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EXCESS OF LOSS REINSURANCE : REINSURER PRICING AND RELATED PITFALLS

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par:

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Preface.

This dissertation has been prepared in partial fulfillment of the requirements for the master degree in actuarial sciences delivered by the Université Catholique de Louvain (UC Louvain).

Key-words *excess-of-loss reinsurance, pricing, interest rates, inflation, claims payment pattern, reinsurer capital*

There are three basic excess-of-loss (XL) reinsurance pricing techniques to determine the expected losses and hence the premium rates : by experience rating, or the use of historical data, by exposure rating, or a procedure used to calculate the reinsurance exposures together with industry data, and finally by frequency/severity rating, or the development of a stochastic model of the covered risk. During the last decades, both the development of personal computers and the improvement of computing calculation speed have encouraged the development of the latter technique. Few papers deal with the practical aspects of pricing XL treaties thanks to a frequency/severity method. Related actuarial literature are published by Walhin et al., dealing with a cash-flow based model (Walhin et al., 2001).

Among the set of variables considered in the pricing techniques, it is clear that inflation and nominal interest rates play a central role. The papers on the subject do not agree on the impacts of macroeconomic rates volatility on XL cover pricing. However they agree that the cedant claims payment pattern and reinsurer capital allocation in a prudential context such as Solvency II are also important factors, despite a scarcity of literature on the matters, and particularly on their impacts on pricing.

In this study, we develop a 'building block' pricing tool coded in R able of capturing the effects of the aforementioned key pricing factors. We propose methods for calibrating and modelling each of these factors within each block of the tool. Each block is presented and thoroughly analyzed. Finally we provide several illustrative examples of the tool and then challenge the positions of papers discussed in literature review.

Acknowledgment

It has already been four years since I started my studies in actuarial sciences. Working and studying at the same time has not always been an easy journey, but today I have no regrets about the choice I made, and I am proud and glad to work as an Actuary. This Master thesis concludes these academic studies but probably not the end of my research in this fascinating field. Words cannot express my gratitude to the professors I met for her invaluable knowledge sharing. Special thanks to my supervisor Jean-François Walhin, Professor at UC Louvain, for taking the time to give me numerous feedbacks but also sharing his expertise. I am also grateful to my previous employer Mazars Belgium, for giving me the chance to follow such studies, and to my current employer Federale Assurance, for giving me the opportunity to mentor me on this Master thesis. Lastly, I would be remiss in not mentioning my relatives, especially my parents and my partner. Their belief in me has kept my spirits and motivation high throughout the Master.

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Chapter 1

Introduction

During the past two years, there has been a disruption in the reinsurance market. Reinsurance conditions, which used to keep improving because of high reinsurers capacity and both low interest and inflation rates environment, have started hardening again. On one hand, pandemics, wars, global warming and natural catastrophes (Cat Nat) had macroeconomic consequences which are used as arguments by reinsurers to negotiate again their terms and conditions. On the other hand, more and more strict regulatory frameworks, such as Solvency II in Eurozone, require reinsurers to carefully think about their capital needs and allocation through years when pricing their contracts, which influences their prices upwards.

Interest and inflation rates may be significant risk factors for a reinsurer writing 'long-tail' lines of business. Excess-of-loss reinsurance treaties are the main contracts active in such lines of business. Our study aims to produce a comprehensive cash flow-based simulation model to price excess-of-loss reinsurance contracts. The model is based on a building block approach, and aims at capturing changes in key factors for such pricing : risk-free interest and inflation rates, claims payment pattern and reinsurer capital allocation method.

The dissertation is structured as follows:

- First part reminds some backgrounds about reinsurance, known as a quite specific and complex field in actuarial sciences. The scope of the study is defined and the objectives of the Thesis are set.
- Second part deals with literature review on excess-of-loss cover pricing, including the impact of related clauses, but recaps the research works about the related importance of exogenous factors, cedant claims payment patterns, reinsurer capital allocation on such pricing.
- Third part develops the methodology and data used for building the model. Each 'block' constituting the model is presented and thouroughly analyzed. Final section of this part deals with the final pricing model, and we use it to challenge positions of some papers discussed in literature review.
- Fourth and final part concludes about the achievements of the Thesis, the findings from the model, and details possible improvements to the latter and associated new study opportunities.

Chapter 2

Background

2.1 Reinsurance : definition and benefits

2.1.1 Definition

Simply stated, reinsurance can be defined as the insurance of insurers. It is a specific insurance sector which is not widely known. The main reason why policyholders ignore the reinsurers is simply that they have no contractual agreements with them. The insurer is indeed liable to the policyholder : it has to pay if case of insured claims. Then, if a reinsurance contract allows it, the insurer will call on the cover the contractually agreed portion of the claims paid. Therefore the reinsurer is not known to the policeholders for having partially taken over this compensation.

The Belgian Law with respect to reinsurance activities (2009), amended by the Belgian Law of 2016 (Solvency II Law, 2016) defines reinsurance as the transfer of all or part of a risk underwritten by an insurer to another insurance undertaking or reinsurance undertaking, without the original insurer losing its capacity as an insurance debtor to the insured. The insurance company that purchases reinsurance is called the cedant or the ceding company. It cedes part of its business to the reinsurer, which accepts it.

For risk management reasons, the reinsurer may also be required to cede part of its risks to other reinsurers via so-called retrocession. The reinsurer which buys retrocession is then called the retroceding company, while the reinsurer that accepts the business is called the retrocessionaire. Note an alternative risk transfer may also be the cession of the part of its risks on the financial markets thanks to the mechanism of securitisation.

2.1.2 Benefits

The European Insurance Directive (2013) states reinsurance as an essential financial activity, as it enables direct insurers, by facilitating a wider spread of risk at the level of the insurer, to increase their underwriting and coverage capacity while reducing their cost-of-capital (EU-LEX, 2013). In addition, it plays a fundamental role in financial stability, since, as major financial intermediaries and institutional investors, reinsurers make a decisive contribution to the financial solvency and stability of the direct insurance markets and the financial system in general.

According to Swiss Re, the main benefits of reinsurance may be classified into two categories (Swiss Re, 2010):

- Benefits of reinsurance for customers
  - Reducing claims volatility : by smoothing the impact of unexpected large losses, for which the well-known law of large numbers does not properly work, reinsurers helps stabilising insurer results and reducing its balance sheet volatility. Such decrease in volatility makes the company more attractive for investors.
  - Providing capital relief : it can be easily proven that reinsurance is a viable alternative to economic capital for insurance undertakings.
  - Being a source of knowledge and information : on top of the aforementioned points, reinsurers advice their clients in pricing and managing risk, and may support them in the development of new products and expansion of their business in new geographical area.
- Benefits of reinsurance for society
  - Transferring risk enables economic growth : reinsurers play a global role in risk diversification, support major projects and help reducing the protection gap.
  - Setting incentives for risk-adequate behaviour : reinsurer are pioneers in promoting risks prevention. They also massively invest in research on emerging risks such as climate change.

2.2 Types of Reinsurance

Reinsurance agreement may be split into two categories : facultative or obligatory. Either facultative or obligatory reinsurance may be proportional or non proportional. In the following sections we remind these various forms of reinsurance. Several illustrative examples of these reinsurance types may be found in the book written by Walhin (Walhin, 2012).

2.2.1 Facultative versus Obligatory

Facultative Falcultative reinsurance are risk per risk reinsurance since it applies for large, complex and single risks, such as a fire policy on a large corporate office building. It gives an opportunity for the reinsurer to analyse in detail the offered risks, and therefore administrative costs are heavy. Then the reinsurer has the faculty to accept or refuse them. Likewise, the insurer is free to choose which risks it wants to reinsure.

Obligatory Obligatory reinsurance covers entire portfolios. In such an agreement named treaty, both parties are obliged to cede or to accept any risk in the scope of the treaty. Such form of reinsurance applies to whole portfolios of risks that are similar, such as motor insurance portfolios.

Note that some hybrid forms of reinsurance may theoretically exist : one speaks about fac-ob, in which the insurer has the faculty to accept to cede its risk while the reinsurer is obliged to accept any risk, and ob-fac, the opposite situation. However these forms are rarely used in practice. Table 2.2.1 summarises the different forms of reinsurance.

	Reinsurer (accepts)	
Insurer (cedes)	Facultative	Obligatory
Facultative	Fac	Fac-ob
Obligatory	Ob-Fac	Treaty

Table 2.2.1: Different forms of reinsurance

2.2.2 Proportional reinsurance

Under proportional reinsurance, the insurer and the reinsurer share premiums and losses by a ratio, a cession rate denoted α, defined in their agreement. Therefore a predetermined part of each and every risk is transferred to the reinsurer: α of the premium is ceded to the reinsurer, and the reinsurer pays off α of the loss if any.

There are two sorts of proportional reinsurance : Quota-Share (QS) and Surplus. The cession rate α does not vary across the QS reinsured portfolio whereas it varies across the surplus reinsured portfolio, depending on the insured sum of the considered risk.

2.2.3 Non-Proportional reinsurance

Non-proportional reinsurance defines the liability of the reinsurer as the excess above a given deductible with a maximal offered capacity. In other words, the ceding company and the reinsurer agree on the reinsured liability (aggregate) claim by (aggregate) claim. Contrary to proportional reinsurance, the reinsurance premium does not depend on the cession rate calculated for each risk. Reinsurance can be defined as an eXcess-Of-Loss (XL) reinsurance or a Stop-Loss (SL) reinsurance.

XL reinsurance XL Reinsurance is closely related to the notion of claim. It can be either per risk - portfolio of cars, houses, buildings...- or per event - all the claims caused by the same event, like a windstorm, hail...- are aggregated and then the priority and the limit are applied).

SL reinsurance SL Reinsurance is per aggregate claims over a given period (typically one year). In a stop loss reinsurance the reinsurer is liable for the (yearly) aggregate claims exceeding a threshold (priority) with a maximal liability (the limit). Often the priority and limit are expressed as a percentage of the premium.

2.3 Scope of the study and related challenges

2.3.1 Scope

Insurance portfolio Let consider an insurance portfolio during one year. In this study, we model large losses. Let N be a random variable representing the number of claims or claims frequency, with a discrete probability function (pf) $P[N = n] = P(n)$. Let X_i be the size or the severity of the i^{th} of these claims, for $i = 1, ..., N$, with a probability density function (pdf) $P[X_i = x_i] = p(x_i)$.

It is known that XL reinsurance covers the part of each claims exceeding a so-called priority (or deductible) P with a maximum limit (or liability) L :

$$Z_i = min(L, max(0, X_i - P)). \tag{2.3.1}$$

One says we have an XL reinsurance for the layer L-P xs P. Let now S denotes the reinsured aggregate claims :

$$S = \sum_{i=1}^{N} Z_i. \tag{2.3.2}$$

Most of the time, for modelling large losses, N is Poisson distributed while X_i is Pareto distributed. We will consider these distributions in this study.

Reinsurance portfolio In our study, we consider a reinsurance portfolio of M yearly-based XL treaties. Denote $S^{(m)}$ the reinsured aggregate claims of the m^{th} treaty ($m = 1, ..., M$). In total the reinsurer covers :

$$S_M = \sum_{m=1}^{M} S^{(m)} \tag{2.3.3}$$

Dependence structure We will model dependence structures due to exogenous factors (e.g. macroeconomic factors) for the M yearly-based XL treaties. Both a claim number and severity dependence structure will be considered, if explicitly stated. All insurance portfolio claims frequency and severity will keep being considered independent from each others for all treaties as we keep working within the collective risk model (see Klugman et al., 2012, for more details). These structures are modeled thanks to copulas. A copula is a mathematical function used in statistics to describe the dependency between two or more random variables. It is a function that links marginal distributions to a joint distribution, and it does so by capturing the dependence structure among the variables. Copulas can be used to model a wide range of dependencies that may not be captured by traditional correlation measures. The concept of copula is based on the Sklar's theorem, which states that any joint distribution can be decomposed into a copula function and the corresponding marginal distributions. The copula function is typically defined on the unit hypercube, which allows for easy transformation to any other range of values. The most commonly used copulas include the Gaussian copula, and the t-copula, and these are the two copulas we will work with. More details about copulas may be found in McNeil et al. (McNeil et al., 2015). We will proceed as follows :

- Claims number dependence :
  1. Specification of the 'frequency' copula. Generation of a uniform $M \times N_{sim}$ marginals matrix, M being the size of the portfolio and N_{sim} the number of Monte-Carlo simulations.
  2. Application of the quantile transform theorem to all generated uniform marginals : set a $M \times N_{sim}$ matrix of Poisson-distributed random variables, the simulated number of claims of the whole reinsurance portfolio.
- Claims severity dependence :

1. Specification of the 'severity' copula. Generation of a $M \times N_{sim}$ uniform marginals matrix, M being the size of the portfolio and N_{sim} the number of Monte-Carlo simulations.
2. Application of the quantile transform theorem to all generated uniform marginals : set a 3-dimensional $M \times N_{sim} \times max(n)$ matrix of Poisson-distributed random variables, which are the simulated severity of claims of the whole reinsurance portfolio, but for all 'layers' of claims numbers. This assumes no change in the dependence structure between severities whatever the number of claims is.

Lines of Business This study will focus on XL reinsurance treaties for long-tail Lines of Business, typically a motor third party liability. A so-called Line of Business (LoB) may be defined as a class of insurance activity. The Solvency II Directive (2015) imposes that insurance and reinsurance undertakings to segment their insurance and reinsurance obligations into homogeneous risk groups, and as a minimum by lines of business. EIOPA specifies that the assignment of an insurance or reinsurance obligation to a line of business shall reflect the nature of the risks relating to the obligation. Annexes I and II of the Directive respectively provides the list of non-life and life LoB (EU-LEX, 2015).

