$\begin{aligned} d\ln P(t,t+\tau) &= f(t,t)dt - \int_t^{t+\tau} df(t,s)ds \\ &= \left(r_t - \int_t^{t+\tau} \gamma(t,s)ds\right)dt - \left(\int_t^{t+\tau} \sigma(t,s)^\top ds\right) d\tilde{W}_t. \end{aligned} \tag{2.50}$$

Using Itô's lemma, with the state variable $\ln P(t,t+\tau)$, alongside the definition of $\gamma(t,s)$ and the condition 2.31 we can show that $P(t,t+\tau)$ is a geometric Brownian motion:

$$\frac{dP(t,t+\tau)}{P(t,t+\tau)} = \left(r_t - \int_t^{t+\tau} \sigma(t,s)^\top \theta_t ds\right)dt - \left(\int_t^{t+\tau} \sigma(t,s)ds\right)^\top d\tilde{W}_t$$

Let us make an (strong) assumption that the drift of $\ln P(t,t+\tau)$ is stationary does not depend upon t, which provides good empirical results:

$$r_t - \int_t^{t+\tau} \gamma(t,s)ds \approx g(\tau).$$

We also assume that:

- we sample $d \geq p$ bond prices at $n+1$ equidistant times $\{t_0, ..., t_n\}$;
- Δ is the interval between two successive sampling times;
- $\{\tau_1, ..., \tau_d\}$ are the maturities of the bond prices.

First, we calculate the first order differences of log-bond prices:

$$y_i(\tau_j) = \ln P(t_{i+1}, t_{i+1}+\tau) - \ln P(t_i, t_i+\tau_j)$$

for $i = 1, ..., n$ and $j = 1, ..., d$.

According to equation [2.50], the vector $y_i = (y_i(\tau_j))_{j=1,...,d}$ may be seen as a realization of a multivariate random variable $Y = \{Y_1, ..., Y_d\}$ with

$$Y_j = g(\tau_j)\Delta - \left(\int_t^{t+\tau_j} \sigma(t,s)^\top ds\right)\left(\tilde{W}_{t+\Delta} - \tilde{W}_t\right) \quad j = 1, ..., d. \tag{2.51}$$

which emphasizes that functions $\sigma(.,.)$ define the covariance matrix of Y. Therefore we can deduce the $d \times d$ empirical covariance matrix Σ of $(y_i)_{i=1,...,n}$, that happens to be positive definite and symmetric:

$$\Sigma = \Psi\Lambda\Psi$$

where Ψ is the $d \times d$ matrix of normed right eigenvectors, $\Psi = (\Psi_1, ..., \Psi_d)$ of the

covariance matrix.
The $\Psi_k = (\Psi_{k,1}, ..., \Psi_{k,d})^\top$ are vectors of dimension d and $\Lambda = \text{diag}(\lambda_1, ..., \lambda_d)$ is the diagonal matrix of ordered eigenvalues such that:

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_d.$$

The occurrence of Y form a cloud of points in a d dimensions space. A new orthonormal basis is then defined by the eigenvectors and the variance of the cloud of points is maximal in each axis direction.
In the new basis, the y_i's (observations) coordinates are given by:

$$\begin{pmatrix} \Psi_1^\top y_i \\ . \\ . \\ . \\ \Psi_1^\top y_i \end{pmatrix}$$

for $i = 1, ..., n$.
This orthonormal basis has $\bar{y} = (\bar{y}_1, ..., \bar{y}_n)^\top = \frac{1}{n}\sum_{i=1}^n y_i$, the estimate of Y's expectation, as its origin. The estimate of variance along each axis is however the associated eigenvalues:

$$\mathbb{E}\Big((\widehat{\Psi_j^\top Y - \Psi_j^\top \mathbb{E}(Y))^2}\Big) = \lambda_j \quad j = 1, ..., d.$$

The sum of variances along the axis $\sum_{j=1}^d \lambda_j$ gives the total variance. In practice, a projection of the cloud of $(y_i)_{i=1,...,n}$ in a subspace spanned by the two or three biggest eigenvalues is sufficient to explain more than 90% of the total variance. In a case where the projection on a subspace of dimension p is done, the distribution of Y can be approximated by:

$$Y_j \approx \bar{y_j} + \sum_{k=1}^p \Psi_{j,k}\sqrt{\lambda_k}X_k \tag{2.52}$$

for $j = 1, ..., d$ and where X_k are independent normal random variables, $N(0, \sqrt{\Delta})$.
Comparing equations 2.50 and 2.52, we can deduce that an estimator $\hat{\sigma}(t, s) = (\hat{\sigma}_k(t, s))_{k=1,...,p}$ of $\sigma(t, s)$ must minimize the spreads between $\Psi_{j,k}\sqrt{\lambda_k}$ and $-\int_t^{t+\tau_j} \sigma_k(t, s)ds$,

for all $j \in 1, ..., d$:

$$\hat{\sigma_k}(.,.) = \arg\min_{\sigma_k(.,.)} \sum_{j=1}^{d} \Big(- \int_t^{t+\tau_j} \sigma_k(t,s)ds - \Psi_{j,k}\sqrt{\lambda_k} \Big)^2 \tag{2.53}$$

for $k = 1, ..., p$.

In practice, a functional form for $\sigma_k(t,s)$ is chosen beforehand and its parameters are estimated by least square minimization of the criterion above.

2.2.5 Valuation Formula : 2-factor HJM model

In the section 2.1.4 we calculated the valuation factors for the 1-factor Hull & White model.

In this part, we shall proceed in the same way in order to determine the valuation factors for the HJM 2-factor model using the same assumptions:

- $r^c(t) = r_k(t) + m$;
- $p(t, t+k) = e^{-k\ r_k(t)}$.

This way, $r_k(t) = \log(p(t, t+k))$ and together with the 2-factor HJM ZC bond price in equation [2.47] we get:

$$r_k(t) = \frac{-\log(A(t,t+k))}{k} \Big(B_1(t,t+k)X_1(t) + B_2(t,t+k)X_2(t) \Big)$$

with $\log(A(t,t+k)) = A'(t,t+k)$ in the following steps. From the equation [1.6], the valuation factor per 1 unit of fund value at $t = 0$ in this model becomes

$$\begin{aligned} V(0,T) &= E_0^Q \left[e^{\int_0^T (r_k(t)+m-r(t))dt} \right] = \exp(mT)\ E_0^Q \left[e^{\int_0^T (r_k(t)-r(t))dt} \right] \\ &= \exp(mT)\ E_0^Q \Bigg[\exp\Bigg(\int_0^T -\Big(\frac{A'(t,t+k)}{k} (B_1(t,t+k)X_1(t) + B_2(t,t+k)X_2(t)) \Big) \\ &\qquad\qquad - r(t)dt \Bigg) \Bigg] \\ &= \exp(mT)\ \exp\left(\int_0^T -\frac{A'(t,t+k)}{k} dt \right)\ \exp(A^*(0,T)) \\ &\qquad\qquad \exp(-\frac{1}{2}\upsilon^*(k))\ \underbrace{E_0^Q \left[e^{-\left(\int_0^T \gamma_1 X_1(t) + \gamma_2 X_2(t) \right)} \right]}_{\exp(\frac{1}{2}\upsilon^*(k))} \\ &= \exp(mT)\ \exp\left(\int_0^T -\frac{A'(t,t+k)}{k} dt \right)\ \exp(A^*(0,T)) \end{aligned} \tag{2.54}$$

where

$$\begin{aligned}
\gamma_j &= \frac{1 - B_k(a_j)}{k} \quad \text{for } j = 1, 2; \\
B_k(a_j) &= \frac{1 - e^{-a_j k}}{a_j} \quad \text{for } j = 1, 2; \\
A'(t, t+k) &= \log \frac{p(0, t+k)}{p(0, t)} + \frac{1}{2}(v(k) + v(t) - v(t+k)); \\
A^*(t, t+k) &= \log \frac{p(0, t+k)}{p(0, t)} + \frac{1}{2}(v^*(k) + v(t) - v(t+k)),
\end{aligned}$$

and

$$\begin{aligned}
v(k) = &\frac{\sigma_1^2}{a_1^2}\left(k - 2B_k(a_1) + B_k(2a_1)\right) + \frac{\sigma_2^2}{a_2^2}\left(k - 2B_k(a_2) + B_k(2a_2)\right) \\
&+ \frac{2\rho\sigma_1\sigma_2}{a_1 a_2}\left(k - B_k(a_1) - B_k(a_2) + B_k(a_1 + a_2)\right),
\end{aligned}$$

$$\begin{aligned}
v^*(k) = &\frac{\gamma_1^2\sigma_1^2}{a_1^2}\left(k - 2B_k(a_1) + B_k(2a_1)\right) + \frac{\gamma_2^2\sigma_2^2}{a_2^2}\left(k - 2B_k(a_2) + B_k(2a_2)\right) \\
&+ \frac{2\gamma_1\gamma_2\rho\sigma_1\sigma_2}{a_1 a_2}\left(k - B_k(a_1) - B_k(a_2) + B_k(a_1 + a_2)\right).
\end{aligned}$$

with v and v^* deterministic functions of the parameters.

2.2.6 Practical case

2.2.6.1 2-factor HJM model parameters' estimation

We shall now fit the HJM model to the Zero-Coupon yields bootstrapped from Belgian linear bonds rates (OLOs) retrieved from the National Bank of Belgium statistics' website [1]. The period was set between January 2 2015 and November 29 2019 with tenors going from 1 year to 30 years; i.e $d = 30$.

Let us consider a model with $p = 2$ (2 Brownian motions) and forward rates which are solutions of

$$df(t, T) = \alpha(t, T)dt + \sigma_1(t, T)dW_1(t) + \sigma_2(t, T)dW_2(t) \tag{2.55}$$

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[1] https://statbel.fgov.be

with $\sigma_1(.)$ and $\sigma_2(.)$ being separable and exponentially decaying volatilities:

$$\begin{aligned} \sigma_1(t, t+\tau) &= \sigma_1 e^{-a_1\tau}, \\ \sigma_2(t, t+\tau) &= \sigma_2 e^{-a_2\tau}. \end{aligned} \tag{2.56}$$

The integration of these volatility functions will help us find the 4 parameters $(a_1, a_2, \sigma_1, \sigma_2)$:

$$\begin{aligned} \int_t^{t+\tau} \sigma_1(t,s)ds &= \sigma_1\left(\frac{1-e^{-a_1\tau}}{a_1}\right), \\ \int_t^{t+\tau} \sigma_2(t,s)ds &= \sigma_2\left(\frac{1-e^{-a_2\tau}}{a_2}\right). \end{aligned} \tag{2.57}$$

The estimates of the first parameters $\hat{\sigma_1}$ and $\hat{a_1}$ are obtained by minimizing the equation [2.53] found above

$$\hat{\sigma_1}, \hat{a_1} = \arg\min_{\sigma_1, a_1} \sum_{j=1}^{30} \Big(- \sigma_1\left(\frac{1-e^{-a_1\tau_j}}{a_1}\right) - \Psi_{j,1}\sqrt{\lambda_1}\Big)^2 \tag{2.58}$$

where Ψ_1 is the normed eigenvector related to the largest eigenvalue λ_1 of the covariance matrix Σ.
The remaining parameters' estimates $\hat{\sigma_2}$ and $\hat{a_2}$ are also obtained as follow

$$\hat{\sigma_2}, \hat{a_2} = \arg\min_{\sigma_2, a_2} \sum_{j=1}^{30} \left(- \sigma_2\left(\frac{1-e^{-a_2\tau_j}}{a_2}\right) - \Psi_{j,2}\sqrt{\lambda_2}\right)^2 \tag{2.59}$$

with Ψ_2 being the normed eigenvector related to the second largest eigenvalue λ_2 of the covariance matrix Σ.
The table 2.6 below shows the estimated values of the 4 parameters and the estimated correlation $\hat{\rho}_{1,2}$

Parameters	Estimates
$\hat{\sigma_1}$	0.0003076277
$\hat{\sigma_2}$	0.0002275381
$\hat{a}_1$	-0.026619
$\hat{a}_2$	0.595228
$\hat{\rho}_{1,2}$	-0.4965

Table 2.6 – Parameters' estimates for volatility functions (2.56)

We can see that $\hat{\sigma_1}$ and $\hat{\sigma_2}$ are respectively around 3 and 2 basis points while on the contrary the first mean reversion $\hat{a_1}$ is negative with an absolute value close to

3% and the second mean reversion $\hat{a_2}$ is positive with a significant absolute value around 60%. The estimate of the correlation $\hat{\rho_{1,2}}$ was found afterwards and appears to be negative with an absolute value close to 50%.
The figure 2.5, comparing the (negative) integrated volatilities [2.57] to their empirical counterparts, confirms the very good fit provided by function in the equation [2.56].