Reinsurance cover can be purchased for various lines of business, including life and non-life. Non-life reinsurance provides financial protection against natural catastrophes, accidents, theft, and fire. This type of reinsurance includes property, casualty, credit and bond, personal accident, travel, and health. Property reinsurance covers damages to assets, while casualty reinsurance protects against claims arising from liabilities. Credit reinsurance protects against the risk of insolvency of customers, personal accident reinsurance covers claims from bodily injuries and accidental death, and health reinsurance covers expenses related to provided health care. Life reinsurance provides financial protection in case of death.

Lines of business are divided into short-tail and long-tail reinsurance. Short-tail business refers to claims which are paid during the term of the policy or shortly after its expiration, such as property insurance. Long-tail business refers to claims that can take years to settle, such as motor third party liability (MTPL), general third party liability, or personal accident. Long-tail liabilities result in high incurred but not reported (IBNR) claims, which are claims that have occured but have not yet been reported to the reinsurer.

2.3.2 Challenges

2020s context During the last 15 years, reinsurance price never stopped decreasing. This trend was mainly driven by high capacity in the reinsurance market fostering competitive prices. 2020 and 2021 were years of COVID-19 pandemics coupled with several major 'Cat Nat' events. The 2020 AON Climate and Catastrophe Insight Annual Report provides an analysis of these global natural disaster events and their economic impacts (AON, 2020). The report notes that 2020 was one of the three warmest years on record and the sixth consecutive year with above-average temperatures, resulting in a record number of climate-related disasters. The report highlights that there were 416 natural disaster events in 2020, resulting in 268 billion in economic losses, of which 97 billion was insured. The report focuses on a number of major events in 2020, including wildfires in the United States and Australia, hurricanes in the Atlantic Basin, and flooding in China. It notes that the Atlantic hurricane season was particularly active in 2020, with a record-breaking 30 named storms. AON highlights the increasing frequency and severity of natural disaster events, and the urgent need for action to address climate risk and promote resilience. This unprecedented recognition of compounded extremes, or how concurrent events can lead to major global large losses, logically leads us to believe that non-proportional reinsurance treaties, and specifically XL treaties, already have and will have a key role to play in the coming years.

Current Challenges 2023 Reinsurance Renewal in Europe proved to be one of the most challenging ever experienced at January 1. This is mainly due to the current post-COVID-19 economical context (high inflation and quite low interest rates). Both reinsurers and cedants worked hard to establish a new market equilibrium, leading to severe price increase, and drastic changes of cedants reinsurance program structures imposed by market conditions - increase in retention level driven by reinsurers requirement to have attachment points above a minimum level in terms of return periods, no more free/prepaid reinstatements and reduction of amount of reinstatement, no more flat rates and higher limits to account for inflation...

2.4 Objectives of the study

Research Target Our study aims to produce a comprehensive cash flow-based simulation model that can be easily run in R, in order to price excess-of-loss reinsurance contracts. In other words, we aim to develop a tool that could be either used by a reinsurer to give him an initial range of rates for such treaties, or by the ceding insurance company or the broker to challenge the reinsurer quoted price.

Research questions A volatile, high inflation and low interest rate environment, like the one we are currently experiencing, is often used as an argument by reinsurers to rise their XL cover quotes. Research questions are then : is the price asked by reinsurer always justified ? What is the price sensitivity to the marketable clauses in such environment ?

Chapter 3

Literature Review

3.1 Pricing XL reinsurance treaties

There are several basic reinsurance pricing techniques. Flower et al. defines the three most important techniques used in the market : by experience rating, by exposure rating and by frequency/severity rating (Flower et al., 2006). Experience rating is a methodology using contract-specific historical loss and exposure data to select expected losses and then deduces the required premium rates for a reinsurance contract. An example of experience rating is the burning cost method. Exposure rating is a practice that uses the actual reinsurance exposures combined with industry data to select expected losses and required premium rates. A example of exposure rating is the use of Increased Limit Factors curves. Frequency/severity rating is a method of developing a stochastic model to simulate the potential loss experience based on estimates of the number and size of reinsured losses together with the reinsurance treaty terms, which gives a distribution for the recoveries and leads to a price. The key assumptions might be based on the contract experience, on more general industry data, or a combination of both. Therefore this latter method may be seen as a mix of experience and exposure rating.

Walhin et al. discussed the practical aspects of pricing XL treaties thanks using a frequency/severity method (Walhin et al., 2001). A comprehensive methodology was developed for pricing excess-of-loss treaties with or without clauses included. This methodology was derived under the framework of collective risk model although it could have been extended within the individual risk model. The methodology consists of a cash flow-based model. Limitations of the article are dual: the limit of computer processing speeds at that time led to the use of Panjer algorithm instead of the use of Monte-Carlo simulations, and, as we will discuss later, some key pricing parameters were arbitrary fixed for illustrative examples. The authors showed that a cash flow model based on frequency/severity method should be primary used for pricing reinsurance cover in long-tail business.

3.2 About the importance of certain clauses typically encountered in XL cover and impacting reinsurer profitability

Index Clauses Index or stability clauses[1] in liability reinsurance contracts, such as MTPL reinsurance contracts, limit the inflation risk by increasing both the retention and limit by the amount of inflation over the period until all recoveries are settled (Flower et al., 2006). These clauses usually index the retention and limit with a named inflation index, such as Belgian Consumer Price Index (CPI), to the time of each payment of each claim. There are three types of index clauses used in practice: full indexation, severe inflation clause (SIC), and franchise inflation clause (FIC). Full indexation transfers all inflation risk to the insured, while SIC and FIC provide mechanisms for sharing the inflation risk between the insurer and reinsurer based on defined percentages of increase in the reference index. Kladivko and Zimmermann developed a stochastic model for valuing an XL reinsurance treaty with an index clause, covering insurance products that ensure regular inflation-adjusted payments until the victim's death or a specified age (Kladivko, K. and Zimmermann, P., 2014). These products are commonly found in LoB MTPL or worker's compensation. They found that, on reasonably large accidents, the impact of this clause on the reinsurer's share may be very substantial and is generally quite robust. This confirms it is important to include an index clause in actuarial XL reinsurance pricing model in order to provide sound support for managerial decisions (Zimmermann, 2012).

Reinstatements Reinstatement premiums are charged for reinstating the cover after a reinsurance loss payment, and are expressed as a percentage of the basic ceded premium. These premiums can be highly significant to the technical rate, reducing both the net expected reinsured loss cost and the volatility in this net loss cost (Flower et al., 2006). A contract may have different percentages for different reinstatements, with each reinstatement allowing for an additional limit of coverage. Sundt is considered to be a pioneer in pricing excess-of-loss reinsurance with reinstatements (Sundt, 1993). In his paper, he provided a practical calculation of pure premiums for XL reinsurance with reinstatements based on best European actuarial practices, i.e. the pure premium and standard deviation principles.

Sliding scales Sliding scales refer to a retrospective premium calculation method used in reinsurance contracts where the premium is determined based on the current claims experience or loss ratio. Levi proved that financial income plays a big role for the calculation of the minimum premium and maximum premium of a sliding scale for non-proportional treaties (Levi, 1988). Walhin performed a review of Charles Levi's work thanks to the cash-flow based model developed two years before (Walhin et al., 2001), and agreed with Levi's conclusions (Walhin, 2003). He stated that ignoring the fact that the reinsurance premium is not fully paid at the beginning of the contract may lead to poor underwriting decisions. Besides, Walhin et al. showed that XL premium rates are very sensitive to the contractually agreed first year of premium adjustment of the sliding scale clause (Walhin et al., 2001). Campana and Ferretti argued the retrospective premium can be considered as a proper method for compensating the insured for loss experience that was not considered in the original premium, and developed a method for pricing such premium (Campana and Ferretti, 2022). They analytically showed that the expected retrospective premium should be equal to the expected value of the loss exceeding the premium, while the variance of the retrospective premium depends on the variance of the loss exceeding the premium and the insured's risk aversion.

[1] Clause de stabilité (FR) / Stabiliteitsclausule (NL)

3.3 About the importance of macroeconomic factors on XL reinsurance pricing

It is well known that macroeconomic factors have a significant impact on non-proportional reinsurance prices.

As pointed out by Ball and Staudt, amongst these factors, inflation, is the key driver of XL reinsurance price (Ball and Staudt, 2001). According to Statbel, inflation is defined as the quantitative measure of the rate between the value of the consumer price index for a given month and the index for that same month in the previous year (Statbel, 2022). In other words, inflation measures the speed at which the general price level is changing over years. Some scientists agree that changes in XL reinsurance technical premiums are related to inflation (Kladivko, K. and Zimmermann, P., 2014). The price of non-proportional reinsurance coverage is indeed estimated based on the discounted future cash flow of the reinsurer, and the effect of inflation is reflected in the amount of money that the reinsurer will pay in the future. With the inflation, this amount will be higher in future years and therefore will generate a higher reinsurance premium rate. According to Fackler, inflation is a critical concern for the reinsurer, especially in long-tail businesses such as in third-party liability (Fackler, 2011).

It is well known in macroeconomics that interest rates are correlated to inflation (Blanchard and Sheen, 2013). In period of high inflation, interest rates tend to fall, leading then to deflation period. This is why not only inflation but also deflation should also be carefully considered by reinsurers. Low interest rates have indeed a significant impact on long-tail reinsurance businesses, affecting both reserving financing due to unrealised profits and consequently underwriting performance.

Current best actuarial practice is to index and develop past claims to estimate future losses, and then adjust for present value using a flat deterministic inflation rate. It would then make sense to replace the deterministic inflation by a stochastic model, in order to capture extreme scenarios i.e. scenarios that rarely happen, but once it happens, many reinsurers may face underwriting issues. According to Flower et al., the impacts on volatility can be very significant so there can be direct implications for the size of risk load. However some actuaries do not fully agree with this statement (Flower et al., 2006). Thanks to their stochastic model, Kladivko and Zimmermann found that the difference in obtained technical premiums between stochastic and deterministic approach negligible and concluded a deterministic approach is an acceptable alternative for valuation (Kladivko, K. and Zimmermann, P., 2014). Similarly, Walhin et al. concluded XL rates are not much sensitive to similar changes on both interest and inflation rates (Walhin et al., 2001).

3.4 About the importance of the cedant claims payment pattern

For a reinsurer covering long tail risks, there is a long delay between the date of the accident causing the loss and the dates of payments by the insurance company. Therefore the modelling of the speed of payment is very important. This payment delay, which can be of several decades, has a huge impact on the price of the claim, notably due to inflation, as discussed in the previous section. A reinsurer that misjudges the cedant future payments makes a mistake in its price and puts its market competitiveness at risk. Underestimating the price has a negative impact on the results, and trigger reinsurer underwriting risk, while overestimating the price makes the reinsurer more expensive than the competition (Cummins et al., 2021).

Despite being an important topic in XL reinsurance, there is, to the knowledge of the author, very little literature about practical cedant claims payment pattern modelling. This seems to be mainly due to the absence of sufficient data, especially in high excess layers, needed for calibrating such model (Flower et al., 2006). It was only very recently that Sriram et al. introduced a Dirichlet model for forecasting outstanding liabilities of non-life insurers (Sriram and Shi, 2019). The Dirichlet model is also good at modeling claims payment patterns because it provides a natural way to model the random clustering of payments, which is a common characteristic in (re)insurance data. Moreover, the Dirichlet distribution has several excellent mathematical properties and many advantages in terms of practical implementation : we will come back to these points in section 4.4. Further details may be also found in Huang and Van Der Merwe et al. (Huang, 2005 and Van Der Merwe and De Waal, 2005).

3.5 Reinsurer capital allocation

In European Union, strong prudential regulations such as Solvency II are in place to ensure reinsurers to have a sufficient solvency margin to mitigate systemic risk (EU-LEX, 2015). Despite being a key topic in reinsurance pricing, there is, to the knowledge of the author, little literature about reinsurer capital allocation modelling. To date, most papers focus on the impact of reinsurance on the ceding company, dealing with optimal reinsurance from the cedant point of view. The reinsurer capital allocation was nonetheless raised by Walhin et al. (Walhin et al., 2001). The paper discussed the importance of capital allocation to excess-of-loss treaties and highlights that an appropriate capital allocation may help in developing an appropriate pricing strategy. The paper suggested various methods of allocating capital such as the allocation of capital proportional to the expected loss, or to the risk premium, or even equal to the Tail-Value-at-Risk. The paper concluded that the allocation of capital is a critical element in the pricing of XL treaties. This is because capital allocation is obviously a way to manage risk and ensure that the insurer has sufficient funds to meet its obligations. The authors emphasize that capital allocation should be viewed as an integral part of the pricing process, and that reinsurers should consider a variety of factors, such as the risk profile of the treaty and the insurer's overall risk appetite, when allocating capital. In their reinsurance pricing model, Schirmacher et al. used a capital approach with a 99% TVaR level of risk and also emphasized the importance of a prudent but efficient capital approach in XL pricing (Schirmacher et al., 2005).

Scientists also agree that reinsurers should seriously consider macroeconomic factors such as inflation when determining their capital buffer. According to De Ravin, failure to properly consider inflation in capital needs can result in mispricing and potentially lead to reinsurer insolvency (De Ravin and Fowlds, 2010). Additionally, an increase in inflation can deplete reserve buffers, causing financial issues for insurers and reinsurers, especially those with insufficient reserving.

Chapter 4

Methodology and Data

4.1 Overall Strategy

In order to answer the research questions from Section 2.4, and considering the discussions from previous chapter 3, the research strategy is based on a so-called building block approach. In other words, the original pricer developed by Walhin et al. (2001) is considered as a starting point, denoted Block 1. Three intermediary blocks are then added to this first block : a short-term risk-free and inflation rate model (Block 2), a payment pattern model (Block 3) and a capital allocation model (Block 4), as we have seen in chapter 3 that these parameters are of high importance in term of pricing impact. Finally all these blocks are aggregated : this leads to the Building Block Pricing Model (Block 5). Figure 4.1.1 provides an overview of the Building Block Approach.

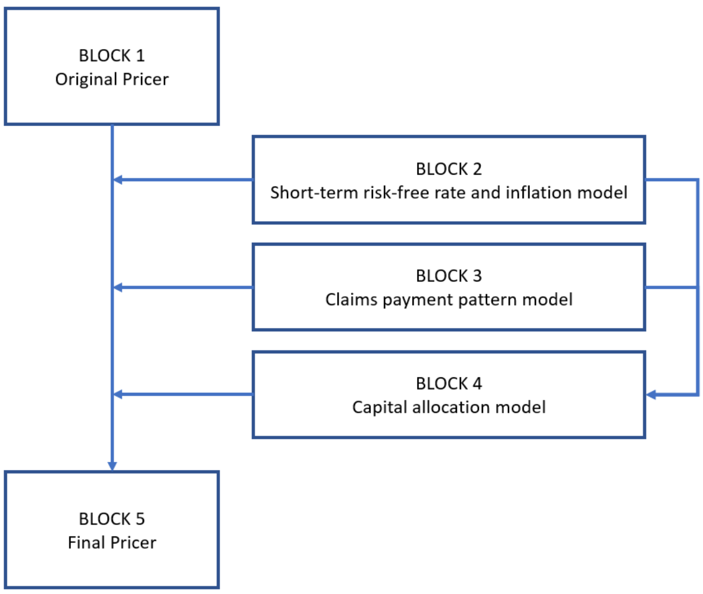

Figure 4.1.1: Building Block Approach

In the following sections, the different blocks are detailed one after the other. Followed methodologies and used data are explained. Some intermediate findings are also provided.

4.2 BLOCK 1 : Original pricer

4.2.1 Approach

The first step is to replicate Walhin et al. (2001) original pricing model into R code. For achieving this, the related article was thouroughly read and analysed, and the methodology described in this article was implemented into R.

4.2.2 Theory

Theoretical developments from Walhin et al. (2001) are taken up here below and adapted to our case study.

4.2.2.1 Claims development model

Inflation A model for the influence of inflation and superimposed inflation on the distribution of paid losses is considered. Assume the random variables X and N, defined in section 2.3, have already been estimated based on past data. A model is proposed for claims development assuming that payments occur at times $t_1, t_2, ..., t_n$ according to a given claims payment pattern $c(t_1), ..., c(t_n)$ where t_n is the largest time up to which the losses are completely paid. The claims payment pattern is estimated on past data and adjusted for potential changes in the future (see later, Block 3). The assumption is made that $t_0 = 0$, the origin of time in the model. The proposed model takes into account the fact that in a long-tail business, it is not realistic to assume that the loss is paid in one installment the same year as the premium inception. Future payments for a loss X_i will undergo future inflation, defined by the index $\inf(t_0), \ldots, \inf(t_n)$. Thus the future payments for a loss X_i write:

$$X_i\left(t_j\right) = c\left(t_j\right) X_i \frac{\inf\left(t_j\right)}{\inf\left(t_0\right)}, \quad j = 1, \ldots, n. \tag{4.2.1}$$

High losses undergo higher inflation than classical inflation, known as superimposed inflation. Thus $\text{supinf}\left(t_0\right), \ldots, \text{supinf}\left(t_n\right)$ is the more adequate index to use for future payments. The future payments for a loss X_i then write:

$$X_i\left(t_j\right) = c\left(t_j\right) X_i \frac{\text{supinf}\left(t_j\right)}{\text{supinf}\left(t_0\right)}, \quad j = 1, \ldots, n. \tag{4.2.2}$$

The cumulative paid losses may be written :

$$X_i^{\Sigma}\left(t_j\right) = \sum_{k=1}^{j} X_i\left(t_k\right), \quad j = 1, \ldots, n. \tag{4.2.3}$$

The evolution of the cumulative paid loss for the reinsurer then writes:

$$X_i^{\Sigma Re}\left(t_j\right) = \min\left(L, \max\left(0, X_i^{\Sigma}\left(t_j\right) - P\right)\right), \quad j = 1, \ldots, n. \tag{4.2.4}$$

In an ideal situation the cedant claims manager is able to calculate exact reserves for each loss:

$$R_i\left(t_j\right) = X_i^{\Sigma}\left(t_n\right) - X_i^{\Sigma}\left(t_j\right), \quad j = 1, \ldots, n. \tag{4.2.5}$$

In reality, systematic deviations from exact reserves are possible, and a pattern of deviation of reservation (overstatement or understatement) is assumed to be observed: $d(t_1), \ldots, d(t_n)$, where $d(t_j) = 100\%$ if there is no deviation of reservation at time t_j. The incurred loss, i.e. loss paid plus outstanding, may now be written:

$$I_i(t_j) = X_i^{\Sigma}(t_j) + d(t_j) R_i(t_j), \quad j = 1, \ldots n. \tag{4.2.6}$$

From the evolution of this incurred loss, it is now possible to derive the evolution of the incurred loss for the excess-of-loss reinsurer:

$$I_i^{Re}(t_j) = \min(L, \max(0, I_i(t_j) - P)), \quad j = 1, \ldots, n. \tag{4.2.7}$$

Note reinsurer could (and often does) assess itself the pattern of deviation of reservation $d(t_j)$ for $j = 1, \ldots, n$.

4.2.2.2 Index clause model

In case of fixed priority and limit, the reinsurer would bear all future inflation during the claims development between the date of accident and the date of the final settlement. Therefore this clause is commonly - even always - requested by the reinsurer to share future inflation with the cedant. Although there are various types of stability clause, the most-used clause is the "date of payment" one, based on the principle that both priority and limit are indexed each future development year with a ratio of the sum of actual payments over the sum of inflation index adjusted payments:

$$\text{ratio} = \frac{\text{sum of actual payments}}{\text{sum of adjusted payments}} \tag{4.2.8}$$

In other words, at each t_j the previously defined ratio writes, if reserves are ignored :

$$\text{ratio}(t_j) = \frac{\sum_{k=1}^{j} X_i(t_k)}{\sum_{k=1}^{j} X_i(t_k) \frac{inf(t_0)}{inf(t_k)}}, \quad j = 1, \ldots, n. \tag{4.2.9}$$

Often there is a margin, i.e. the payments will be adjusted only if inf shows an evolution larger than the margin. Let us assume that the margin is ω. With our notations we find

$$\begin{aligned} \nu(t_k) &= 1 \text{ if } \frac{inf(t_k)}{inf(t_0)} \leq 1 + \omega, \\ &= \frac{inf(t_0)}{inf(t_k)} \text{ if } \frac{inf(t_k)}{inf(t_0)} > 1 + \omega, \\ \Rightarrow \text{ratio}(t_j) &= \frac{\sum_{k=1}^{j} X_i(t_k)}{\sum_{k=1}^{j} X_i(t_k) \nu(t_k)}, \quad j = 1, \ldots, n. \end{aligned} \tag{4.2.10}$$

A severe inflation clause (SIC) works similarly to a margin, but only the excess of inflation exceeding the margin is considered :

$$
\begin{aligned}
\nu\left(t_k\right) &= 1 \text{ if } \frac{inf\left(t_k\right)}{inf\left(t_0\right)} \leq 1+\omega, \\
&= \frac{inf\left(t_0\right)(1+\omega)}{inf\left(t_k\right)} \text{ if } \frac{inf\left(t_k\right)}{inf\left(t_0\right)} > 1+\omega, \\
\Rightarrow \text{ratio}\left(t_j\right) &= \frac{\sum_{k=1}^{j} X_i\left(t_k\right)}{\sum_{k=1}^{j} X_i\left(t_k\right)\nu\left(t_k\right)}, \quad j=1,\ldots,n.
\end{aligned} \tag{4.2.11}
$$

In case of a margin, the influence of the index clause is reduced and the reduction is even more important when there is a SIC. This also means that the priority and the limit of the treaty will change in the future according to the clause. We will thus have future priorities and limits: $P\left(t_1\right), P\left(t_2\right), \ldots, P\left(t_n\right)$ and $L\left(t_1\right), L\left(t_2\right), \ldots, L\left(t_n\right)$ instead of singles P and L with:

$$
\begin{aligned}
P\left(t_j\right) &= \text{ratio}\left(t_j\right) P, \quad j=1,\ldots,n, \\
L\left(t_j\right) &= \text{ratio}\left(t_j\right) L, \quad j=1,\ldots,n.
\end{aligned} \tag{4.2.12}
$$

Based on contractual terms, or on common market practice, the stability clause may be applied based on incurred losses instead of on paid losses. In this case the correction term writes:

$$
\text{ratio}\left(t_j\right) = \frac{d\left(t_j\right) R_i\left(t_j\right) + \sum_{k=1}^{j} X_i\left(t_j\right)}{d\left(t_j\right) R_i\left(t_j\right)\nu_j + \sum_{k=1}^{j} X_i\left(t_k\right)\nu_k}, \quad j=1,\ldots,n. \tag{4.2.13}
$$

The evolution of the cumulative paid loss and of the incurred loss for the reinsurer finally writes :

$$
\begin{aligned}
X_i^{\Sigma Re}\left(t_j\right) &= \min\left(L\left(t_j\right), \max\left(0, X_i^{\Sigma}\left(t_j\right) - P\left(t_j\right)\right)\right), j=1,\ldots,n, \\
I_i^{Re}\left(t_j\right) &= \min\left(L\left(t_j\right), \max\left(0, I_i\left(t_j\right) - P\left(t_j\right)\right)\right), j=1,\ldots,n.
\end{aligned} \tag{4.2.14}
$$

4.2.2.3 Interests sharing clause model

In long claims development, legal interests are expected to be paid, and the interests sharing clause allows them to be shared proportionally between the cedant and the reinsurer based on their liability, if the principal is shared between both parties. A proportion δ of the incurred loss represents the interests, but it is difficult to estimate it accurately. With our notations, if $I_i\left(t_j\right)$ is the incurred loss, the part of the principal is $(1-\delta)I_i\left(t_j\right)$. The liability of the reinsurer in the principal is:

$$
\min\left(L\left(t_j\right), \max\left(0, (1-\delta)I_i\left(t_j\right) - P\left(t_j\right)\right) j=1,\ldots,n. \tag{4.2.15}
$$

The liability of the reinsurer in the legal interests is:

$$
\delta I_i\left(t_j\right) \frac{\min\left(L\left(t_j\right), \max\left(0, (1-\delta)I_i\left(t_j\right) - P\left(t_j\right)\right)\right.}{(1-\delta)I_i\left(t_j\right)}, j=1,\ldots,n. \tag{4.2.16}
$$

The incurred loss for the reinsurer becomes:

$$
I_i^{Re}\left(t_j\right) = \frac{1}{1-\delta}\min\left(L\left(t_j\right), \max\left(0, (1-\delta)I_i\left(t_j\right) - P\left(t_j\right)\right)\right), j=1,\ldots,n. \tag{4.2.17}
$$

Similarly we obtain the cumulative paid loss of the reinsurer:

$$X_i^{\Sigma Re}\left(t_j\right) = \frac{1}{1-\delta} \min\left(L\left(t_j\right), \max\left(0, (1-\delta) X_i^{\Sigma}\left(t_j\right) - P\left(t_j\right)\right)\right), \; j = 1, \ldots, n. \tag{4.2.18}$$

4.2.2.4 Annual aggregate liability of the reinsurer

Two clauses often assort an XL treaty : the Annual Aggregate Deductible (AAD) *Aad* and the Annual Aggregate Limit (AAL) *Aal*, which are respectively a deductible and limit on the annual aggregate liability S of the reinsurer. Define $S_{X^{\Sigma Re}}\left(t_j\right), j = 1, \ldots, n$ the aggregate cumulative payments at time t_j and $S_{I^{Re}}\left(t_j\right), j = 1, \ldots, n$ the aggregate incurred losses at time t_j :

$$\begin{aligned} S_{X^{\Sigma Re}}\left(t_j\right) &= \min\left(Aal, \max\left(0, \sum_{i=1}^{N} X_i^{\Sigma Re}\left(t_j\right) - Aad\right)\right), \; j = 1, \ldots, n, \\ S_{I^{Re}}\left(t_j\right) &= \min\left(Aal, \max\left(0, \sum_{i=1}^{N} I_i^{Re}\left(t_j\right) - Aad\right)\right), \; j = 1, \ldots, n. \end{aligned} \tag{4.2.19}$$

The incremental payments are then given by :

$$\begin{aligned} \mathrm{Paid}\left(t_1\right) &= S_{X^{\Sigma Re}}\left(t_1\right), \\ \mathrm{Paid}\left(t_j\right) &= S_{X^{\Sigma Re}}\left(t_j\right) - S_{X^{\Sigma Re}}\left(t_{j-1}\right), \quad j = 2, \ldots, n, \end{aligned} \tag{4.2.20}$$

and the loss reserves are :

$$\mathrm{Reserve}\left(t_j\right) = S_{I^{Re}}\left(t_j\right) - S_{X^{\Sigma Re}}\left(t_j\right), \quad j = 1, \ldots, n. \tag{4.2.21}$$

This situation is the one where the reinsurer follows the information given by the cedant. Another situation might be that the reinsurer books the ultimate loss in such a way that it avoids overstatement and / or understatement of the ceding company's reserves. In this case the loss reserves write:

$$\mathrm{Reserve}\left(t_j\right) = S_{X^{\Sigma Re}}\left(t_n\right) - S_{X^{\Sigma Re}}\left(t_j\right), \quad j = 1, \ldots, n. \tag{4.2.22}$$

We are now able to retrieve the average aggregate payments and average aggregate reserves for the reinsurer: $\mathbb{E}\,\mathrm{Paid}(j)$ and $\mathbb{E}\,\mathrm{Reserve}(j)$, $j = 1, \ldots, n$.