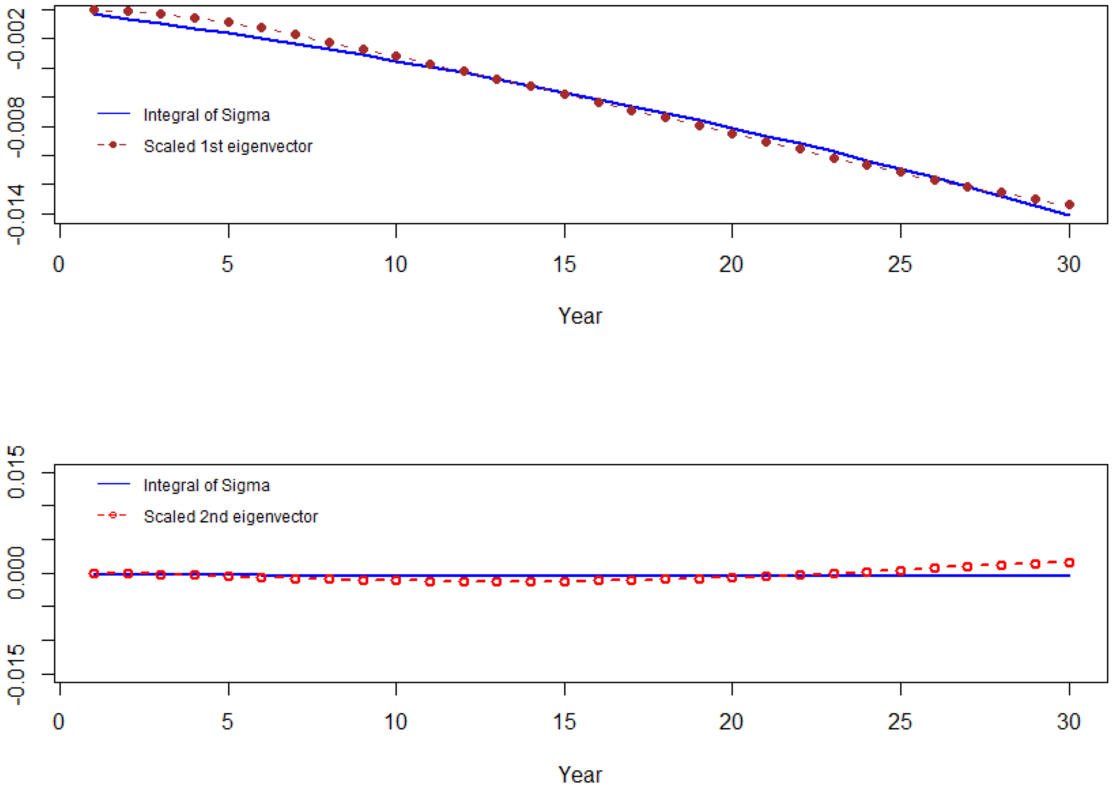


Figure 2.5 – Comparison of $-\int_t^{t+\tau_j} \sigma_k(t,s)ds$ and $\Psi_{j,k}\sqrt{\lambda_k}$ for $k = 1, 2$ and $j = 1, ..., 30$

2.2.6.2 Valuation factors V(0,T)

In this part we shall do the same thing as in section 2.1.5.2, i.e we shall calculate the valuation factors $V(0,T)$, for $T = \{5, 10, 15, 20, 25, 30\}$, using the equation [2.54] and the 2-factor HJM model's estimated parameters : $\hat{\sigma_1} = 0.03\%$, $\hat{\sigma_2} = 0.02\%$, $\hat{a}_1 = -0.0266$, $\hat{a}_2 = 0.5952$ and a negative correlation $\hat{\rho}_{1,2} = -49.65\%$.
The crediting rates [2.22] were equal to the k-year spot rates $r_k(t)$ in the first three cases, meaning that $m = 0$ whereas the last two cases had non-null margins : $m = 100$ basis points for the $r_5(t)$ and $m = 150$ basis points for the $r_1(t)$, as seen in the table 2.7.

		Valuation factor					
k	$r_k(t)$	5	10	15	20	25	30
10	0.03%	1.000633	1.001270	1.001931	1.002663	1.003569	1.004845
20	0.44%	1.011002	1.022130	1.033405	1.044877	1.056659	1.068968
30	0.46%	1.011624	1.023388	1.035313	1.047451	1.059913	1.072919
5+100BP	-0.35%	1.042222	1.086233	1.132127	1.180045	1.230202	1.282963
1+150BP	-0.52%	1.063962	1.132020	1.204460	1.281626	1.363971	1.452141

Table 2.7 – Valuation factors $V(0,t)$ per 1€ of account balance at March 23 2021, with HJM parameters: $\hat{\sigma_1} = 0.03\%$, $\hat{\sigma_2} = 0.02\%$,$\hat{a}_1 = -0.0266$, $\hat{a}_2 = 0.5952$ and $\hat{\rho}_{1,2} = -49.65\%$

The results are a representation of a market consistent valuation of the interest rate guarantee, per 1€ of each participant's account balance as at **March 23, 2021**.

Overall, there is an extremely small difference (0.0267%) compared to the values found in the 1-factor H&W model case, in the table 2.3. That difference is quite null for small maturities and grows slowly as the maturities increase.

CHAPTER 3

FUNDING METHODS FOR CASH-BALANCE PLANS

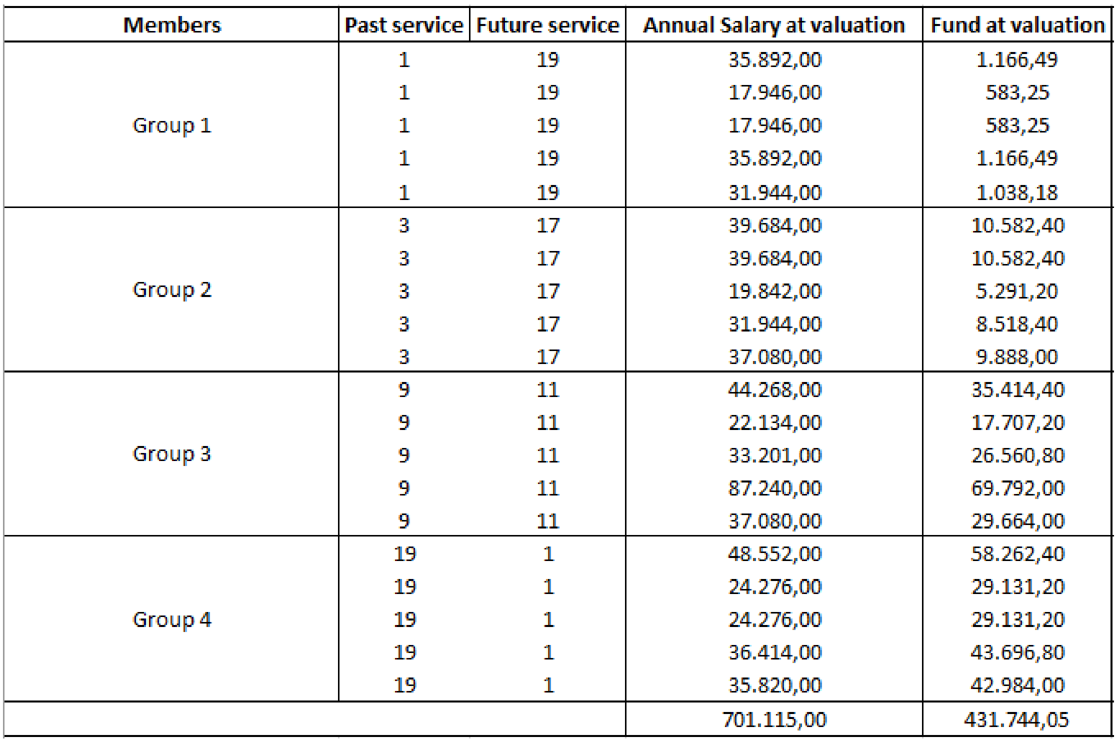
3.1 Defining the Accrued Benefit

As mentioned above in the introduction part, the CB pension plan is classified as a DB plan under the U.S regulation. Under the DB plan, there exist accruals-based valuation methods that value the liability that relates to past services and the two usual approaches are: the **Traditional Unit Credit** (TUC) and the **Projected Unit Credit** (PUC). About the differences between the two approaches resulting in the calculation/definition of the Actuarial Liability (AL_t) and the Normal Cost (NC_t):

- AL_t: the future service or indexation of past service benefits from future salary growth is allowed in the actuarial liability and included in the liability valuation of the PUC method but not in the TUC method;
- NC_t: the normal contributions fund the additional benefit arising from extra service during the contribution period for both methods and they also fund the increase in the past service benefits arising from salary growth in the TUC method.

Under the CB plan, there exist several ways/methods to interpret the unit credit principles. Three particular methods, of which two are similar to the TUC and PUC methods, were used by Hardy, M.R. and Saunders, D. and Zhu, X. (2014 [5]) to describe and calculate the AL_t along with the NC_t.
Here we shall first split into 2 cases: one where salaries don't evolve over time (Constant salary S_t) and one where the salaries evolve through time. In the first case, the two methods similar to the TUC and PUC methods will be introduced and

used. In the second one, we shall work on one hand with a constant salary evolution and on the other hand with a stochastic salary evolution through time.
Regarding the population studied, we opted for 20 (fictive) individuals divided into 4 groups according to the number of years already spent in the enterprise (past service) as well as the number of years left (future service), as seen on the figure 3.1. The time horizon (years to retirement) T goes from 1 to 19. The fund at valuation was fixed arbitrary and the annual salaries were calculated/estimated using the 2018 Belgian gross median salaries for full-employed people retrieved from the National Bank of Belgium[1]. Last but not least, the currency in use is the Euro €.



Members	Past service	Future service	Annual Salary at valuation	Fund at valuation
Group 1	1	19	35.892,00	1.166,49
	1	19	17.946,00	583,25
	1	19	17.946,00	583,25
	1	19	35.892,00	1.166,49
	1	19	31.944,00	1.038,18
Group 2	3	17	39.684,00	10.582,40
	3	17	39.684,00	10.582,40
	3	17	19.842,00	5.291,20
	3	17	31.944,00	8.518,40
	3	17	37.080,00	9.888,00
Group 3	9	11	44.268,00	35.414,40
	9	11	22.134,00	17.707,20
	9	11	33.201,00	26.560,80
	9	11	87.240,00	69.792,00
	9	11	37.080,00	29.664,00
Group 4	19	1	48.552,00	58.262,40
	19	1	24.276,00	29.131,20
	19	1	24.276,00	29.131,20
	19	1	36.414,00	43.696,80
	19	1	35.820,00	42.984,00
			701.115,00	431.744,05

Figure 3.1 – Presentation of the 20 individuals divided in 4 groups

3.2 Constant salary S_t

Before going on with the first two methods, let us start by defining and reminding some notations:

- F_t: notional amount of a participant's fund at valuation date t,
- S_t: salary at t,
- c: notional contribution rate into participant's fund,

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[1] https://statbel.fgov.be

- $\upsilon_r(k)$: valuation discount factor from k back to the valuation date (for a valuation interest r) and if the discount is done using the prevailing yield curve it becomes the price of a k-year unit zero-coupon bond $\upsilon_i(k) = e^{r_k(t)}$ where $r_k(t)$ is the k-year spot rate at t,
- NC_t: normal contribution at t,
- $r^c(t)$: crediting rate in year t,
- $\tilde{r}^c_{t+k}$: assumed crediting rate in year $t+k$,
- $V_i(t,T)$: valuation factor, value at t of the CB benefit due at T, for $i = 1, 2$ ($1 =$ 1-factor H&W model ; $2 =$ 2-factor HJM model),
- T: retirement time/year.