4.2.2.5 Paid reinstatements

The paid reinstatements clause, introduced in chapter 3, is based on the principle that an annual aggregate limit Aal can be expressed as a multiple of the limit L. In other words, if $Aal = (K+1)L$, there are K reinstatements and thus the capacity C = L-P of the L-P xs P treaty may be consumed K times. Each reinstatement may be free or paid. Note that all free reinstatements is equivalent to a pure annual aggregate limit. Paid reinstatements mean that the cedant has to be pay an additional premium to reinstate the layer. This reinstatement premium may be calculated either *prorata capita* of the amount consumed or *prorata temporis*, i.e. taking into account the time remaining until the next reinsurance renewal. Since the *prorata temporis* approach is not widely used, we will consider only *prorata capita* reinstatement premiums. This premium is random since it depends on the aggregate claims S, and is given by:

$$P_{reins,rand} = \frac{P_{reins,init}}{l} \sum_{k=1}^{K} c_k min(L, max(0, S - (k-1)L)) \tag{4.2.23}$$

with c_k the fraction of the initial premium $P_{reins,init}$ for the K^{th} reinstatements. The retained risk of the reinsurer is then :

$$S_{reins} = min(S, (K+1)L) - P_{reins,rand} \tag{4.2.24}$$

Within the model, the random reinstatement premium is calculated each year t_j and thus depends on $S_{X^{\Sigma Re}}(t_j)$, and not $S_{I^{Re}}(t_j)$. It is then given by:

$$P_{reins,rand}(t_j) = \frac{P_{reins,init}}{L} \sum_{k=1}^{K} c_k min(L, max(0, S_{X^{\Sigma Re}}(t_j) - (k-1)L)) \tag{4.2.25}$$

with c_k the fraction of the initial premium $P_{reins,init}$ for the K^{th} reinstatements. The cumulative retained risk of the reinsurer is observed each year t_j and is therefore given by:

$$S_{reins}(t_j) = min(max(S_{X^{\Sigma Re}}(t_j) - P, 0), L) - P_{reins,rand}(t_j) \tag{4.2.26}$$

Then the incremental reinsurer payments per year post-reinstatement, $Paid_{reins}(t_j)$, are :

$$\begin{aligned} Paid_{reins}(t_1) &= S_{reins}(t_1) \\ Paid_{reins}(t_j) &= S_{reins}(t_j) - S_{reins}(t_{j-1}) \ \ (j = 2, ...n) \end{aligned} \tag{4.2.27}$$

4.2.2.6 Sliding Scale Premium

Sliding scale premium clause, already introduced in chapter 3, is often requested by the cedant arguing the aggregate claims will not be significant or difficult to forecast. In such case, the reinsurance premium is :

$$P_{ss} = \begin{cases} P_{ss,min} \quad if \quad S \leq \frac{P_{ss,min}}{f} \\ fS \quad if \quad \frac{P_{ss,min}}{f} < S < \frac{P_{ss,max}}{f} \\ P_{ss,max} \quad if \quad S \geq \frac{P_{ss,max}}{f} \end{cases} \tag{4.2.28}$$

with f a loading coefficient representing the reinsurer (admin and claims) expenses and cost-of-capital. The random part of this reinsurance premium is then : $P_{ss,rand} = P_{ss} - P_{ss,min}$. Thus the retained risk of the reinsurer is :

$$S_{ss} = S - P_{ss,rand} \tag{4.2.29}$$

Within the model, the cumulative random premium is calculated as follows :

$$P_{ss}(t_j) = \begin{cases} P_{ss,min} \quad if \quad S_{I^{Re}}(t_j) \leq \frac{P_{ss,min}}{f} \\ fS_{I^{Re}}(t_j) \quad if \quad \frac{P_{ss,min}}{f} < S_{I^{Re}}(t_j) < \frac{P_{ss,max}}{f} \\ P_{ss,max} \quad if \quad S_{I^{Re}}(t_j) \geq \frac{P_{ss,max}}{f} \end{cases} \tag{4.2.30}$$

Typically $f = \frac{100}{80}$ or $f = \frac{100}{70}$. The cumulative retained risk of the reinsurer at t_j is :

$$S_{ss}(t_j) = S_{I^{Re}}(t_j) - P_{ss,rand}(t_j) \tag{4.2.31}$$

Suppose pa_{start} the first year for which a premium adjustment is contractually agreed, the expected future incremental premium adjustments may be then deduced by decumulating the expectation of the above-defined cumulative random premium for $t_j > pa_{start}$.

4.2.2.7 A cash flow model

General principle Since the losses are not paid in one instalment, a cash flow model, dealing with future cash flows and capturing the concepts of solvency margin and cost-of-capital in the context of reinsurer business valuation, is now presented. The business is considered to be worth its value if the net present value of all future cash flows, including the capital allocation and its release, is at least positive. Thus this value creation satisfies the shareholders' requirements.

$$\sum_{j=0}^{n} \frac{CF_{total}\,(t_j)}{(1+coc)^{t_j}} > 0. \tag{4.2.32}$$

The inequality of a positive value signifies that the expected return from the business is greater than the cost-of-capital, leading to value creation for the shareholders. When a reinsurer wants to write some businesses, it has to provide a solvency margin, or some allocated capital C. Let us assume that the return the shareholders demand from this capital is coc, the cost-of-capital. Then the above equation means that a treaty is acceptable if the net present value of all future cash flows, including the variations in allocated capital, is positive.

Definition of the cash flows Assume that all cash flows arise at equally-spaced times, on annual basis. All cash flows relate to losses occurring at times $t_j = j - 0.5$, $j = 1, \dots n$, middle of each year, which is reasonable for uniformly spread yearly payments. There are three types of cash flows related to losses:

- paid losses: $PL(j-0.5),\ j = 1, \dots, n$:

$$PL(j-0.5) = -\mathbb{E}\,\mathrm{Paid}\,(t_j) \tag{4.2.33}$$

- variation of the loss reserve: $VR(j-0.5),\ j = 1, \dots, n$:

$$\begin{aligned} &RES(j-0.5) = \mathbb{E}Reserve\,(t_j)\,, \\ &VR(0.5) = -RES(0.5),\ and\,VR(j-0.5) = RES(j-1.5) - RES(j-0.5), \end{aligned} \tag{4.2.34}$$

- return on reserve: $IR(j-0.5), j = 1, \dots, n$:

$$\mathrm{IR}(0.5) = 0,\ and\ \mathrm{IR}(j-0.5) = rRES(j-1.5) \tag{4.2.35}$$

Assume that interests on the reserves are paid with a one year delay, we can now define the aggregate cash flow at the middle of the year:

$$CF(j-0.5) = PL(j-0.5) + VR(j-0.5) + IR(j-0.5), \quad j = 1, \dots, n. \tag{4.2.36}$$

The related pure technical rate may be then defined as :

$$TR = \frac{TP}{EPI} \tag{4.2.37}$$

where TP is the technical or pure premium for the treaty, and EPI is the cedant's estimated premium income. TP may be written as follows:

$$TP = \sum_{j=1}^{n} -PL\,(j-0.5) = \sum_{j=1}^{n} \mathbb{E}S_{X^{Re}}\,(t_j) = \mathbb{E}S_{X^{\Sigma Re}}\,(t_n) = \mathbb{E}S_{I^{Re}}\,(t_n)\,. \tag{4.2.38}$$

Note this is the sum of the future average paid losses at different moments in the future. Within the cash flow model, the above definition is not very satisfactory because it is the sum of non-discounted cash flows. Assume that the premium may be invested in risk-free assets yielding a return r. Then we introduce the discounted technical rate, denoted $TR_{discount}$, calculated as follows :

$$TR_{discount} = \frac{DTP}{EPI} \tag{4.2.39}$$

where the discounted technical premium is calculated as follows :

$$DTP = \sum_{j=1}^{n} \frac{-PL(j+0.5)}{(1+r)^{j+0.5}} \tag{4.2.40}$$

The DTP is a premium obtained at time 0 that is not influenced by over(under)statement of loss reserves. Let us now introduce the technico-financial premium (TFP). The TFP takes into account the variation of reserves and interests on them, with discounted cash flows at cost-of-capital:

$$TFP = \sum_{j=1}^{n} \frac{CF(j+0.5)}{(1+coc)^{j+0.5}}. \tag{4.2.41}$$

The technico-financial rate TFR may be then calculated as follows :

$$TFR = \frac{TFP}{EPI} \tag{4.2.42}$$

Assume that all other cash flows occur at the beginning of the year: $t_j = j, j = 0, 1, \ldots, n$. These cash flows are:

- commercial premium ($CP(j)$). The premium is usually incepted at time 0, but sometimes there is a minimum deposit premium at time 0, and the balance is paid at time 1. The reality may be more complex. Premium cash flows at times other than 0 and 1 are not excluded, especially in the presence of random clauses.

- brokerage ($B(j)$). Brokerage, if any, is classically a percentage of the commercial premium. It will thus be deducted at times premiums are paid.

- retrocession ($R(j)$). A cost of retrocession can be modeled as a percentage of the commercial premium minus a fraction of the paid losses. This percentage represents the rate demanded by the retrocessionnaire on commercial premiums, while the fraction represents the share of the average claims the retrocessionnaire will pay. Note this is a crude Quota-Share approximation: the influence of the retrocession could be further investigated. However it is out-of-scope of this study.

- administrative expenses ($AE(j)$). The administrative expenses of a reinsurer can be of two types: fixed expenses and proportional expenses, with the latter associated with the management of the treaty and based on paid losses. Proportional expenses are assumed to be paid during the course of the treaty, which may result in expenses cash flows for all times j.

- variation and return in the allocated capital ($VC(j)$ and $IC(j)$). Capital is allocated to run the business but is given back to shareholders at the latest by the end of the development. The allocated capital may vary based on the rule of allocation, such as returning after a certain number of years or based on the evolution of the loss reserves. The resulting variations in allocated capital are captured each time j by $VC(j)$. While capital is allocated there is a cash flow of interest on it at a return rate l, denoted $IC(j)$ for all times j.

Let us now define the cash flows at integer times:

$$\begin{aligned} &CF(j) = CP(j) + B(j) + R(j) + AE(j) + VC(j) + IC(j) \\ &j = 0, 1, \ldots, n \end{aligned} \tag{4.2.43}$$

It remains to deal with the tax cash flows. Define the taxable profit at times j by :

$$\begin{aligned} &\text{TaxProfit } (j) = CP(j) + B(j) + R(j) + AE(j) + IC(j), \quad j = 0, 1, \ldots, n, \\ &\text{TaxProfit } (j - 0.5) = PL(j - 0.5) + VR(j - 0.5) + IR(j - 0.5), \quad j = 1, \ldots, n. \end{aligned} \tag{4.2.44}$$

The tax cash flows are then:

$$\begin{aligned} &\text{Tax}(j) = \tau \,\text{TaxProfit}(j), \quad j = 0, \ldots, n, \\ &\text{Tax}(j - 0.5) = \tau \,\text{TaxProfit}(j - 0.5), \quad j = 1, \ldots, n. \end{aligned} \tag{4.2.45}$$

where τ is an average tax rate. It assumes all cash flows, including financial return, to be taxed at the same rate. This is obviously not always true and specific corrections are easy to include in the model according to the tax regime of the reinsurer's domicile.

Computation of the commercial premium Starting from equation 4.2.32, the treaty will be acceptable if:

$$\sum_{j=0}^{n} \frac{CF(j) - \text{Tax}(j)}{(1 + coc)^j} + \sum_{j=1}^{n} \frac{CF(j - 0.5) - \text{Tax}(j - 0.5)}{(1 + coc)^{j-0.5}} > 0. \tag{4.2.46}$$

Therefore the initial commercial premium ($CP(j)$ at time 0 and 1) will be deducted such that the above condition is at least satisfied. Mathematically we aim to find the root of the function given in equation 4.2.46. For achieving this, the bisection method is implemented. Also known as the dichotomy method, it is a root-finding algorithm that is used to find the roots of a given function. It is based on the intermediate value theorem, which states that if a continuous function has values of opposite sign at two points, then it must have at least one root between those two points. The bisection method starts by bracketing the root, that is, by identifying two points where the function changes sign. It then divides the interval between these two points in half and evaluates the function at the midpoint. Depending on the sign of the function at the midpoint, it then either repeats the process with the lower half or the upper half of the interval, until the desired level of accuracy is achieved. Despite being relatively slow, the advantage of the bisection method is that it is guaranteed to converge to a root, provided that the function is continuous and changes sign on the interval being searched. This is the case of our function, which takes as input variable the initial commercial premiums (all other parameters being set) and as output variable the net present value of the cash-flow model.

The commercial rate CR may be then calculated as follows :

$$CR = \frac{CP}{EPI \times Share} \tag{4.2.47}$$

where $Share$ stands for the share of the reinsurer in the treaty, which weights both the cash-flows and the premium rates. It is indeed best market practice to have several insurers sharing a given treaty, for obvious solvency reasons. More details may be found in Walhin et al. (Walhin et al., 2001).

4.2.3 Data

Recall our aim for Block 1 is to replicate the Walhin et al (2001) pricing model into R code. In order to ensure that the model is properly implemented, the various technical, technico-financial and commercial rates, defined in the previous section and obtained in the numerical applications from Walhin et al. (2001), have been compared with the rates obtained from our R code. For achieving this, the same dummy data as in Walhin et al. (2001) will be used. Table A.1.1 of Appendix A shows this data set.

4.2.4 Findings

We firstly check that our Monte-Carlo simulations properly converge and we get the number of simulations needed for achieving this. We select as convergence indicator the TR relative convergence gap, i.e. the relative difference in pure Technical Rate (TR) value between a number of simulations N'_{sim} and N_{sim}, such that $N'_{sim} > N_{sim}$. Our convergence criteria is thus deemed met when the TR relative convergence gap is below 1% after 2 consecutive numbers of simulations. Figure 4.2.1 illustrates examples of Monte-Carlo simulations convergence checking for a pure treaty, a treaty with AAD and finally one with AAL.

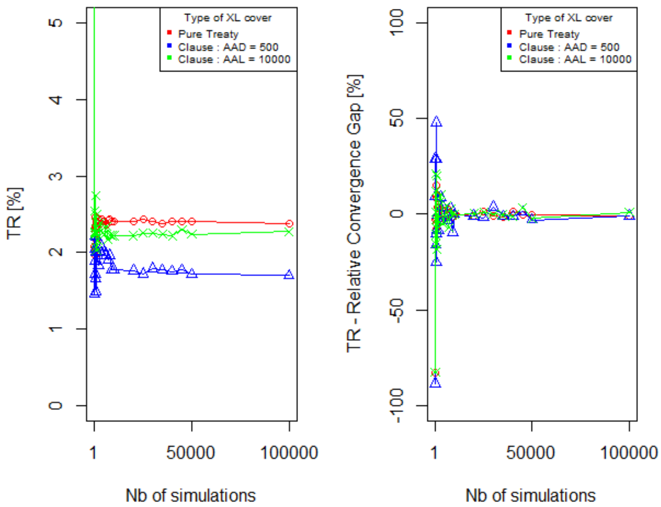

Figure 4.2.1: BLOCK 1 : Monte-Carlo simulations convergence checking - Some examples

One may clearly observe that our convergence criteria is satisfied for $N_{sim} = 100\,000$. In what follows, we will then perform our simulations thanks to this convergence criteria. We will consider 100 000 simulations. Table 4.2.1 provides an overview of the obtained prices of treaties without and with several different clauses. Note that the reinstatement clause consists of 3 paid reinstatements at 100%, whereas the sliding scale clause is active after 4 years with a minimum (and initial) rate R_{min} set to 2% and a loading factor f set to $\frac{100}{70}$. The rates are compared with the ones disclosed in Walhin et al. (Walhin et al., 2001).

Type of XL cover	**Rates [%]**	**Walhin et al. (2001)** Panjer algorithm [lattice span = 25]	**Master Thesis** Monte-Carlo algorithm [Nsim = 100K]	**Diff**	**Diff [%]**
Pure Treaty	TR	2.28	2.38	- 0.10	- 4%
	TFR	1.97	2.07	- 0.10	- 5%
	CR	3.24	3.18	0.06	2%
Clause : AAD = 500	TR	1.63	1.70	- 0.07	- 4%
	TFR	1.42	1.49	- 0.07	- 5%
	CR	2.51	2.58	- 0.07	- 3%
Clause : AAL = 10000	TR	2.28	2.27	0.01	0%
	TFR	1.97	1.96	0.01	0%
	CR	3.23	3.18	0.05	2%
Clause : Paid Reinstatements	CR	2.56	2.60	- 0.04	- 2%
Clause : Sliding scale	R max	4.79	4.84	- 0.05	- 1%

Table 4.2.1: BLOCK 1 : Findings

Considering the difference in the underlying implemented algorithms (Panjer versus Monte-Carlo), the obtained results are deemed satisfactory, all gaps being below 5%.

4.3 BLOCK 2 : Short-term interest and inflation rate modelling

4.3.1 Objectives and approach

Objectives In the core pricer developed in Block 1, both the risk-free interest rate and the inflation rate are arbitrary fixed and considered flat. The aforementioned model may then improved by computing, instead of a flat deterministic rate structure, both an historical short term nominal and an historical short term inflation rate stochastic model. An historical short term real interest rate model is also deducted thanks to the Fisher formula, for illustrative purpose.

Approach In this section we detail the methodologies used for modelling the historical short-term nominal interest rate, inflation rate but also real interest rates. We explain our calibration methods and provide some forecasts from the obtained models. The dependence between both historical short-term nominal interest and inflation rates is also considered.

4.3.2 Short-term nominal interest rate modelling

4.3.2.1 Theoretical background

Model Vasicek proposed a generalised one-factor short rate model under the historical probability measure (Vasicek, 1977). The Vasicek model assumes that the spot short-term interest rate follows an Ornstein-Uhlenbeck process with constant coefficients. The mean reversion property of the model causes the interest rate to be pulled towards its long run mean. A coefficient λ determines the speed of adjustment of the interest rate towards its long run level. Economic arguments support mean reversion in finance, where high rates lead to a slowdown in the economy and a decrease in demand for funds. On the contrary, when rates are low, borrowers tend to demand more funds, leading to a rate increase (Blanchard and Sheen, 2013). Note that such interest rate model is able to capture negative rates, which is an important property considering the rates from the past few years.

$$di_t = \lambda\left(\mu - i_t\right)dt + \sigma dW_i^{\mathbb{P}}(t) \tag{4.3.1}$$

Maximum Likelihood Estimation This method involves searching for the set of parameter values that make the observed sample most likely in a specified model. In other words, this means finding the values of the parameters that are most likely to have produced the data we may observe (Klugman et al., 2012). The log-likelihood function $L(\theta)$ on the Vasicek model is given by:

$$\begin{aligned} L(\theta) = L\left(\lambda, \mu, \sigma^2\right) = &-\frac{n}{2}\log\left(\frac{\sigma^2}{2\lambda}\left(1-e^{-2\lambda\Delta t}\right)\right) - \frac{n}{2}\log 2\pi \\ &- \frac{\lambda}{\sigma^2\left(1-e^{-2\lambda\Delta t}\right)}\sum_{i=1}^{n}\left(i_{t_i} - i_{t_{i-1}}e^{-\lambda\Delta t} - \mu\left(1-e^{-\lambda\Delta t}\right)\right)^2 \end{aligned} \tag{4.3.2}$$

where t_i is the i^{th} time step of the sampling data. Using this log-likelihood function, we can then derive each estimator for the parameters λ, μ and σ^2. If i_{t_0} is the first given interest rate in the time series and i_{t_n} is the current (last observed) interest rate, the parameters λ, μ and σ^2 in the Vasicek models are given by the following maximum-likelihood estimators:

$$\begin{aligned}
\hat{\lambda} &= -\frac{1}{\Delta t}\log\left(\frac{n\sum_{i=1}^{n} i_{t_i} i_{t_{i-1}} - \sum_{i=1}^{n} i_{t_i}\sum_{i=1}^{n} i_{t_{i-1}}}{n\sum_{i=1}^{n} i_{t_{i-1}}^2 - \left(\sum_{i=1}^{n} i_{t_{i-1}}\right)^2}\right) \\
\hat{\mu} &= \frac{1}{n\left(1-e^{-\hat{\lambda}\Delta t}\right)}\left(\sum_{i=1}^{n} i_{t_i} - e^{\hat{\lambda}\Delta t}\sum_{i=1}^{n} i_{t_{i-1}}\right) \\
\hat{\sigma}^2 &= \frac{2\hat{\lambda}}{n\left(1-e^{-2\hat{\lambda}\Delta t}\right)}\sum_{i=1}^{n}\left(i_{t_i} - i_{t_{i-1}}e^{-\hat{\lambda}\Delta t} - \hat{r}\left(1-e^{-\hat{\lambda}\Delta t}\right)\right)^2
\end{aligned} \tag{4.3.3}$$

where $\Delta t = t_i - t_{i-1}$, $\forall i = 1,\ldots,n$. The standard errors of the above estimates are then obtained by computing the inverse of the information matrix at the values of these estimates. Further details, including proofs of the above equations, are provided in Appendix B.

4.3.2.2 Practice : data used, calibration and forecasting method

Data Dataset of monthly EURIBOR 1-month rates from December 2002 to December 2022 has been downloaded from the ECB website (ECB Statistical Data Warehouse, 2022). A time frame of 20 years was chosen as it is deemed a sufficient and necessary amount of data for calibrating a model which will serve for our future cash flows discounting. Figure 4.3.1 shows the rates and their monthly variations.

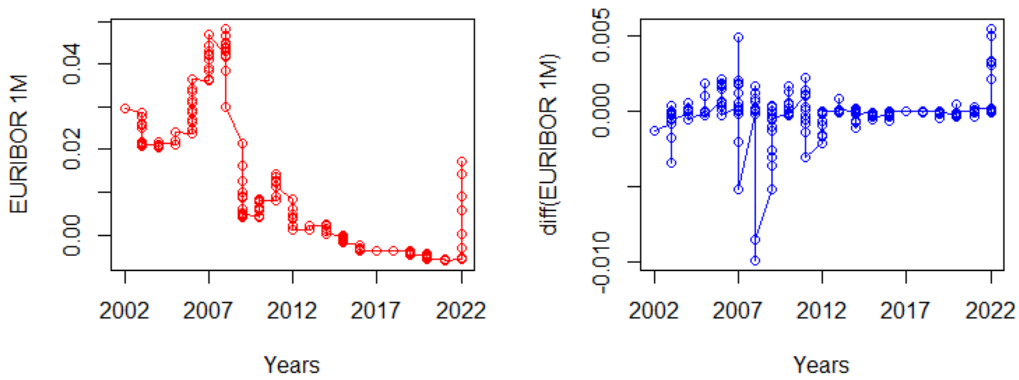

Figure 4.3.1: EURIBOR 1-month rates and their related monthly differences, from December 2002 to December 2022

Parameters calibration A Maximum-Likelihood Estimation of the parameters, as previously described, is implemented into R. Table 4.3.1 exhibits the obtained statistics. Note the estimate of μ is not statistically significantly different from 0, thus it will be set to 0.

Forecasting In order to forecast the short-term nominal interest rate, an Euler discretisation is carried out - see Burden and Faires for more details about Euler's method (Burden and

	Estimate	Std. Error	t-value	Pr(>t)
μ	0.00298	0.012	0.249	0.617
λ	0.099	0.0622	15.9	6.64e-05 ***
σ	0.00529	0.000241	21.9	2.86e-06 ***

Table 4.3.1: Vasicek model parameters calibration outcome

Faires, 2000). A Vasicek Euler scheme is deducted from equation 4.3.1 and then implemented into R :

$$\tilde{i}_{t+\Delta t} = i_t + \lambda\,(\mu - i_t)\,\Delta t + \sigma\sqrt{\Delta t}\epsilon^{\mathbb{P}}_{i,t} \tag{4.3.4}$$

Figure 4.3.2 shows the forecast of nominal interest rates 30 years ahead. Only 100 scenarios have been considered for sake of clarity.

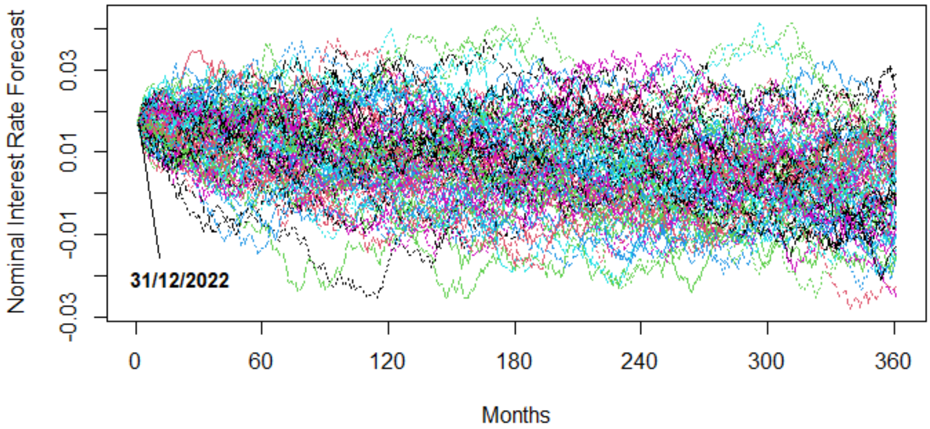

Figure 4.3.2: Nominal interest rate forecasting (h = 360 months)

4.3.3 Short-term inflation rate modelling

4.3.3.1 Theoretical background

Model Ahlgrim et al. proposed a short-term inflation rate model under the historical probability measure (Ahlgrim et al., 2005). As for the Vasicek model, this is a type of Ornstein-Uhlenbeck one. As suggested by Kladivko and Zimmerman, this model may be improved by adding a Pearson's correlation between inflation and short term nominal rates (Kladivko, K. and Zimmermann, P., 2014). Note we could model the dependence between both rates in a more complex way - using copulas for instance. The short term inflation rate is modelled as follows :

$$dq_t = a_q(\mu_q - q_t)dt + \sigma_q\, dZ^{\mathbb{P}}_q(t) \tag{4.3.5}$$

where μ_q is an unconditional - and long term - mean of the inflation rate, a_q determines a speed at which the short-rate reverts to its unconditional mean, and σ_q is the volatility of the inflation rate. The Brownian motion $dZ_q^{\mathbb{P}}(t)$ can be expressed in terms of two independent Brownian motions $dW_i^{\mathbb{P}}(t)$ and $dW_q^{\mathbb{P}}(t)$:

$$dZ_q^{\mathbb{P}}(t) = \rho\, dW_i^{\mathbb{P}}(t) + \sqrt{1-\rho^2}\, dW_q^{\mathbb{P}}(t) \tag{4.3.6}$$

where ρ determines the Pearson's correlation between the interest and inflation rate.

This Ornstein-Uhlenbeck process may be discretised into a auto-regressive (AR(1)) process for which the parameters are maximum-likelihood estimated :

$$\hat{q_{t+1}} = \mu_q(1 - e^{-a_q}) + \hat{q}_t e^{-a_q} + \epsilon_q(t, t+1) \tag{4.3.7}$$

with $\epsilon_q \sim \mathcal{N}(0, \sigma_{\epsilon_q}^2 = \sigma_q^2 \frac{[1-e^{-2a_q}]}{2a_q})$. A proof may be found in Ahlgrim et al. (Ahlgrim et al., 2005).