Moreover, we assume that there are no exits before T and the NC are paid in full at the start of each year.

Now we shall detail the two methods with their respective AL_t and NC_t formulas:

- *Past service, with credited interest to retirement* [**Method 1**]: in this method, the AL_t will be the cost of securing the retirement benefit through the capital markets at the valuation date and the NC_t the cost of the new notional input in the participant's account. Furthermore, the AL_t and the NC_t both use a $T-t$ year horizon hence this method is similar to the **PUC** approach.

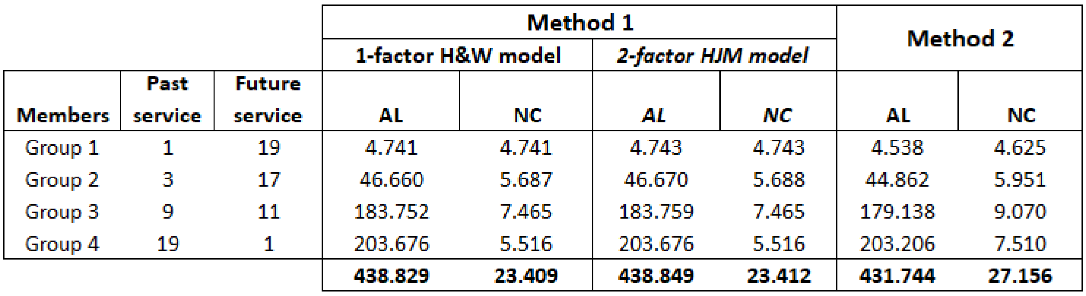
$$\begin{aligned} AL_t &= F_t V_i(t,T) \\ NC_t &= cS_t V_i(t,T) \end{aligned} \tag{3.1}$$

- *Past service, no future credited interest* [**Method 2**]: here the AL_t represents the dissolution liability at when the benefits are well-established and the NC_t funds, at the same time, the increase in F_t from the new notional input and the new input from the credited interest in the year t to $t+1$. This method is analogous to the **TUC** approach since the AL_t and the NC_t are set on a one-year horizon.

$$\begin{aligned} AL_t &= F_t \\ NC_t &= cS_t + (F_t + cS_t)\left((1 + r^c(t))\upsilon_r(1) - 1\right) \end{aligned} \tag{3.2}$$

3.2.1 Practical case

With all the information gathered from the figure 3.1 above and using the formulas of the methods 1 & 2, we calculated the total AL_t and NC_t with a fixed contribution rate $c = 3.25\%$ and a crediting rate equal to the 20-year spot rate (0.44% as at March 23, 2021), as seen below in the figure 3.2. For the first method, we calculated the AL_t & NC_t using both the $V_1(0,T)$, i.e the 1-factor H&W model's valuation factors, and the 2-factor HJM model's valuation factors $V_2(0,T)$.

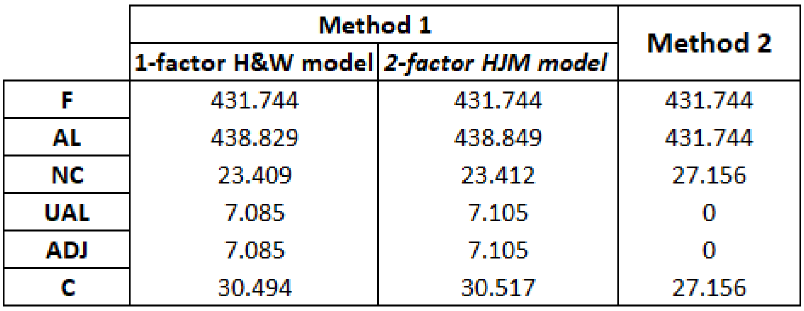


			Method 1				Method 2	
			1-factor H&W model		*2-factor HJM model*			
Members	Past service	Future service	AL	NC	*AL*	*NC*	AL	NC
Group 1	1	19	4.741	4.741	4.743	4.743	4.538	4.625
Group 2	3	17	46.660	5.687	46.670	5.688	44.862	5.951
Group 3	9	11	183.752	7.465	183.759	7.465	179.138	9.070
Group 4	19	1	203.676	5.516	203.676	5.516	203.206	7.510
			438.829	**23.409**	**438.849**	**23.412**	**431.744**	**27.156**

Figure 3.2 – AL_t and NC_t values for the two methods with a contribution rate $c = 3.25\%$ and using a 20-year spot rate $r_k(t) = 0.44\%$

In method 1, the results found using either valuation factors are very close where there exist small and negligible differences: 20 € between the actuarial liabilities and 3 € between the normal costs.

We can go on and calculate the Unfunded AL_t ($UAL_t = AL_t - F_t$), the Adjustment ($ADJ_t = UAL_t$) and the total cost/contributions ($C_t = NC_t + ADJ_t$), as seen below in the figure 3.2

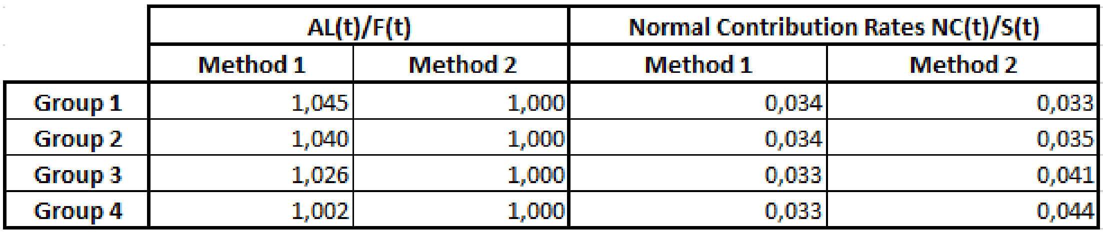


	Method 1		Method 2
	1-factor H&W model	*2-factor HJM model*	
F	431.744	431.744	431.744
AL	438.829	438.849	431.744
NC	23.409	23.412	27.156
UAL	7.085	7.105	0
ADJ	7.085	7.105	0
C	30.494	30.517	27.156

Figure 3.3 – Values of the UAL_t, ADJ_t and C_t for the two methods

The total cost C is strictly positive and significant in both methods meaning that the employer will have to spare some extra amounts: around 30.5K € (7K € of unfunded liability on top of 23.4K € of normal costs) in the first method using either valuation factors and 27K € in the second one.

Furthermore, we can calculate the AL_t/F_t ratio and the normal contribution rates (NC_t/S_t) for each group depending on the time left in the plan for the two methods, as seen in the figure 3.4



	AL(t)/F(t)		Normal Contribution Rates NC(t)/S(t)	
	Method 1	**Method 2**	**Method 1**	**Method 2**
Group 1	1,045	1,000	0,034	0,033
Group 2	1,040	1,000	0,034	0,035
Group 3	1,026	1,000	0,033	0,041
Group 4	1,002	1,000	0,033	0,044

Figure 3.4 – Comparison of the two methods using AL_t/F_t and NC_t/S_t (Normal contribution rates) as criteria

For the AL_t/F_t ratio, we get the valuation factors in the first method and that ratio is equal to 1 for all the groups in the second method. We observe a slow decrease in the first method's normal contribution rates the more we approach the maturity/exit whereas it is the other way around in the second method where new participants (Group 1) start with low NC rates (3.31%) and increase to reach 4.43% for participants with one year left in the plan. It can all be seen in the figure 3.5 which is read from right to left according to number of years (T) to maturity/exit; i.e, new participants have 19 years of future service hence their NC rate is equal to 3.31% in the second method for instance.

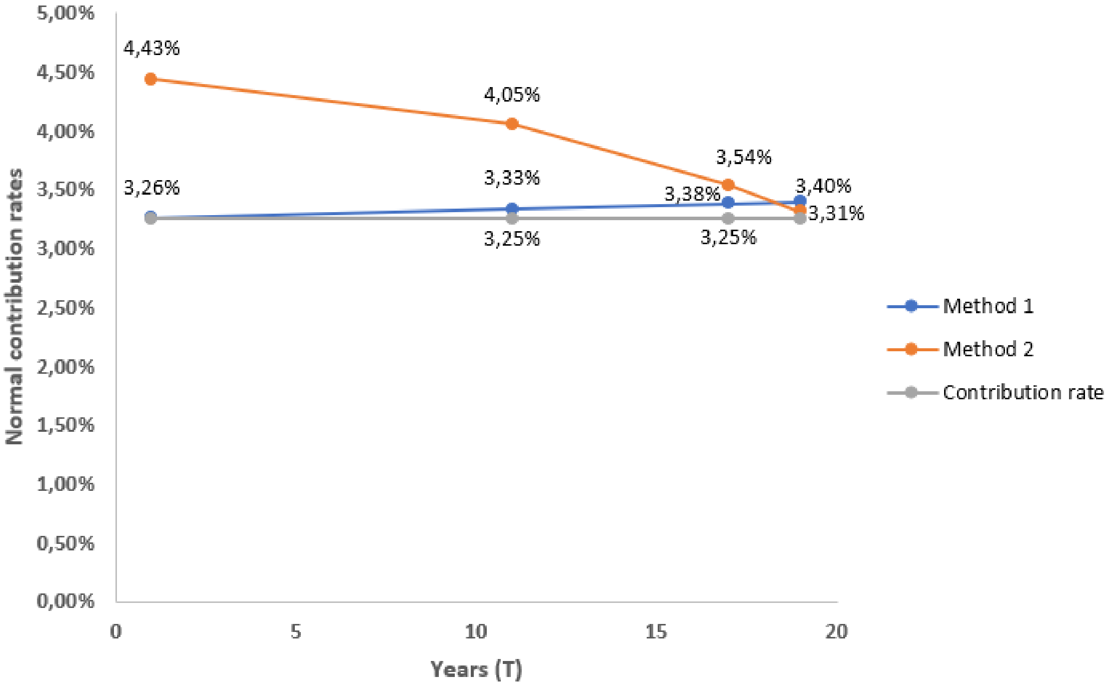


Figure 3.5 – Graphical representation of the normal contribution rates for method 1 & 2

3.3 Stochastic salary S_t

For this part, we start with a constant salary increase and then use a stochastic/time-varying increase. Moreover, we assume that the salary increase is equal to the inflation rate: known, assumed or estimated.
The Euro area inflation rate for the month of June 2021 [2] (= 1.9%) will be of use in the constant inflation part and the 1-factor H&W inflation rate model described above in section 2.1.2 will be used in the stochastic inflation part.

3.3.1 Constant inflation rate i

Here we shall use another method proposed by Hardy, M.R. and Saunders, D. and Zhu, X. (2014 [5]) and similar to the **PUC** approach which, contrarily to method 1, projects the salary growth and the final benefit. The assumptions used in the section above still apply and the method can be defined as follow:

- *Full service, credited interest to retirement, accrued pro rata to service* [**Method 3**]: this method isn't really accruals-based because it incorporates future notional inputs which are dependent to future service but is commonly used in real life. Here, the full final benefit, assumed to accrue linearly over the service period of the participant, is projected via a model for future salaries and a deterministic assumption for future crediting rates. Moreover, the credited interest rate takes, under the CB pension plan, the role that salary growth has under the DB plan; meaning that it acts as the indexation process for the benefit while the participant is still an active member. However this indexation happens to be exogenous under the CB plan whereas it is controllable by the sponsor under the DB plan.

$$\begin{aligned} AL_t &= \frac{n}{n+T-t}\tilde{F}(T)\upsilon_r(T-t) \\ NC_t &= \frac{1}{n+T-t}\tilde{F}(T)\upsilon_r(T-t) \end{aligned} \tag{3.3}$$

where $\tilde{F}(T)$ is the projected final benefit and is equal to:

$$\begin{aligned} \tilde{F}(T) = F_t(1+r_t^c)(1+\tilde{r}_{t+1}^c)(1+\tilde{r}_{t+2}^c)...(1+\tilde{r}_{T-1}^c) \\ + \sum_{k=0}^{T-1} cS_{t+k}(1+\tilde{i}_{t+k})(1+\tilde{i}_{t+k+1})...(1+\tilde{i}_{T-1}) \end{aligned} \tag{3.4}$$

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[2] https://ec.europa.eu/eurostat

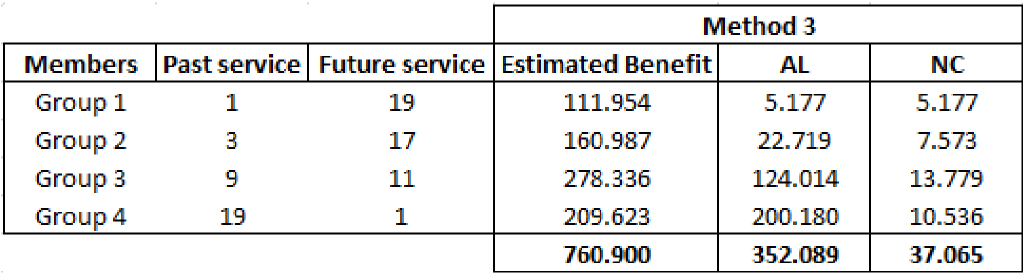
with $\tilde{i}_{t+k}$ being the assumed inflation rate in year $t+k$.

3.3.1.1 Practical case

The individuals and the groups remain the same as seen above in the figure 3.1. With a salary growth rate equal to 1.9% and assuming future crediting rates to be deterministic, the projected final benefit $\tilde{F}(T)$ defined in equation [3.4] can be rewritten as, for each individual $p = 1, ..., 20$

$$\tilde{F}_p(T) = F_p(t)(1+r_t^c)^{T_p-t_p} + cS_{t,p}\left(\frac{(1+r_t^c)^{T_p-t_p} - (1+i)^{T_p-t_p}}{1-\frac{1+i}{1+r_t^c}}\right) \tag{3.5}$$

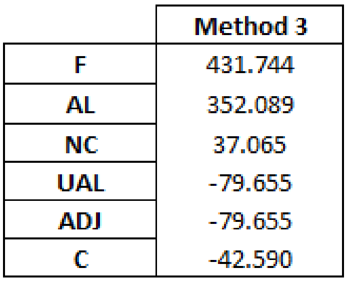
where i is the constant salary growth rate and r_t^c is the crediting interest rate equal to the 20-year spot rate (0.44%).
With all the information gathered from the figure 3.1 above and using the adequate formulas, we calculated the total estimated benefits, AL_t and NC_t as seen below in the figure 3.6



			Method 3		
Members	Past service	Future service	Estimated Benefit	AL	NC
Group 1	1	19	111.954	5.177	5.177
Group 2	3	17	160.987	22.719	7.573
Group 3	9	11	278.336	124.014	13.779
Group 4	19	1	209.623	200.180	10.536
			760.900	**352.089**	**37.065**

Figure 3.6 – $\tilde{F}(T)$, AL_t and NC_t values for the third method with a contribution rate $c = 3.25\%$ and the salary growth rate $i = 1.9\%$

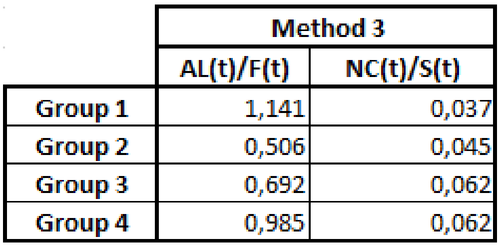
We can go on and calculate the Unfunded AL_t ($UAL_t = AL_t - F_t$), the Adjustment ($ADJ_t = UAL_t$) and the total cost/contributions ($C_t = NC_t + ADJ_t$), as seen below in the figure 3.6



	Method 3
F	431.744
AL	352.089
NC	37.065
UAL	-79.655
ADJ	-79.655
C	-42.590

Figure 3.7 – Values of the UAL_t, ADJ_t and C_t for the third method

The total cost is negative and very significant meaning that the employer would save up to 42.6K€. Let us now calculate the AL_t/F_t ratio and the NC rates for each group depending on the time left in the plan, as seen in the figure 3.8



	Method 3	
	AL(t)/F(t)	NC(t)/S(t)
Group 1	1,141	0,037
Group 2	0,506	0,045
Group 3	0,692	0,062
Group 4	0,985	0,062

Figure 3.8 – AL_t/F_t and NC_t/S_t (Normal contribution rates) for the third method

We can clearly see that this method is flawed. For instance, if we take the third group's AL_t/F_t ratio equal to 0.692, it means that the actuarial liability is approximately 31% less than the participant's termination benefit which is very unsafe and unwise. This happens to be the case for 3 out of 4 groups. On top of that, the NC rates are quite high compared to the other methods and evolve in the opposite direction as the first method (which is the most similar), as seen in the figure 3.9.

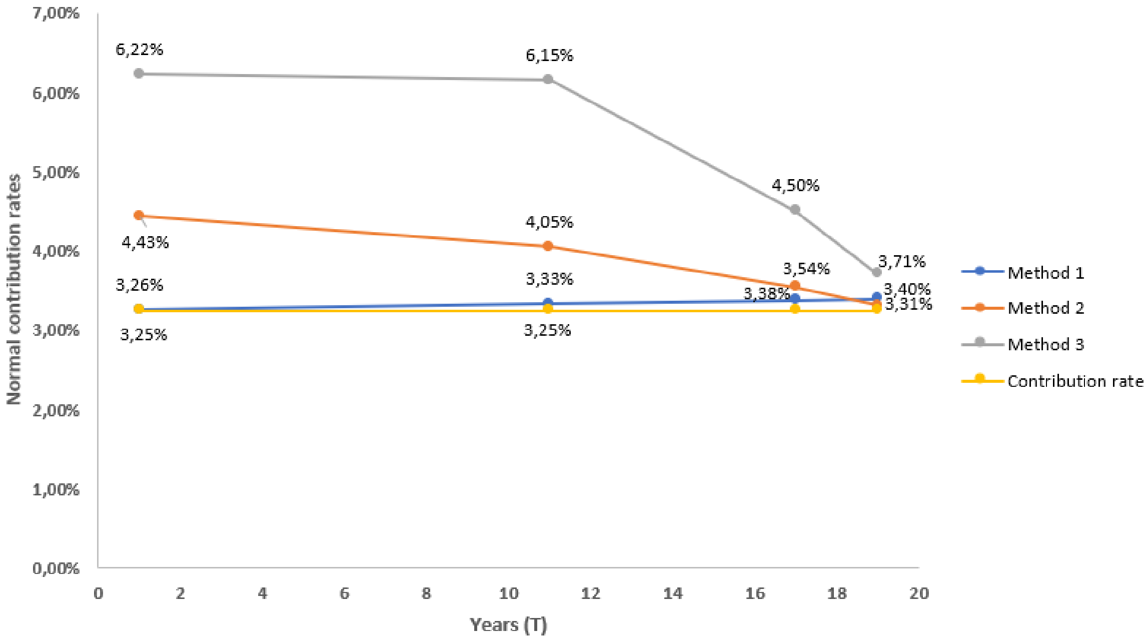


Figure 3.9 – Graphical representation of the normal contribution rates

Due to assumptions based on the future, the results of this method are hard to explain and quite incomparable since both the AL_t/F_t ratios and the NC rates have future-based parameters hence results vary according to the assumptions made.

3.3.2 Time-varying inflation rate $i(t)$

Let us now consider the case where the inflation rate varies over time and follows the 1-factor H&W inflation rate model $i(t)$ described in the section 2.1.2. Moreover, since there exists a link/correlation between this inflation rate model and the 1-factor H&W interest rate model $r(t)$, we shall use the appropriate valuation factors $V_1(0, T)$ and only work with the 1-factor case for the first method. We shall also consider the second method along with the first one and not use the third.
The parameters of the inflation rate model having been estimated in the section 2.1.5.1, we used the $i(t)$ formula found in equation [2.12] and made 1000 simulations of the 20-year inflation rate $i_{20}(t)$, as seen on the figure 3.10.

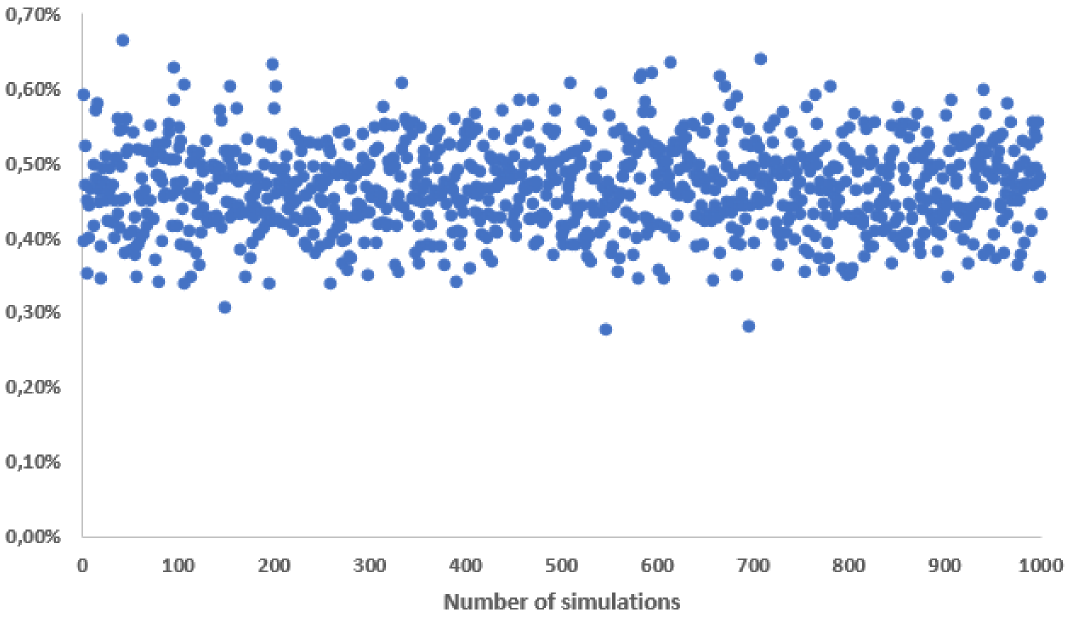


Figure 3.10 – 1000 simulations of the H&W 20-year inflation rate $i_{20}(t)$

After the simulation, we made some statistics about the 20-year inflation rate as seen in the table 3.1 and the CDF & PDF of the rate can be found in Appendices .7.