Parameters calibration Short-term inflation $\hat{q_t}$ may be considered as a continuous compounding rate for Consumer Price Index (CPI) (Ahlgrim et al., 2005):

$$CPI(T) = CPI(t)\ \exp \int_t^T q_u\, du \tag{4.3.8}$$

Then we have :

$$ln\,(\frac{CPI(T)}{CPI(t)}) = \int_t^T q_u\, du \tag{4.3.9}$$

Differentiating the previous equation between T-1 and T, and following Ahlgrim et al. (2005) approach, an approximation is deducted :

$$d[ln(\frac{CPI(T)}{CPI(T-1)})] \approx \hat{q_T} \tag{4.3.10}$$

Note that the development is done here on a yearly scale. In practice, we deal with monthly data, then we may differentiate the previous equation between $T - 12$ (months) and T. This process is also called seasonal differentiating.

4.3.3.2 Practice : data used, calibration and forecasting method

Data Dataset of monthly Belgian CPI from December 2002 to December 2022 has been downloaded from the StatBel website (Statbel, 2022). A time frame of 20 years was chosen for the same reason and consistency with the short-term nominal rate dataset. We compute the short-term inflation rate by seasonally differentiating the logarithm of monthly CPI. Figure 4.3.3 shows the obtained inflation between December 2002 and December 2022, and its monthly variation.

Parameters calibration It is clear that the inflation itself is not a stationary process. Thus in what follows we will then fit an AR(1) process on the variation of the inflation and use the approximation detailed in the theoretical part. Box test performed on fitted residuals provides the results detailled in Table 4.3.2.

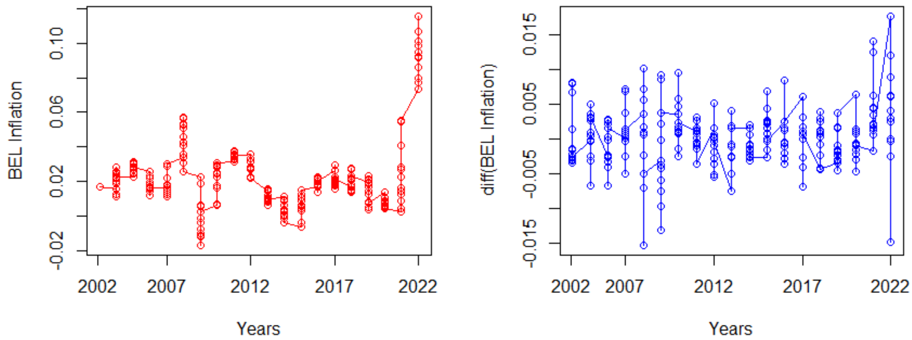

Figure 4.3.3: Belgian inflation rates and its monthly variation from December 2002 to December 2022

Box-Test on Residuals	Value
X-squared	12.721
df	4.4293
p-value	0.01762

Table 4.3.2: Box test outcome

The following parameters have been estimated : $a_q = 1.35$ and $\sigma_{\epsilon_q} = 0.007$. Note that μ_q is set to 2% based on ECB predictions (ECB Statistical Data Warehouse, 2022).

	Estimate	Standard Error
AR(1) coefficient	0.26	0.07
Intercept	Set to 0	N/A
Innovation mean	0.00026	<10e-5
Innovation volatility	0.0041	<10e-5

Table 4.3.3: Inflation model parameters calibration outcome

Pearson's correlation Correlation between EURIBOR 1-month rate and short-term inflation has been captured by simply computing the correlation matrix between both rates. The coefficient is equal to -0.07, which is negative and macro-economically intuitive (Blanchard and Sheen, 2013).

Forecasting In order to forecast the short-term inflation rate, an Euler discretisation of the equation 4.3.5 is performed:

$$\tilde{q}_{t+\Delta t} = \mu_q(1 - e^{-a_q \Delta t}) + q_t e^{-a_q \Delta t} + \sqrt{V(t, t+\Delta t)}[\rho\, \epsilon^{\mathbb{P}}_{i,t} + \sqrt{1-\rho^2}\, \epsilon^{\mathbb{P}}_{q,t}] \tag{4.3.11}$$

where $V(t, t+\Delta t) = \sigma_q^2 \frac{[1-e^{-2a_q \Delta t}]}{2a_q}$. Figure 4.3.4 shows the forecast of the inflation rate 30 years ahead and for 100 scenarios. One may observe the influence of the long-term inflation rate, set to 2%.

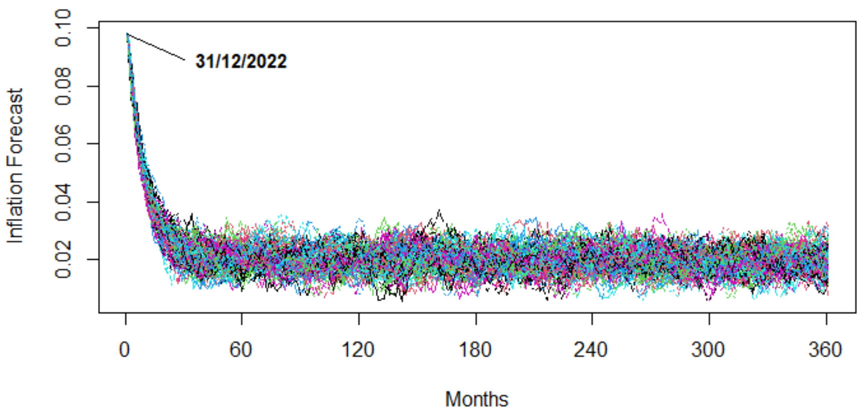

Figure 4.3.4: Inflation rate forecasting (h=360 months)

4.3.4 Short-term real interest rate modelling

4.3.4.1 Theoretical background

Model The Fisher equation expresses the relationship between nominal interest rates and real interest rates under inflation (Fisher, 1977). This can be approximated as follows :

$$r_t \approx i_t - q_t \tag{4.3.12}$$

4.3.4.2 Practice : data used, calibration and forecasting method

Parameters calibration Note there is no calibration to perform since the real interest rate may be directly obtained by computing the difference between the nominal interest rate and the short-term inflation rate.

Forecasting The following equation has been directly computed :

$$\tilde{r}_{t+\Delta t} = \tilde{i}_{t+\Delta t} - \tilde{q}_{t+\Delta t} \tag{4.3.13}$$

Figure 4.3.5 shows the forecast of the real interest rates 30 years ahead and again for 100 scenarios.

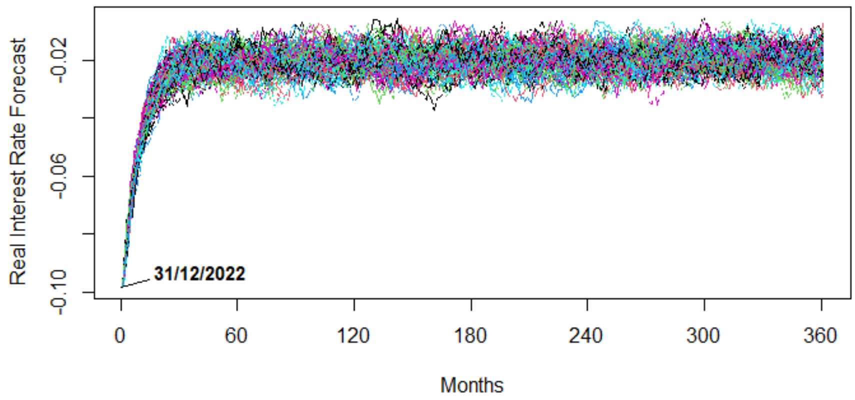

Figure 4.3.5: Real interest rates forecasting (h = 360 months)

4.4 BLOCK 3 : Claims Payment Pattern Modelling

4.4.1 Objectives and Approach

Objectives A limitation of the Block 1 pricing model is also that Claims Payment Pattern (CPP), i.e. the speed at which the ceding insurance companies will pay the claim, is fixed. For a reinsurer covering long tail risks, i.e. contracts where there is a long delay between the date of the accident causing the loss and the dates of payment by the insurance company, the modelling of the speed of payment is very important. This payment delay, which can be several decades, has a significant impact on the expected loss. This is mainly due to the final settlement date that will determine how long inflation applies. A reinsurer that misjudges the cedant's future payments makes a mistake in its price and puts its market competitiveness at risk. Underestimating the price has a negative impact on the results, while overestimating the price makes the reinsurer more expensive than the competition. The aforementioned cash-flow model may then improved by computing, instead of a deterministic payment pattern vector, a more sophisticated stochastic model, in which CPP is assumed to be Dirichlet distributed.

Approach In this section, we firstly provide some insights about the available data set, then we present the CPP model and its parameterisation, and finally we perform a forecasting exercise based on the obtained model.

4.4.2 Dataset: Overview and preliminary analysis

4.4.2.1 Overview

A run-off triangle of As if Belgian MTPL inflated claims paid from 2005-2018 (14 development years) has been retrieved from AON, with their Inflated Ultimate estimated thanks to actuarial best practice.

4.4.2.2 Preliminary analysis

The set contains a total of 277 claims. The claims have several key features, which are analysed in the following paragraphs.

Claims per Accident Year As shown in Figure 4.4.1, we observe the data are almost uniformly distributed. On average there are 19 claims per year of occurrence. Accident Year 2018 is the one with the less claim amount (7) but we might suppose the data retrieved are not covering this year fully because no specific event might explain the difference between this most recent accident year and the past ones.

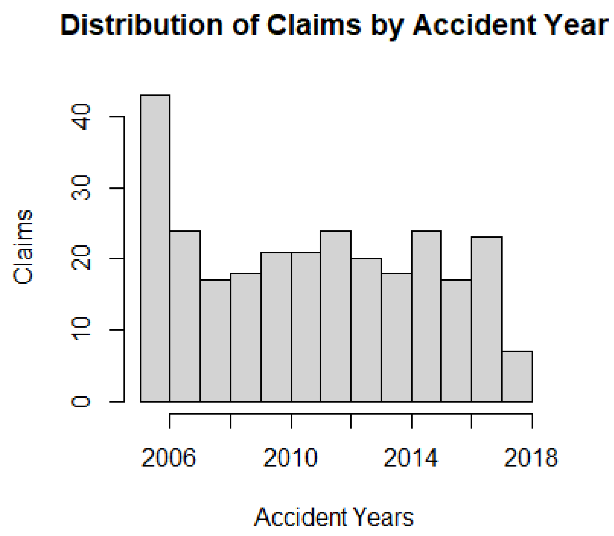

Figure 4.4.1: Distribution of Claims per Accident Year

Claims per Open/Closed claims See Figure 4.4.2. Almost all claims are open to date (272 over 277). Only 5 claims are closed, which makes it difficult to capture any trend in a closing date with sufficient statistical evidence. We will come back to this point later on.

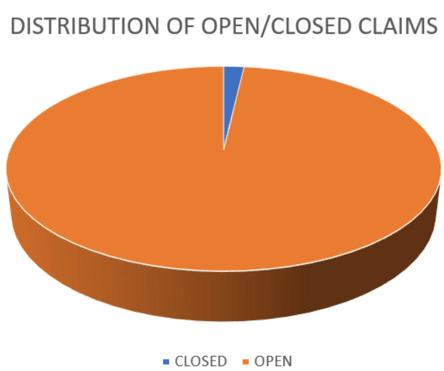

Figure 4.4.2: Distribution of Claims per Status (Open/Closed)

Claims per Inflated Ultimate Bucket As shown in Figure 4.4.3, more than half of the claims belong to the window [2.5 MEUR, 4 MEUR] (148 over 277). Most of the remaining claims have Inflated Ultimate which are between 4 and 10 MEUR (119 over 277). Few of them are above 10 MEUR (10 over 277), which is not a surprise as these extreme amounts are very rare events. Note that closed claims are the ones with relatively low Ultimates, below 6 MEUR (most of them are below 4 MEUR).

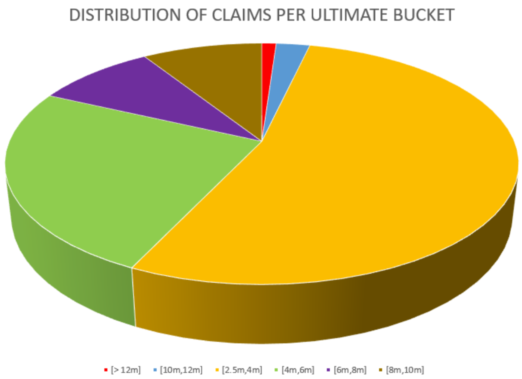

Figure 4.4.3: Distribution of Claims per Ultimate Bucket

Now we try to explore some trends in the payment pattern. For achieving this, we calculate the ultimate weighted average of the payment patterns per variable of interest. Again, the factors Accident Year, Claim Status and Inflated Ultimate are subsequently considered.

CPP per Accident Year As exhibited in Figure 4.4.4, we might observe a very interesting trend, which is that the payment patterns do not seem to be much dependent of the Year of Occurrence. This will be considered later when developing the model.