	Mean	Standard deviation	VaR 95%	VaR 99.5%
$i_{20}(t)$	0.468%	0.058%	0.561%	0.622%

Table 3.1 – Statistics on the 1000 simulations made for the 20-year inflation rate $i_{20}(t)$

Now that we have made enough simulations for $i_{20}(t)$, we can study the evolution of the costs and the fund for the different groups, using the first two methods, in

1-year time $(t+1)$ but valued in $t=0$ (i.e, multiplied by the 1-year discount factor $\upsilon_r(1)$). The different variables can now be defined as

$$\begin{aligned}\tilde{S} &= S_t(1+i_{20}(t))\upsilon_r(1),\\ \tilde{F} &= F_t + c\tilde{S}.\end{aligned} \tag{3.6}$$

From the respective equations, we can recalculate the actuarial liabilities and the normal costs, both in $t+1$, in order to observe the impact of the introduction of a time-varying inflation/salary growth using the methods 1 and 2.
In $t+1$, the oldest participants (Group 4) will be paid off with an amount equal to the actuarial liability; i.e, according to the method used and assuming that the participants leave at the end of the year t_+ (or $t+1_-$), their benefits will be equal to the following amounts

	Method 1	Method 2
Benefits paid	203.676	203.206

Table 3.2 – Benefits paid in € to the Group 4 participants in $t+1$, valued at $t=0$

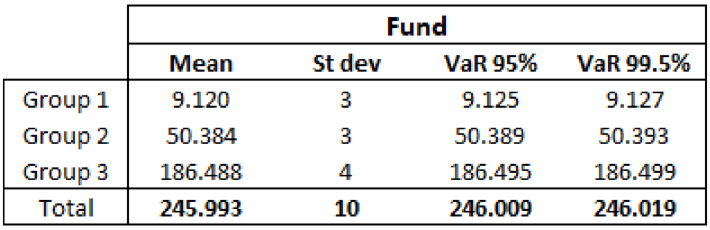
The AL_t and NC_t equations for the first method in equation [3.1] can be rewritten, in $t+1$ but valued in $t=0$, as follow

$$\begin{aligned}AL &= \tilde{F}V_1(t+1,T)\\ NC &= c\tilde{S}V_1(t+1,T)\end{aligned} \tag{3.7}$$

with $V_1(t+1,T)$ being the 1-factor H&W model's valuation factors. By doing the same thing for the second method's equations in [3.2] we get

$$\begin{aligned}AL &= \tilde{F}\\ NC &= c\tilde{S} + (F_t + cS_t)\left((1+i(20))\upsilon_r(1) - 1\right)\end{aligned} \tag{3.8}$$

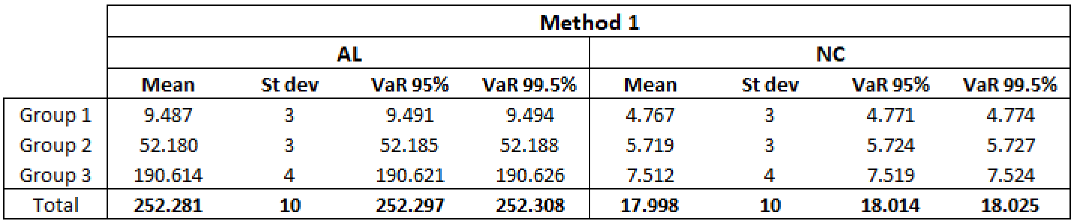
The figure 3.11 shows different statistics on the present value of the simulated fund values in 1-year time for the three remaining groups and in total.



	Fund			
	Mean	St dev	VaR 95%	VaR 99.5%
Group 1	9.120	3	9.125	9.127
Group 2	50.384	3	50.389	50.393
Group 3	186.488	4	186.495	186.499
Total	**245.993**	**10**	**246.009**	**246.019**

Figure 3.11 – Statistics on the present value of the simulated fund values $\tilde{F}$ in 1-year time

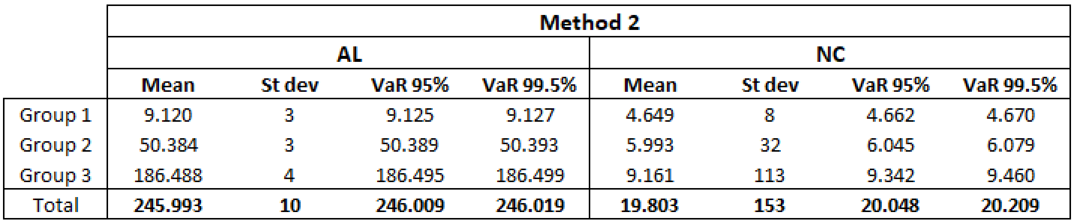
It shows that the simulated fund has a mean value of 245.993 € with a standard deviation of 10 €. Moreover, the value at risk with a probability of 95% and 99.5% are equal to 246.009 € and 246.019 € respectively; they are relatively close and are all greater than the mean.
After finding the fund's statistics, we can determine the same statistics for the AL and NC for the two methods.
The method 1 statistics of the AL and NC can be found in the figure 3.7



	Method 1							
	AL				NC			
	Mean	St dev	VaR 95%	VaR 99.5%	Mean	St dev	VaR 95%	VaR 99.5%
Group 1	9.487	3	9.491	9.494	4.767	3	4.771	4.774
Group 2	52.180	3	52.185	52.188	5.719	3	5.724	5.727
Group 3	190.614	4	190.621	190.626	7.512	4	7.519	7.524
Total	**252.281**	**10**	**252.297**	**252.308**	**17.998**	**10**	**18.014**	**18.025**

Figure 3.12 – AL and NC statistics under method 1

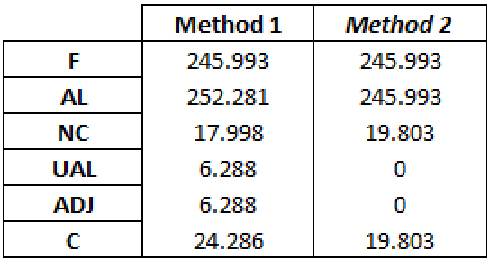
We observe nothing in particular compared to the fund except the fact that the AL mean value is greater than the one of the fund; this was also the case in the deterministic part in section 3.2.1.
As for the method 2, as seen in the figure 3.8, the standard deviation of the normal costs is big (comparing to the first method for instance). This may be due to the fact that it normally projects the increase of the fund and salary in 1-year time but using crediting rates as opposed to inflation rate/salary growth rate in this case.



	Method 2							
	AL				NC			
	Mean	St dev	VaR 95%	VaR 99.5%	Mean	St dev	VaR 95%	VaR 99.5%
Group 1	9.120	3	9.125	9.127	4.649	8	4.662	4.670
Group 2	50.384	3	50.389	50.393	5.993	32	6.045	6.079
Group 3	186.488	4	186.495	186.499	9.161	113	9.342	9.460
Total	**245.993**	**10**	**246.009**	**246.019**	**19.803**	**153**	**20.048**	**20.209**

Figure 3.13 – AL and NC statistics under method 2

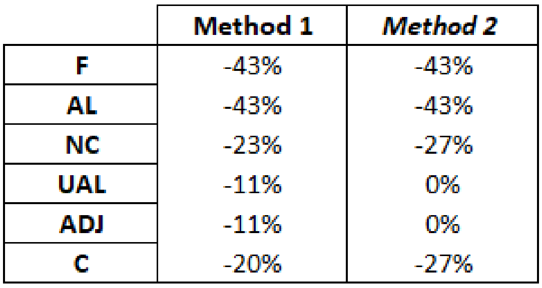
A comparison between these two methods can be made by calculating the total cost C under each method as seen in the figure 3.14



	Method 1	***Method 2***
F	245.993	245.993
AL	252.281	245.993
NC	17.998	19.803
UAL	6.288	0
ADJ	6.288	0
C	24.286	19.803

Figure 3.14 – Values of the UAL, ADJ and C for the two methods

We can see that the total cost C decreased compared to the values found in the figure 3.3, in the deterministic part. But, we cannot say or conclude by seeing these values that the employer made a gain. For that, we calculated the variations between t and $t+1$ of the different variables (valued in $t = 0$) and we got the following results



	Method 1	***Method 2***
F	-43%	-43%
AL	-43%	-43%
NC	-23%	-27%
UAL	-11%	0%
ADJ	-11%	0%
C	-20%	-27%

Figure 3.15 – Evolution of the different criteria from t to $t+1$

Now we see clearly that the employer did not make a gain since the fund and the AL both decreased by 43% in the two methods but the total cost C decreased by just 20% in the first method and by 27% in the second one which happens to be more beneficial in this case.

CHAPTER 4

CONCLUSION

The main purpose of the Cash-balance pension plan was to capture the advantages of the Defined contribution plan but still be regulated as the Defined benefit plan. That is why a few years after its introduction by the Bank of America, in the late 1990s, many companies switched from DB plans to CB plans instead of terminating them since they were costly and not so easy to explain to the employees. However, the CB plan is not that cheap and has some inconveniences such as the "wearaway" feature for instance even though it has some advantages like its attractiveness to young people and the flexibility.

The aim of this study was to analyze the CB pension and contributions using the methods and principles of financial economics to give a market consistent evaluation of the CB liability's costs.
First, we defined and derived the valuation factor $V(0,T)$ which is the market consistent value of a CB benefit at time t and due at T, with $t < T$. Since the valuation of the CB benefit can be done using market consistent model of the term structure of interest rates, we used on one hand the 1-factor H&W model and on the other hand the 2-factor HJM model in the determination of the different valuation factors. In the H&W model case, we estimated and calibrated the parameters using different Euro area data as at March 23 2021 for both the interest rate model and the inflation rate model which would be used later on in the stochastic part of the funding methods section. The valuation factors' results show positive but not very significant values which are all greater than 1; meaning that the interest rate guarantee has a positive market value and the surplus over 1 is the cost of that

interest rate guarantee or, in other words, the additional amount to be added to the fund coming from the pockets of the plan sponsor. The parameters' sensitivities showed that only the changes in spot rates had significant impacts on the valuation factors: positive variations lead to an increase of valuation and negative variations reduced the valuation factors, sometimes below 1 (i.e the plan sponsor doesn't have any extra cost hence is beneficial). As for the HJM model, the parameters were estimated using the Belgian linear bonds rates (OLOs), with a period set between 02/01/2015 and 29/11/2019 and tenors going from 1 year to 30 years. The values of the valuation factors came out very close to the ones in the H&W model, with a small difference equal to 0.0267%.

Then, after defining the population to be studied (20 imaginary individuals cut up into 4 groups according to their past & future services) and having arbitrary fixed their salaries, we split the funding method section into two parts: the constant salary and the stochastic salary parts.

In the first part, we introduced and defined two methods, method 1 & 2, which are similar to the well-known Projected Unit Credit (PUC) and the Traditional Unit Credit (TUC) respectively. Under the two methods, the total costs were positive and very significant with the first method having the highest value; i.e, some extra cash has to be spared to meet the liabilities. The valuation factors were used in the first method and the outcome is logical since we had already found that the plan sponsor had to put extra money on the table.

In the stochastic part, we started by defining the third method which is similar to the PUC approach but, contrarily to method 1, projects the salary growth and the final benefit. Here we used the Euro area inflation rate for the month of June 2021 ($= 1.9\%$) as the salary growth. However, this method is flawed and was ruled out for being unsafe and unwise.

Finally, we used the 1-factor inflation rate model as the salary growth in the methods 1 and 2 for risk management purposes. We took the 20-year inflation rate and simulated it 1000 times. After computing the values, we made some statistics (mean, standard deviation and Value-at-Risk) for the inflation rate, the actuarial liabilities & the normal costs then calculated the total costs under the two methods. The results show a decrease in the costs for both methods but a deeper inspection showed that the plan sponsor does not necessarily make a gain due to greater decrease in the fund (-43% in both methods) compared to -20% and -27% for the methods 1 and 2 respectively with the second method looking more advantageous for the plan sponsor in this case.

Appendices

.1 Literature review comparison figures

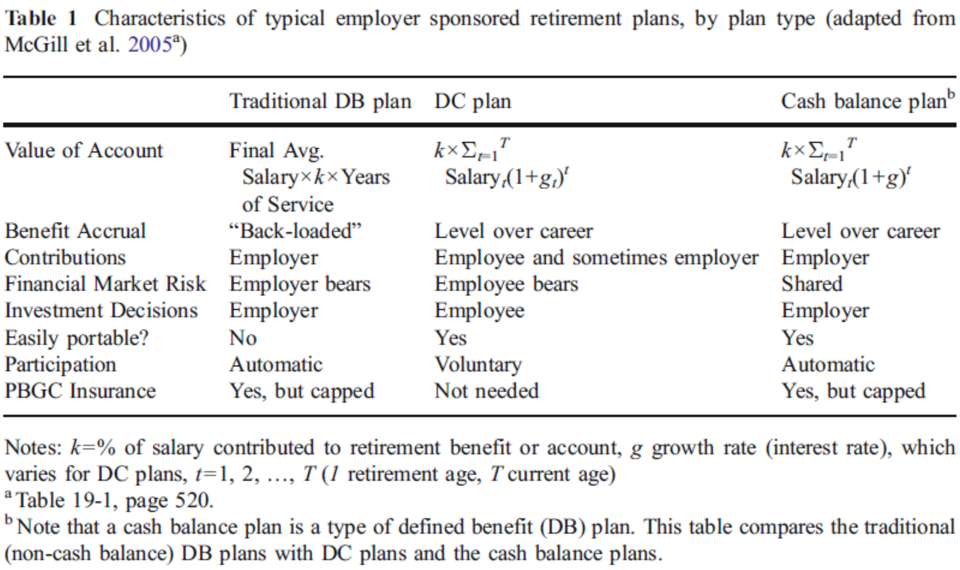


Table 1 Characteristics of typical employer sponsored retirement plans, by plan type (adapted from McGill et al. 2005[a])

	Traditional DB plan	DC plan	Cash balance plan[b]
Value of Account	Final Avg. Salary × k × Years of Service	$k \times \Sigma_{t=1}^{T}$ Salary$_t(1+g_t)^t$	$k \times \Sigma_{t=1}^{T}$ Salary$_t(1+g)^t$
Benefit Accrual	"Back-loaded"	Level over career	Level over career
Contributions	Employer	Employee and sometimes employer	Employer
Financial Market Risk	Employer bears	Employee bears	Shared
Investment Decisions	Employer	Employee	Employer
Easily portable?	No	Yes	Yes
Participation	Automatic	Voluntary	Automatic
PBGC Insurance	Yes, but capped	Not needed	Yes, but capped

Notes: k=% of salary contributed to retirement benefit or account, g growth rate (interest rate), which varies for DC plans, t=1, 2, …, T (*1* retirement age, T current age)

[a] Table 19-1, page 520.

[b] Note that a cash balance plan is a type of defined benefit (DB) plan. This table compares the traditional (non-cash balance) DB plans with DC plans and the cash balance plans.

Figure 1 – Comparison among the DB, DC and CB pension plans; Kapinos K.A. (2009 [6])

Cash balance pension plan conversions and the new economy 303

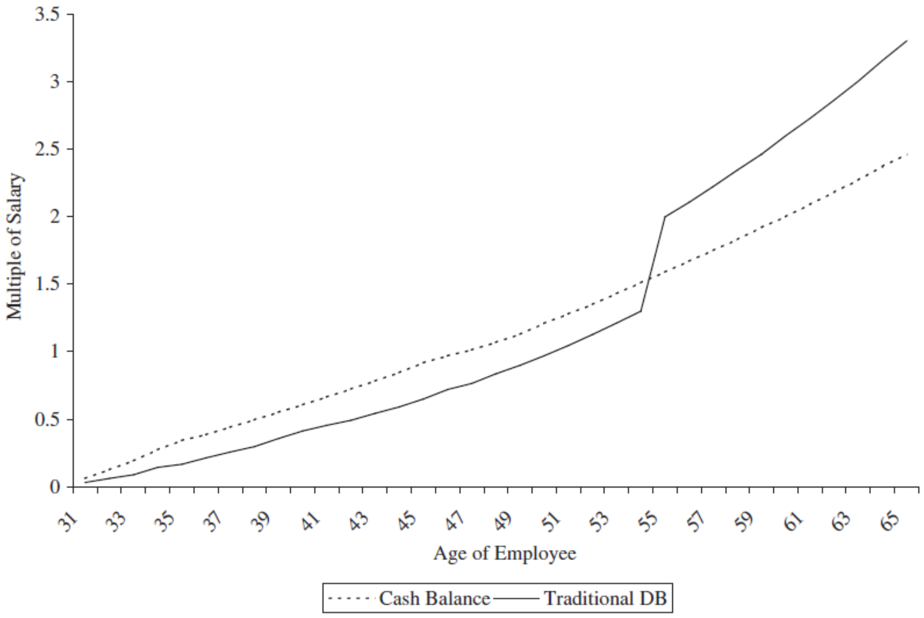


Figure 1. Projected benefit obligation under different pension plans

Figure 2 – Benefits accrual over the years under DB and CB pension plans; Coronado J.L. and Copeland P.C. (2004 [3])

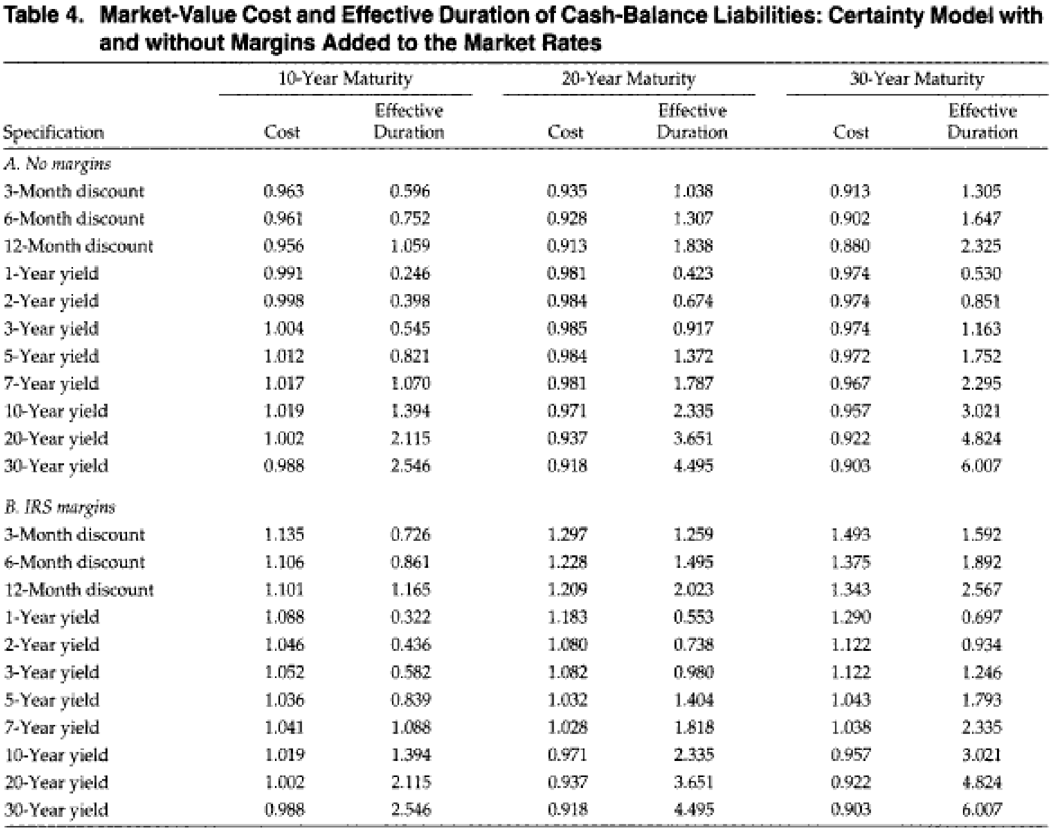


Table 4. Market-Value Cost and Effective Duration of Cash-Balance Liabilities: Certainty Model with and without Margins Added to the Market Rates

	10-Year Maturity		20-Year Maturity		30-Year Maturity	
Specification	Cost	Effective Duration	Cost	Effective Duration	Cost	Effective Duration
A. No margins						
3-Month discount	0.963	0.596	0.935	1.038	0.913	1.305
6-Month discount	0.961	0.752	0.928	1.307	0.902	1.647
12-Month discount	0.956	1.059	0.913	1.838	0.880	2.325
1-Year yield	0.991	0.246	0.981	0.423	0.974	0.530
2-Year yield	0.998	0.398	0.984	0.674	0.974	0.851
3-Year yield	1.004	0.545	0.985	0.917	0.974	1.163
5-Year yield	1.012	0.821	0.984	1.372	0.972	1.752
7-Year yield	1.017	1.070	0.981	1.787	0.967	2.295
10-Year yield	1.019	1.394	0.971	2.335	0.957	3.021
20-Year yield	1.002	2.115	0.937	3.651	0.922	4.824
30-Year yield	0.988	2.546	0.918	4.495	0.903	6.007
B. IRS margins						
3-Month discount	1.135	0.726	1.297	1.259	1.493	1.592
6-Month discount	1.106	0.861	1.228	1.495	1.375	1.892
12-Month discount	1.101	1.165	1.209	2.023	1.343	2.567
1-Year yield	1.088	0.322	1.183	0.553	1.290	0.697
2-Year yield	1.046	0.436	1.080	0.738	1.122	0.934
3-Year yield	1.052	0.582	1.082	0.980	1.122	1.246
5-Year yield	1.036	0.839	1.032	1.404	1.043	1.793
7-Year yield	1.041	1.088	1.028	1.818	1.038	2.335
10-Year yield	1.019	1.394	0.971	2.335	0.957	3.021
20-Year yield	1.002	2.115	0.937	3.651	0.922	4.824
30-Year yield	0.988	2.546	0.918	4.495	0.903	6.007

Figure 3 – The cost and duration of CB pension plans, with and without margins; Brown D.T., Dybvig P.H. and Marshall W.J.(2001 [2])

.2 EURIBOR rates and a one sample path of the H&W short-term interest rate

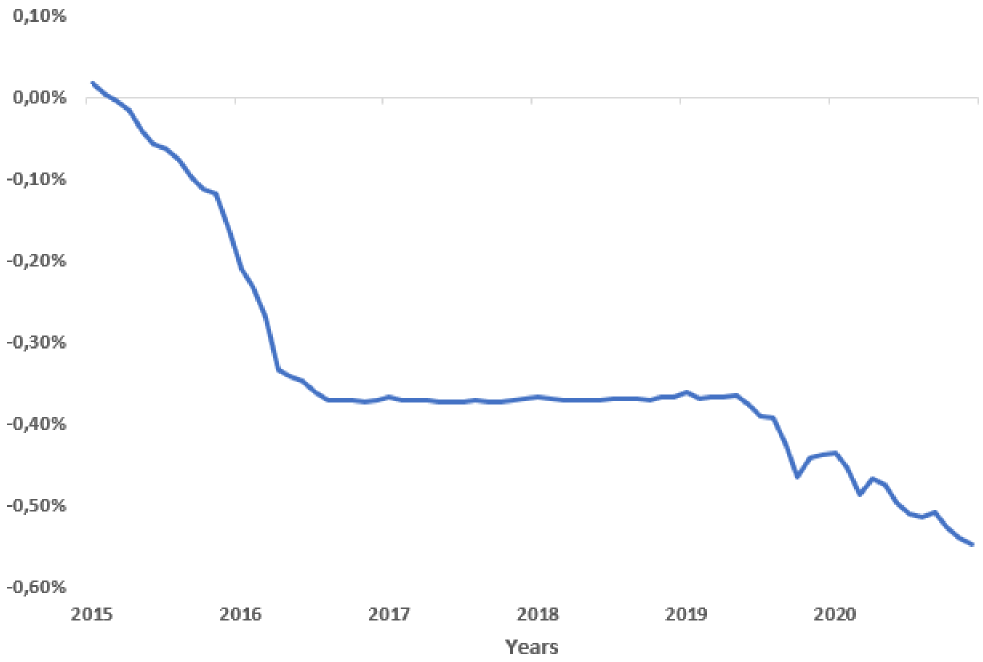


Figure 4 – Euro Interbank Offered Rate(EURIBOR) rates from January 2015 to January 2020