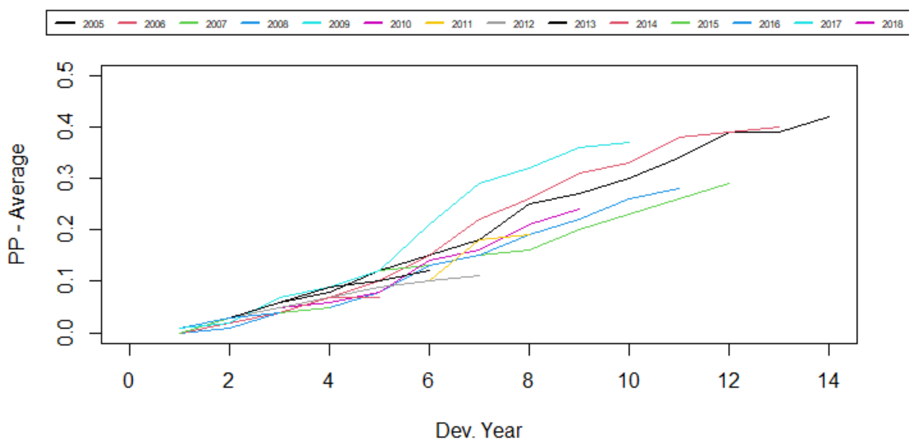

Figure 4.4.4: Ultimate weighted average CPP against DY per Accident Year

CPP per Open/Closed claims As shown in Figure 4.4.5, if we compare the payment patterns ultimate weighted average amongst the closed claims with the one amongst open claims, we might directly observe that the payment pattern for closed claims is very different from the one of the open claims. If open claims show an almost linear trend from 0% to around 40% within a range of 14 years, the closed claims show a polynomial to logarithm trend, with a fast increase of payment from 3 development years onwards. The average payment pattern logically follows the one of the open claims because they have much more weight than the closed ones.

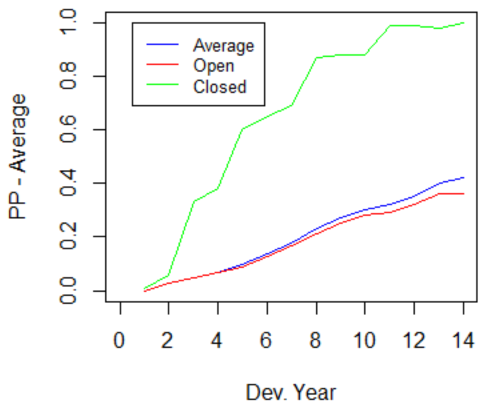

Figure 4.4.5: Ultimate weighted average CPP against DY per Claims Status

CPP per Inflated Ultimate Bucket See Figure 4.4.6. Payment patterns seem to be very dependent of the Inflated Ultimate bucket. The lower the Ultimate is, the higher the speed of the payment. This can be linked to the previous observations. The higher the Inflated Ultimate, the more chance the claim has to be open, and the slower the payment is.

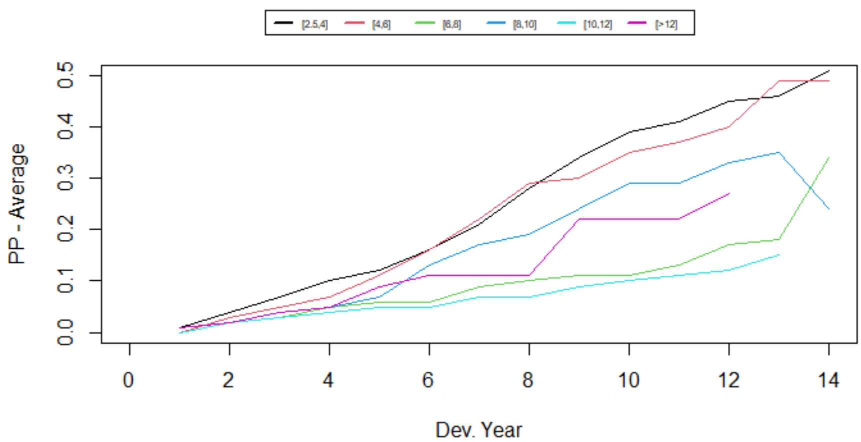

Figure 4.4.6: Ultimate weighted average CPP against DY per Inflated Ultimate Bucket

All of the above trends will be keys when we will set the different models and calibrate them. These are performed in the next sections.

4.4.3 CPP model

4.4.3.1 Theoretical background

Overview The Dirichlet distribution ($Dir(\alpha)$) is a probability distribution used in Bayesian statistics and machine learning to model probabilities over a set of categorical outcomes. It is named after the mathematician Johann Dirichlet and is a multivariate generalization of the beta distribution. The Dirichlet distribution is characterized by a vector of positive shape parameters, which determine the shape of the distribution. Each element in the vector corresponds to a different category or outcome, and the sum of the vector is equal to 1. The distribution represents a probability distribution over the simplex, which is a multi-dimensional generalization of the interval [0,1] (Van Der Merwe and De Waal, 2018).

Probability density function The Dirichlet distribution of order $K \geq 2$ with parameters $\alpha_1, ..., \alpha_K > 0$ has a probability density function with respect to Lebesgue measure on the Euclidean space $\mathbb{R}^{K-1}$ given by :

$$f(x_1, ..., x_K, \alpha_1, ..., \alpha_K) = \frac{1}{B(\alpha)} \prod_{i=1}^{K} x_i^{\alpha_i - 1} \tag{4.4.1}$$

where $\sum_{i=1}^{K} x_i = 1$ and $x_i \in [0, 1]$ for all $i \in [1, K]$. The normalizing constant is the multivariate beta function, which can be expressed in terms of the gamma function, for $\alpha = (\alpha_1, ..., \alpha_K)$

$$B(\alpha) = \frac{\prod_{i=1}^{K} \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{K} \alpha_i)} \tag{4.4.2}$$

The probability density function of the Dirichlet distribution describes the probability of observing a particular vector of categorical outcomes given a set of positive shape parameters.

Expectation and Cross-Moment The expectation of a component of a Dirichlet distribution vector ($Dir(\alpha)$) is given by:

$$\mathbb{E}[X_i] = \frac{\alpha_i}{\sum_{j=1}^{K} \alpha_j} \tag{4.4.3}$$

where X_i is the i-th component of a random variable X following a Dirichlet distribution with parameters vector $\boldsymbol{\alpha}$. It can also be proven that the cross-moment of Dirichlet-distributed random variables is (Van Der Merwe and De Waal, 2018):

$$\mathbb{E}[\prod_{i=1}^{K} X_i^{\beta_i}] = \frac{B(\alpha + \beta)}{B(\alpha)}. \tag{4.4.4}$$

Properties The Dirichlet distribution has a number of useful properties, which will not be detailed in this document, including conjugacy with the multinomial distribution and the fact that its marginal distributions are beta distributions. The density function can be used to compute the likelihood of observed data, and it is often used in Bayesian inference to update prior beliefs about the parameters of a categorical distribution.

4.4.3.2 Parametrisation and calibration

CPP is modelled thanks to a Dirichlet distribution. However calibrating such distribution is quite demanding since the Dirichlet model requires couple of mathematical conditions to be respected (recall equations 4.4.1 and 4.4.2) :

- A fixed closing date (fixed length of α 's vector);
- Data set of $\mathbf{x}$' s to be calibrated which are strictly positive (generally between 0 and 1, despite 1 could have been extended to a larger number). Moreover, missing data are not allowed.

Consequently, before calibrating CPP thanks a Dirichlet distribution, we have to perform the following tasks :

- Model the Horizon (H), or Maximum Development Year Closing (DYC) of the open claims;
- Carry out the extrapolation of the Payment Patterns between the Last Development Year (LDY) and the Horizon (H).

4.4.3.3 Development Year Closing (DYC)

Strategy As noticed in our preliminary analysis, most the claims of the dataset are still open to date. Only very few claims are closed, thus it is difficult to capture any specific reliable trend in a closing date. Therefore an alternative strategy is to model the closing date of open claims. From this model, our aim is to find a maximum acceptable closing date, called Horizon, which we will take as reference for the next steps of our CPP model.

In order to model the open claims' closing date in a simple way, a Poisson law may be chosen :

$$P(DYC = k) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad k = 0, 1, \ldots. \tag{4.4.5}$$

where the positive real number λ is equal to the expected value of DYC and also to its variance. Poisson law is commonly used to model rare events, such as the large claims closing within 14 years, i.e. the last development year (LDY).

We previously observed that payment patterns are very dependent on the inflated ultimate bucket. The lower the ultimate is, the more the payment is fully proceeded. It seems then logical to think that a claim has more chance to close early if its corresponding ultimate is lower, and all the way around. Therefore modelling a claim closing date by a Poisson distribution without making any distinction between the ultimate value could lead to substantial modelling error and thus an erroneous XL cover price. However, in practice, only few claims are closed and all of them have low ultimate values, thus we are facing a lack of sufficient data for the closed claims to follow the above approach. Considering this, once having estimated the parameter λ, the mean expected closing date, a sufficiently large quantile (99.5%) of the Poisson distribution is taken to deduce the Horizon (H). This is deemed being a conservative approach.

Poisson law calibration Remind that the payment patterns do not seem to be much dependent of the accident year. We will therefore focus on the maximum developed claims, which are the ones from 2005 (14 DY). The claims dated from the following accident years should close around the same date. Lack of information imposes us to mostly work with open claims. Thus the following methodology is proposed to calibrate the parameter λ :

- λ is firstly calculated *a priori* thanks to the Maximum Likelihood Estimation (MLE) on the closed claims. It is straightforward to prove that the Poisson parameter is maximum-likelihood estimated as follows : $\hat{\lambda} = \frac{1}{N_{closed}} \sum_{n=1}^{N_{closed}} DYC(n)$ (see proofs in Denuit et al., 2019).
- After getting a first *a priori* estimation, the open claims closing date are estimated thanks to expectation maximization method, as follows:

  - $DYC_{a\,posteriori} = DYC_{observed}$ if the claim is closed, else

  - $\hat{DYC}_{a\,posteriori} = \mathbb{E}[DYC_{observed}|DYC_{observed} > LDY]$ if the claim is open at LDY (=14 years).
- Then λ is again calculated *a posteriori*, thanks to the Maximum Likelihood Estimation (MLE).

Maximum DYC Based on our sample, we obtain $\lambda_{a\,priori} = 10$ then $\lambda_{a\,posteriori} = 16$. We then consider a sufficiently large quantile (99.5%) to deduce the Maximum DYC or Horizon H, which is equal to 30 Years.

4.4.3.4 Payment Patterns Extrapolation

Strategy Now we aim to model the payment patterns knowing H. We will proceed in two steps:

- The first step is to extrapolate the claims payment patterns with an accident year later than 2005 until LDY (14 years). By definition, these claims date from 2006 to 2018 and therefore are younger than the ones from 2005 and do not have full 14 years of development.
- The second step is to extrapolate the DY from LDY (14 years) until H (30 years).

Step 1 Recall we observed that the payment patterns do not seem to be much dependent of the Accident Year. Therefore it was decided to use a simple deterministic Chain Ladder to forecast future payments the next years until LDY (14 years). Details about the deterministic Chain-Ladder model may be found in Mack (Mack, 1993). Note that the inflation is stable between 2005 and 2018 (around 2-3 %) thus inflation is implicitly considered in our Chain Ladder. We gathered all claims per Accident Year and calculated the link ratios for each DY. Then the future losses were computed for each Accident and Development Year (until LDY = 14). Figure C.1.1 - Appendix C - shows the obtained Chain-Ladder model.

Step 2 We also observed that the payment patterns of the closed claims show very different trends than the ones of the open claims. Recall the ones of the closed claims show a polynomial/logarithm trend. They may be easily modeled by the cumulative distribution function of the Log-Logistic law. The cumulative distribution function of the Log-Logistic law is defined the following way, for $x > 0$:

$$F(x) = \frac{x^\beta}{\alpha^\beta + x^\beta} \tag{4.4.6}$$

where $\alpha > 0$ is the scale parameter and $\beta > 0$ the shape parameter of the law. $F(x) \in [0, 1]$ and thus is a convenient function for modelling payment patterns. This function also has an interesting fashionable S-shape which captures the ones observed for the closed claims. Based on the observed data for the closed claims, we aim to estimate the Log-Logistic law parameters by maximum-likelihood with a sufficient statistical reliance. Due to lack of information, we have to perform a re-sampling of the data by simulating the closed claims payment pattern over the development year as a discrete random variable. The obtained empirical cumulative distribution function is then fitted by maximum-likelihood.

Then the payment patterns of the open claims are extrapolated using this log-logistic function, assuming that from 14 years onwards they will follow the payment patterns of the closed claims to reach their Ultimate value between LDY (14 years) and H (30 years). Another way to see this assumption is to assume that the maximal final judgment date will be 15 years after the occurrence of the claim. For achieving this, we firstly find the fictive last (known) development years DY_f for which :

$$P(DY \leq DY_f) = \frac{Last\,Known\,Inflated\,Payment}{Inflated\,Ultimate} \tag{4.4.7}$$

i.e. a quantile of the Log Logistic Law. Then for all $DY_{extr} \in [LDY, H]$ we find all the $F(DY_{extr})$ by applying Bayes' rule:

$$F(DY_{extr} | DY_{extr} > DY_f) = \frac{F(DY_{extr})}{P(DY_{extr} > DY_f)} \tag{4.4.8}$$

We then obtain the extrapolated payment patterns of the open claims between LDY and H.

4.4.3.5 CPP Dirichlet Modelling

Pre-processing In order to model the payment patterns thanks to a Dirichlet law, we finally have to prepare the obtained CPP extrapolated matrix. This is done in two steps.

Step 1 It is possible to occasionally observe a decreasing trend in the payment patterns. This could be interpreted as the fact that an incremental payment proceeded by the insurer was undue, for instance because of a change in judgment issue. Our model does not aim to capture such effect and thus we smooth all negative payments by considering them negligible, the value being subtracted from the next payment, or from the following payments if the next payment is not sufficient. Note that, if the sum of payments is not sufficient to smooth the negative payment, then the claim is considered as zero. With this procedure, the Ultimate value does not change and the cumulative payments are indeed strictly increasing.

Step 2 Payment patterns are cumulative. Thus we finally 'decumulate' the payment patterns.