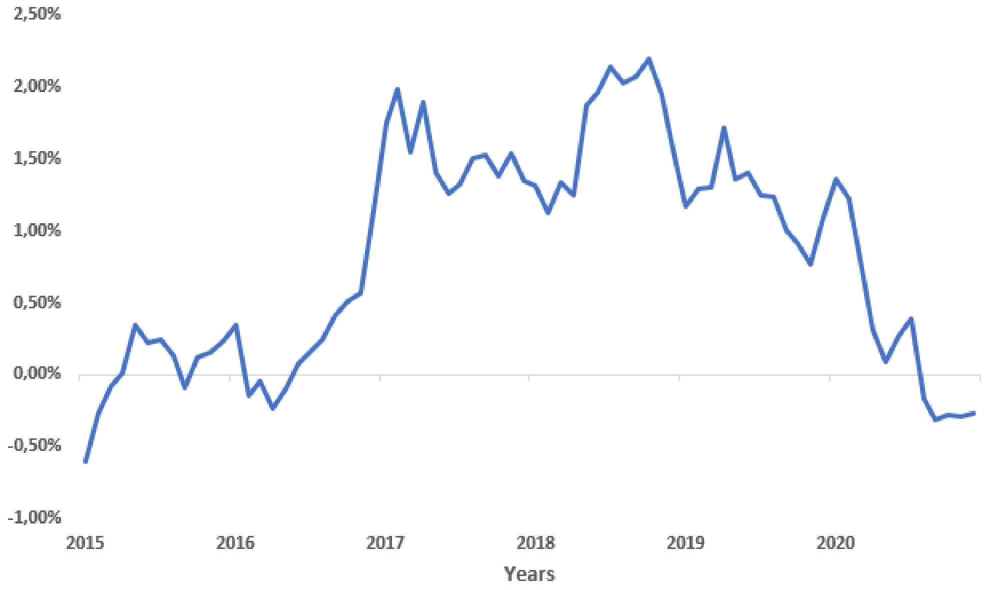


Figure 5 – Euro area historical inflation rates from January 2015 to January 2020

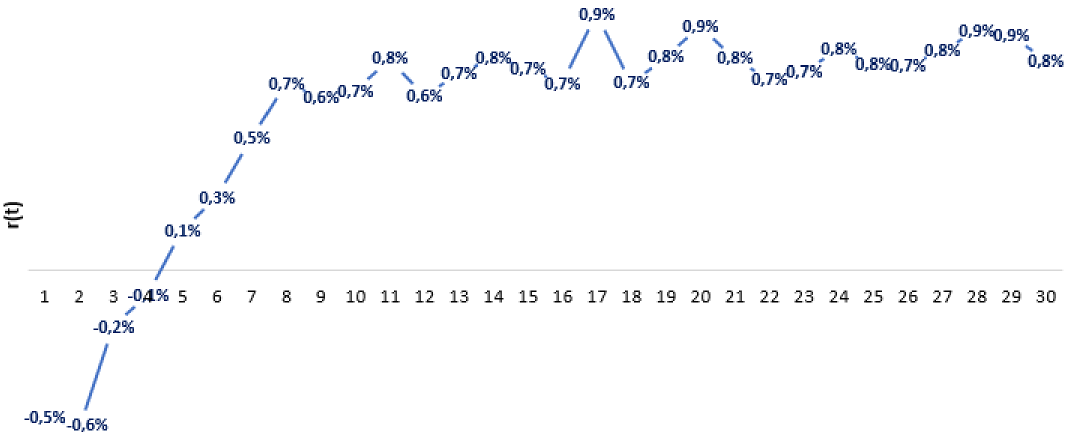


Figure 6 – One sample path of the Hull & White interest rate $r(t)$, for $t = 1, ..., 30$

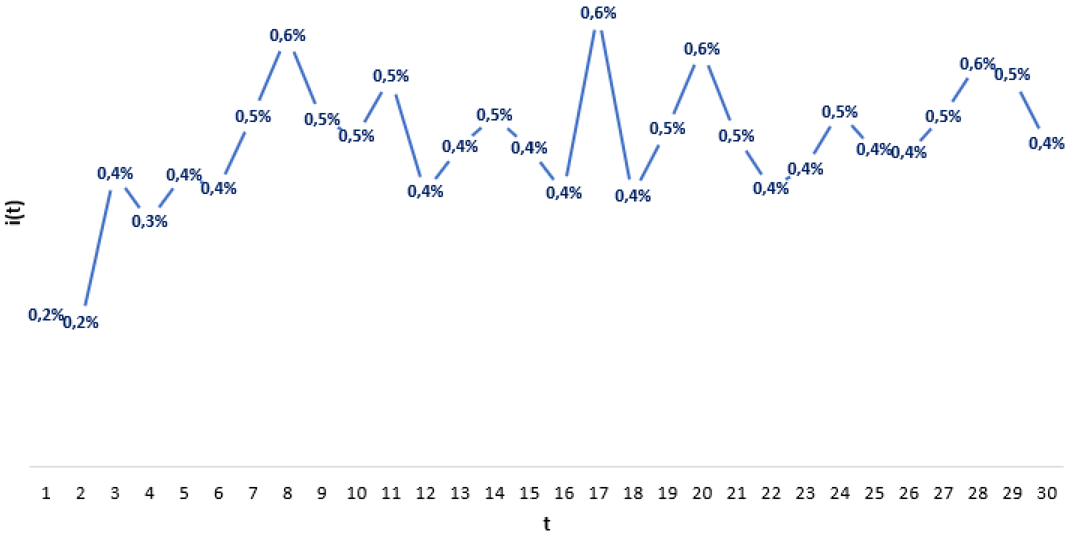


Figure 7 – One sample path of the Hull & White inflation rate $i(t)$, for $t = 1, ..., 30$

.3 Derivation of the HJM 's drift and zc bond price under risk neutral world

Proposition .3.1. *If the market is arbitrage free then the drift of forward rates* $\alpha(t,T)$ *is related to* $\sigma(t,T)$ *by the next relation:*

$$\alpha(t,T) = \sigma(t,T)^\top \int_t^T \sigma(t,u)du = \sum_{i=1}^{p} \sigma_i(t,T)^\top \int_t^T \sigma_i(t,u)du \tag{1}$$

Proof. First of all, this condition on the drift helps ensure that the market is arbitrage free. Then, under the risk neutral measure $\mathbb{Q}$, discounted prices are martingales.
Therefore, from equations 2.29 and 1, the discounted bond price is equal to:

$$\frac{P(t,T)}{B_t} = \exp\left(-\int_0^T f(0,s)ds - \int_0^t \int_u^T \alpha(u,s)dsdu + \int_0^t S(u,t)^\top dW_u\right) \tag{2}$$

where B_t^{-1} is the discount factor in this framework and $S(u,T)$ is a p-vector, integral of $\sigma(u,s)$:

$$S(u,T) = -\int_u^T \sigma(u,s)ds.$$

Moreover, if the market is arbitrage free and complete, there exists a single risk neutral measure Q such that discounted zero-coupon prices are martingales which is verified/fulfilled if and only if the drift of $d\left(\frac{P(t,T)}{B_t}\right)$ is null. By Itô's lemma, this differential is:

$$\frac{d\left(\frac{P(t,T)}{B_t}\right)}{\frac{P(t,T)}{B_t}} = \left(\frac{1}{2}S(t,T)^\top S(t,T) - \int_t^T \alpha(t,s)ds\right)dt + S(t,T)^\top dW_t. \tag{3}$$

The drift is null if and only if:

$$\int_t^T \alpha(t,s)ds = \frac{1}{2}S(t,T)^\top S(t,T)$$

which leads to the proposition/condition .3.1 when derived with respect to T. □

Corollary .3.1.1. *Under the risk neutral measure, the zero-coupon bond is solution of the SDE, for* $t \leq T$*:*

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \left(\int_t^T \sigma(t,s)ds\right)^\top dW_t. \tag{4}$$

Proof. By definition, $P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$ and

$$d\left(-\int_t^T f(t,s)ds\right) = d\ln P(t,T) = f(t,t)dt - \int_t^T df(t,s)ds \tag{5}$$

$$= \left(r_t - \int_t^T \alpha(t,s)ds\right)dt - \left(\int_t^T \sigma(t,s)^\top ds\right)dW_t. \tag{6}$$

Then using the Itô's lemma with the variable $-\int_t^T f(t,s)ds$ leads to:

$$\frac{dP(t,T)}{P(t,T)} = \left(r_t \underbrace{-\int_t^T \alpha(t,s)ds + \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^\top \left(\int_t^T \sigma(t,s)ds\right)}_{=0}\right)dt - \left(\int_t^T \sigma(t,s)ds\right)^\top dW_t \tag{7}$$

where the condition .3.1 was used. □

.4 Derivation of the HJM ZC coupon bond price

In this section we shall consider the case where we have separable and exponentially decaying volatility:

$$\sigma_i(s,t) = \varsigma_i(r(s))e^{-\int_s^t \kappa_i(x)dx} \tag{8}$$

where ς_i and κ_i are deterministic functions for $1 \le i \le p$.

Lemma .4.1. *Let $\sigma_i(t,T)$ be given by [8]. Then the following identity holds:*

$$\sigma_i(s,t+u) = \sigma_i(s,t)e^{-\int_s^{t+u} \kappa_i(x)dx} \tag{9}$$

Proof.

$$\sigma_i(s,t+u) = \varsigma_i(r(s))e^{-\int_s^{t+u} \kappa_i(x)dx} = \left(\varsigma_i(r(s))e^{-\int_s^t \kappa_i(x)dx}\right)e^{-\int_t^{t+u} \kappa_i(x)dx}$$

$$= \sigma_i(s,t)e^{-\int_s^{t+u} \kappa_i(x)dx}$$

□

Lemma .4.2. *Let* $\gamma_i(t,T) = \int_t^T e^{-\int_t^u \kappa_i(x)dx} du$. *Then the following identities hold:*

$$\chi_i(t,T) = \int_t^T e^{-\int_t^u \kappa_i(x)dx} \int_t^u e^{-\int_t^v \kappa_i(x)dx} dv du = \frac{1}{2}\gamma_i^2(t,T); \tag{10}$$

$$\int_t^T \alpha_i(s,u)du = \gamma_i(t,T)\alpha_i(s,t) + \frac{1}{2}\gamma_i^2(t,T)\sigma_i^2(s,t). \tag{11}$$

$$\int_t^T \sigma_i(s,u)du = \gamma_i(t,T)\sigma_i(s,t) \tag{12}$$

with $\alpha_i(s,u) = \sigma_i(s,u)\int_s^u \sigma_i(s,v)dv$.

Proof.

$$\begin{aligned}
\chi_i(t,T) &= \int_t^T e^{-\int_t^u \kappa_i(x)dx} \int_t^u e^{-\int_t^v \kappa_i(x)dx} dv du \\
&= \int_t^T \frac{d}{du}\left[\int_t^u e^{-\int_t^v \kappa_i(x)dx} dv\right]\left[\int_t^u e^{-\int_t^v \kappa_i(x)dx} dv\right] du \\
&= \int_t^T \gamma_i(t,u)\frac{d}{du}\gamma_i(t,u)du = \int_t^T d\left[\frac{1}{2}\gamma_i^2(t,u)\right] \\
&= \frac{1}{2}[\gamma_i^2(t,T) - \gamma_i^2(t,t)] \\
&= \frac{1}{2}\gamma_i^2(t,T)
\end{aligned}$$

since $\gamma_i(t,t) = 0$. For the equation [11], using equation [9], we get:

$$\begin{aligned}
\int_t^T \alpha_i(s,u)du &= \int_t^T \sigma_i(s,u)\int_s^u \sigma_i(s,v)dv du \\
&= \sigma_i(s,t)\int_t^T e^{-\int_t^v \kappa_i(x)dx}\left[\int_s^t \sigma_i(s,v)dv + \int_t^u \sigma_i(s,v)dv\right] du \\
&= \sigma_i(s,t)\int_t^T e^{-\int_t^v \kappa_i(x)dx} du \int_s^t \sigma_i(s,v)dv + \sigma_i^2(s,t)\chi_i(t,T) \\
&= \gamma_i(t,T)\alpha_i(s,t) + \frac{1}{2}\gamma_i^2(t,T)\sigma_i^2(s,t)
\end{aligned}$$

The same arguments are used to prove [12]. ☐

Recalling the forward rate formula in equation [2.27] and the general formula of HJM's zc bond prices $P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$, this latter formula becomes:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(-\sum_{i=1}^{p}\int_t^T \zeta_i(u)du\right) \tag{13}$$

with $\zeta_i(t) = \int_0^t \alpha_i(s,t)ds + \int_0^t \sigma_i(s,t)dW_i(s)$.

Theorem .4.1. *Let $\sigma_i(t,T)$ be given by [8]. Then the bond price is given by the following formula:*

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(-\Phi(t,T) - \Psi(t,T) - \gamma_1(t,T)[r(t) - f(0,t)]\right) \tag{14}$$

where $\gamma_i(t,T) = \int_t^T e^{-\int_t^u \kappa_i(x)dx} du$, for $1 \leq i \leq p$, and

$$\begin{aligned}
\Phi(t,T) &= \frac{1}{2}\sum_{i=1}^{p} \gamma_i^2(t,T)\epsilon_i(t); \\
\Psi(t,T) &= \sum_{i=2}^{p} \zeta_i(t)[\gamma_i(t,T) - \gamma_1(t,T)]; \\
\epsilon_i(t) &= \int_0^t \sigma_i^2(s,t)ds.
\end{aligned}$$

Proof. From the equation [13], we have that:

$$\begin{aligned}
\int_t^T \zeta_i(u)du &= \int_t^T \left[\int_0^t \alpha_i(s,u)ds + \int_0^t \sigma_i(s,u)d\tilde{W}_i(s)\right] du \\
&= \int_0^t \int_t^T \alpha_i(s,u)duds + \int_0^t \int_t^T \sigma_i(s,u)dud\tilde{W}_i(s) \\
&= \int_0^t \left[\gamma_i(t,T)\alpha_i(s,t) + \frac{1}{2}\gamma_i^2(t,T)\sigma_i^2(s,t)\right] ds \\
&\quad + \int_0^t \gamma_i(t,T)\sigma_i(s,t)d\tilde{W}_i(s) \\
&= \gamma_i(t,T)\zeta_i(t) + \frac{1}{2}\gamma_i^2(t,T)\epsilon_i(t)
\end{aligned}$$

where the equations [11] and [12] were used to go from the second equality to the third one. Finally we have

$$\begin{aligned}
\sum_{i=1}^{p} \int_t^T \zeta_i(u)du &= \sum_{i=2}^{p} \left(\gamma_i(t,T)\zeta_i(t) + \frac{1}{2}\gamma_i^2(t,T)\epsilon_i(t)\right) \\
&= \Phi(t,T) + \sum_{i=1}^{p}[(\gamma_i(t,T) - \gamma_1(t,T)) + \gamma_1(t,T)]\zeta_i(t) \\
&= \Phi(t,T) + \Psi(t,T) + \gamma_1(t,T)\sum_{i=1}^{p} \zeta_i(t) \\
&= \Phi(t,T) + \Psi(t,T) + \gamma_1(t,T)[r(t) - f(0,t)]
\end{aligned}$$

□

.5 Derivation of the HJM-H&W ZC coupon bond price

From the theorem 2.2.1 and equation [2.34] we can determine the zero-coupon bond price of the 1-factor HJM with the volatility defined as in equation [2.35]. Let $\varsigma_1(r(s)) = \sigma_1$ =constant and $e^{-\int_s^t \kappa_1(x)dx} = e^{-\int_t^T a_1 ds} = e^{-a_1(T-t)}$, we have

$$\begin{aligned}
\gamma_1(t,T) &= \int_t^T e^{-\int_t^u \kappa_1(x)dx} du = \int_t^T e^{-\int_t^u a_1 ds} du = \left(\frac{1-e^{-a_1(T-t)}}{a_1}\right) = B_1(t,T);\\
\epsilon_1(t) &= \int_0^t \sigma_1^2(s,t)ds = \sigma_1^2\left(\frac{1-e^{-2a_1 t)}}{2a_1}\right);\\
\Phi(t,T) &= \frac{1}{2}\sum_{i=1}^{p}\gamma_i^2(t,T)\epsilon_i(t) = \frac{1}{2}\gamma_1^2(t,T)\epsilon_1(t);\\
\zeta_1(t) &= \int_0^t \alpha_1(s,t)ds + \int_0^t \sigma_1(s,t)dW_1(s);\\
\alpha_1(s,u) &= \sigma_1(s,u)\int_s^u \sigma_1^2(s,v)dv;\\
\Psi(t,T) &= \sum_{i=2}^{p}\zeta_i(t)[\gamma_i(t,T)-\gamma_1(t,T)] = 0.
\end{aligned}$$

Then the zero-coupon price from the equation [2.34] becomes

$$\begin{aligned}
P(t,T) &= \frac{P(0,T)}{P(0,t)}\exp\left(-\Phi(t,T)-\Psi(t,T)-\gamma_1(t,T)[r(t)-f(0,t)]\right)\\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-\frac{1}{2}\gamma_1^2(t,T)\epsilon_1(t) - 0 - \gamma_1(t,T)[r(t)-f(0,t)]\right)\\
&= \frac{P(0,T)}{P(0,t)}\exp\left(-\frac{1}{2}\Big(\frac{1-e^{-a(T-t)}}{a}\Big)^2\sigma^2\left(\frac{1-e^{-2at)}}{2a}\right) + \frac{1-e^{-a(T-t)}}{a}[f(0,t)-r(t)]\right)\\
&= \exp\Bigg\{\underbrace{\log\frac{p(0,T)}{p(0,t)} + f(0,t)\frac{1-e^{-a(T-t)}}{a} - \frac{\sigma^2}{4a}\Big(\frac{1-e^{-a(T-t)}}{a}\Big)^2(1-e^{-2at})}_{A(t,T)}\\
&\qquad\qquad - \underbrace{\Big(\frac{1-e^{-a(T-t)}}{a}\Big)}_{B(t,T)}r(t)\Bigg\}
\end{aligned}$$

meaning that the ZC bond price in this case is the same as the one from the 1-factor Hull & White in equation [2.6].

.6 Derivation of the 2-factor HJM model components : $\boldsymbol{X_1(t)}$ and $\boldsymbol{X_2(t)}$

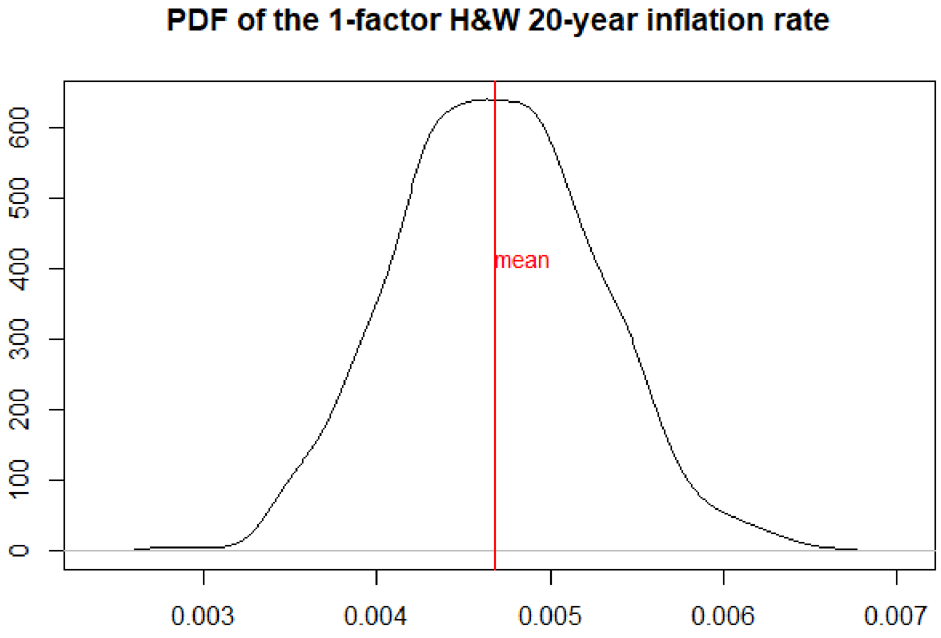
From the equation [2.42] and for $i = 1$ we get

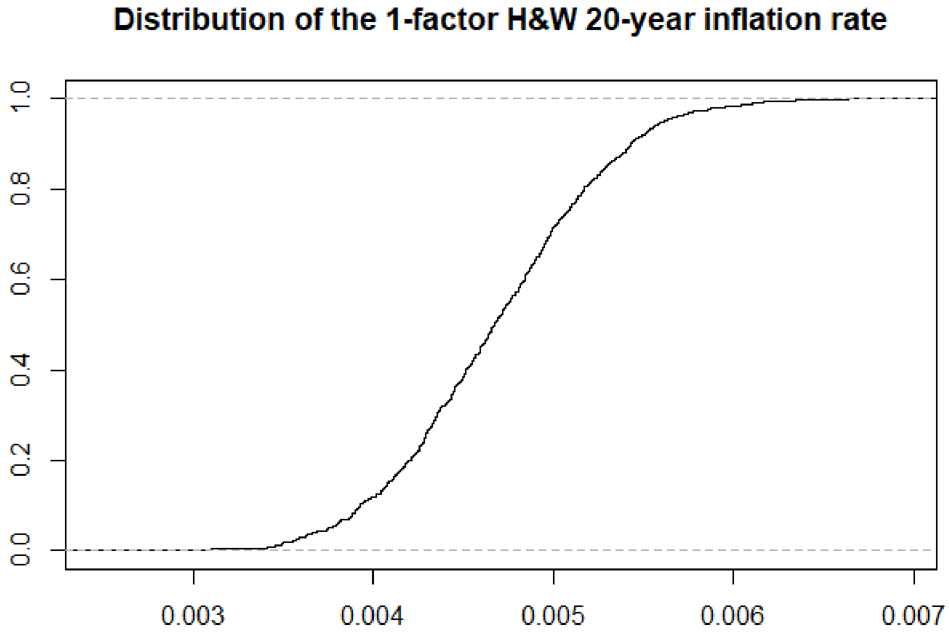
$$
\begin{aligned}
X_1(t) &= X_1(s)e^{-a_1(t-s)} + \int_s^t Z_{1,1}(s,u)e^{-a_1(t-u)}du \\
&+ \int_s^t Z_{1,2}(s,u)e^{-a_1(t-u)}du + \int_s^t \sigma_1 e^{-a_1(t-u)}d\tilde{W}_1^Q(u) \\
&= X_1(s)e^{-a_1(t-s)} + \int_s^t \frac{\sigma_1^2(1-e^{-2a_1(u-s)})}{2a_1}e^{-a_1(t-u)}du \\
&+ \int_s^t \frac{\rho_{1,2}\sigma_1\sigma_2}{a_1+a_2}\left(1-e^{-(a_1+a_2)(u-s)}\right)e^{-a_1(t-u)}du + \int_s^t \sigma_1 e^{-a_1(t-u)}d\tilde{W}_1^Q(u) \\
&= X_1(s)e^{-a_1(t-s)} + \frac{\sigma_1^2(1-e^{-a_1(t-s)})^2}{2a_1^2} + \frac{\rho_{1,2}\sigma_1\sigma_2}{(a_1+a_2)a_1a_2} \\
&\Big(a_2 - (a_1+a_2)e^{-a_1(t-s)} + a_1e^{-(a_1+a_2)(t-s)}\Big) + \int_s^t \sigma_1 e^{-a_1(t-u)}d\tilde{W}_1^Q(u)
\end{aligned}
\tag{15}
$$

then, using the same logic, we get for $i = 2$

$$
\begin{aligned}
X_2(t) &= X_2(s)e^{-a_2(t-s)} + \int_s^t Z_{2,1}(s,u)e^{-a_2(t-u)}du \\
&+ \int_s^t Z_{2,2}(s,u)e^{-a_2(t-u)}du + \int_s^t \sigma_1 e^{-a_2(t-u)}d\tilde{W}_2^Q(u) \\
&= X_2(s)e^{-a_2(t-s)} + \frac{\rho_{1,2}\sigma_1 1\sigma_2}{(a_1+a_2)a_1a_2}\Big(a_1 - (a_1+a_2)e^{-a_2(t-s)} \\
&+ a_2e^{-(a_1+a_2)(t-s)}\Big) + \frac{\sigma_2^2(1-e^{-a_2(t-s)})^2}{2a_2^2} + \int_s^t \sigma_2 e^{-a_2(t-u)}d\tilde{W}_2^Q(u)
\end{aligned}
\tag{16}
$$

.7 The 20-year inflation rate $i_{20}(t)$ distributions





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