Dirichlet Parameters estimation The 'decumulated' CPP matrix is finally ready for Dirichlet law calibration. The Dirichlet distribution parameters are maximum-likelihood estimated (MLE). MLE of Dirichlet distribution parameters have been widely studied by Huang and and Van Der Merwe et al. (Huang, 2005 and Van Der Merwe and De Waal, 2018). Our R code has been implemented via a Fixed Point Iteration approach, which is deemed a convenient method to estimate the Dirichlet parameters. We refer to the aforementioned papers for more details (see Huang, 2005 and Van Der Merwe and De Waal, 2018). The following outcomes were obtained :

i	1	2	3	4	5	6	7	8	9	10
α_i	0.30	0.74	0.79	0.71	0.76	0.86	0.82	0.97	0.99	1.01
i	11	12	13	14	15	16	17	18	19	20
α_i	1.07	1.01	0.67	1.13	0.23	10.00	7.44	5.26	3.72	2.68
i	21	22	23	24	25	26	27	28	29	30
α_i	1.99	1.53	1.22	1.00	0.85	0.73	0.64	0.57	0.52	0.48

Table 4.4.1: MLE of Dirichlet distribution parameters - Outcome

4.4.4 CPP forecating

Once the 30 Dirichlet α's parameters are estimated, the claims payment patterns are then simulated. Generating random variates from the Dirichlet distribution can be done using the Gamma distribution. To generate a random variate $(PP_1, ..., PP_K)$ from a Dirichlet distribution with shape parameters $(\alpha_1, ..., \alpha_K)$ using the Gamma distribution, we may directly use the following formula : $PP_i = \frac{Y_i}{\sum_{j=1}^{K} Y_j}$ where $Y_i \sim Gamma(\alpha_i, 1)$ for $i = 1, ..., K$.

By summing again the payment patterns, the expectation of the cumulative payment patterns within 30 years may be computed. Figure 4.4.7 shows an example of 100 simulations of CPP, the related simulated cumulative payment patterns and its corresponding expectation.

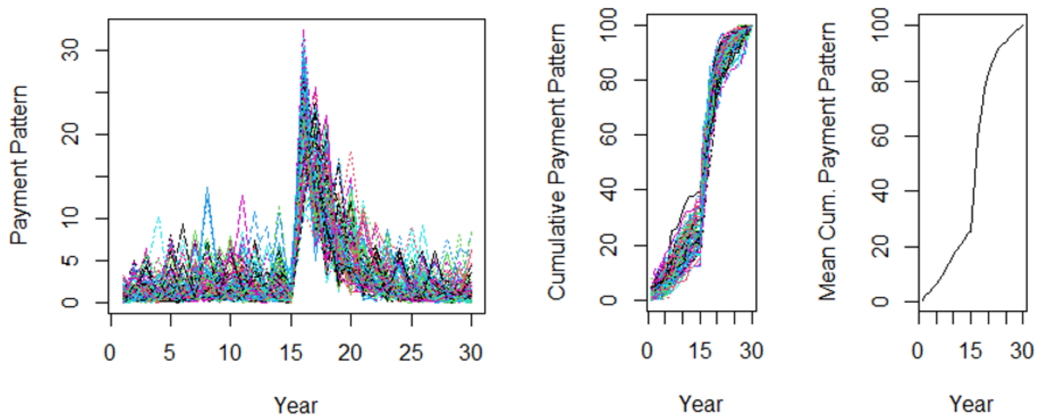

Figure 4.4.7: Simulations of Dirichlet-modelled payment patterns, cumulative payment patterns and related expectation (N=100 simulations)

From Figure 4.4.7, we immediately notice there is a 'jump' of the CPP between 14 and 20 years. This jump is logically due to the method we used and underlying assumption we made when extrapolating our payment patterns from LDY (14 years) until H (30 years). Recall that, after 15 years, we consider that all still open claims at LDY will follow the trend of the payment patterns of the closed claims to reach their Ultimate value between LDY and H. Therefore there is a disruption in the speed of the payment pattern captured by the Dirichlet distribution.

Also note that the expected cumulative payment pattern could be analytically computed using the formula of the expectation of the components of a Dirichlet distribution vector. Recall equation 4.4.3 and the estimated Dirichlet parameters, the analytical value for the expected cumulative payment pattern after 15 years is $\frac{12.06}{50.69} = 23.79\%$, which seems to be the value observed in the third rightmost chart of figure 4.4.7.

4.5 BLOCK 4 : Capital Allocation Modelling

4.5.1 Objectives and Approach

Objectives Amongst the arbitrary set parameters from Block 1 model (see section 4.2), the allocated capital is one of them. In Walhin et al. (2001), it is assumed to be 1.25 times the standard deviation of the ultimate aggregate claims of the reinsurer, net of a fixed retrocession contribution (Walhin et al., 2001). It seems clear that this model may then improved by modelling the reinsurer capital allocation depending on its own portfolio size. The aim of this section is thus to provide an acceptable methodology for calculating the reinsurer capital needs and allocation through future years. Both the impact of the portfolio size and dependence between risks in exogenous factors on capital allocation are analysed. The impacts of the XL cover clauses are also discussed. Several illustrative examples and intermediary findings are provided.

Approach Our approach is based on two risk measures, denoted ρ, inherited from ruin theory. Specifically, we consider the Value-at-Risk (VaR) at 99.5% and the Expected Shortfall (ES) at 99%. We will study the sensitivity of the reinsurer capital allocation to these risk measures. Premium principles calculations, based on European actuarial literature, are applied to compute the pure premiums. We firstly investigate the reinsurer capital needs in a single period framework, then extend them to multiple periods.

4.5.2 Capital Allocation Modelling in a Single Period Framework

4.5.2.1 Simulation of the reinsurer portfolio aggregated claims

Risk model and Insurance Portfolio We refer to section 2.3. The number of claims is Poisson distributed whereas the severity of these claims is Pareto distributed.

Dependence structure Apart from the dependence structure respectively between severities and frequencies, captured by copulas and already defined in section 2.3, the impact of stochastic inflation and thus claims inflation dependence on the reinsurer capital need is studied. The number of claims keeps being assumed independent of their severity. Inflation I will be set to be in the following states of the world during a one-year period :

$$I = \begin{cases} 1.02 \times i_1 & with \quad p_1 = 0.20 \\ 1.03 \times i_2 & with \quad p_2 = 0.20 \\ 1.10 \times i_3 & with \quad p_3 = 0.60 \end{cases} \tag{4.5.1}$$

with p_i the discrete probability mass for each state $i = 1, 2, 3$. It will be generated as a discrete random variable and then used to inflate the simulated severities. For sake of comparability, in the case the inflation is not stochastic, the claims will be inflated with the expectation of the inflation ($E[I] = 1.07$).

Reinsurance portfolio See section 2.3. In what follows we will consider M equal to 1, 10, 100, and 1000.

Monte-Carlo simulations Despite being computationally expensive, Monte-Carlo simulations allow to numerically simulate the reinsurer portfolio aggregated claims under all kind of clauses, including those which lead to random premiums, for which no analytical closed-form formula may be derived. We will keep using data from Walhin et al (2001), see Table D.1.1 of Appendix D (Walhin et al., 2001). In Block 4, the number of Monte-Carlo simulations which satisfies our convergence criteria - relative TR value gap below 1 % - is equal to 20 000 (Figure D.2.1 - Appendix D).

4.5.2.2 Case 1 : Pure XL treaties, without any clause

Pure Premium Calculations The pure premium per treaty is derived using the expected value premium principle for each treaty. Neither loading nor risk margin considered, which in a sense is a conservative assumption with respect to the reinsurer capital calculation :

$$Premium^{(m)} = \mathbb{E}(S^{(m)}). \tag{4.5.2}$$

Then these premium are summed over the entire portfolio to end up with the total pure premium for a given line of business:

$$Premium_M = \sum_{m=1}^{M} Premium^{(m)}. \tag{4.5.3}$$

Allocated Capital Denote S_M the distribution of total reinsured aggregate claims :

$$u = \rho(S_M - Premium_M) \tag{4.5.4}$$

In what follows, ρ is either VaR at 99.5% to be in line with Solvency II requirements or ES at 99% to be in line with Swiss Solvency Test requirements.

4.5.2.3 Case 2 : XL treaties with annual aggregate liability clause

Annual aggregate liability As stated in section 4.2, two clauses often assort an XL treaty, which are the annual aggregate deductible *Aad* and the annual aggregate limit *Aal*. Similarly to Case 1, we compute the reinsurer total loss, pure premiums and allocated capital. Details may be found in Appendix D, section D.3.

Limit Case Checks The robustness of the implementation is challenged by considering a limit case where the aggregate priority is set to 0 and aggregate limit to infinity. The obtained pure premium and reinsurer capital needs are in line with what was expected : these are the same as the ones for the pure treaty. See section D.3 - Appendix D - for more details.

4.5.2.4 Case 3 : XL treaties with paid reinstatements

Paid reinstatements Recall that, from section 4.2, in a *prorata capita* approach, the reinstatement premiums are random since they depend on aggregate claims S. Similarly to Case 1, we compute the corresponding reinsurer total loss, pure premiums and allocated capital. Details may be found in Appendix D, section D.4.

Limit Case Checks The robustness of the implementation is also crucial here as this clause is a very common one in the XL reinsurance market. This clause is challenged in section D.4 of appendix D. In each case the obtained pure premiums and capital needs are in line with what was expected, and several results from Sundt are found again (Sundt, 1993). This provides use confidence that the clause is properly implemented.

4.5.2.5 Case 4 : XL treaties with sliding scale premium

Sliding Scale Premium Recall, from section 4.2, sliding scale premium clause is often requested by the cedant arguing the aggregate claims will not be significant or difficult to forecast. Similarly to Case 1, we compute the corresponding reinsurer total loss, pure premiums and allocated capital. Details may be found in Appendix D, section D.5.

Limit Case Checks The robustness of the clause implementation is challenged by considering a limit case where the sliding scale is a "flat" pure treaty premium scale, i.e. the minimum premium is equal to the maximum premium, equal to a pure premium of a pure treaty, without any clause. The obtained pure premium and capital needs are in line with what was expected. See section D.5 of appendix D for more details.

4.5.2.6 Intermediary Findings

Section D.6 of Appendix D exhibits the results of our study. Below are some take-aways :

Some XL cover clauses logically impact more the capital needs that others. The sliding scale clause is the one which reduces the most the capital needs. A clause with paid reinstatement (ex. 1 @100%) reduces more the capital needs than an equivalent AAL.

Diversification benefits Capital needs per treaty is severely improved when the number of treaties increases. This pooling effect is logically improved by the presence of any clauses.

Sensitivity analysis The capital need seems to be slightly sensitive to the presence of stochastic inflation structure. The capital need is logically sensitive to the considered risk measure, and the sensitivity of the capital need to the presence of stochastic inflation structure increases with the use of ES risk-measure. Finally, the capital need is very sensitive to dependence between risks severities and frequencies due to exogenous factors and captured by the use of copulas.

4.5.3 Capital Allocation Modelling in a Multiple Period Framework

The reinsurer capital need is now modelled over multiple periods. We aim to study a possible capital allocation method which properly captures the reinsurer capital needs each year of claims development.

The philosophy behind our approach is to take a prudent position by looking forward up to the ultimate aggregate cumulative payments, denoted $S_{X^{\Sigma Re}}(t_n)$, where t_n is the horizon or maximal development year of the claims development. Based on ruin theory, the reinsurer may estimate an initial prudent value of the capital needs from the distribution of $S_{X^{\Sigma Re}}(t_n)$. Then, the reinsurer may allocate this capital amount during the lifetime of the claims by amortizing it at the speed of the claims payment.

4.5.3.1 Simulation of the reinsurer portfolio aggregated claims

Risk, insurance and reinsurance portfolio models The risk model keeps being the same as the one defined in the single period framework, i.e. we are considering the same distributions for frequency and severity for all insurance portfolios engaged in the XL cover. The dependence structure models, captured by copulas and due to exogenous factors excluding inflation, are also identical to the ones defined in the single period framework. The claims development model proposed by Walhin et al. (2001) is considered here. In what follows all notations and all values of loss distribution parameters, XL cover and claims development detailed in part "9. Numerical Application" of Walhin et al (2001) will apply here. A reinsurance portfolio of

M yearly-based XL treaties is considered ($M = 1, 10, 100, 1000$). All treaties of the portfolio are deemed the same, meaning they have an Horizon equal to 8 years (see Walhin et al., 2001). Monte-Carlo simulations are again performed : 20 000 simulations also allow to reach convergence.

Inflation dependence structure The impact of claims inflation dependence on the reinsurer capital allocation need is studied. Stochastic inflation is also introduced, with $N_{scenarios}$ scenarios of inflation are considered. In practice, an $Horizon \times N_{scenarios}$ data frame of forecast monthly short-term inflation rate, output from previous Block 2, is imported . Integration of inflation rates is then performed. Recall the following capitalisation and discount factors between t and T :

$$\begin{aligned} U(t,T) &= exp(\int_t^T f_x \, \mathrm{d}x) \\ V(t,T) &= \frac{1}{U(t,T)} \end{aligned} \tag{4.5.5}$$

where f_x could be either the monthly risk-free rate or inflation rate. Then these factors might be computed in R by using the concept of Riemann sum, i.e. approximating the area under a curve by dividing the area into rectangles and adding up the areas of the rectangles - see Burden and Faires for more details about the definition of Riemann sum (Burden and Faires, 2000) :

$$\begin{aligned} U(t,T) &= exp(\sum_{i=1}^{n} f(x_i^*) \, \Delta x_i) \\ V(t,T) &= \frac{1}{U(t,T)} \end{aligned} \tag{4.5.6}$$

where $f(x_i^*)$ is the height of the i^{th} rectangle, Δx_i is its width, and n is the number of rectangles used to approximate the area under the curve defined by the integral $\int_t^T f_x \, \mathrm{d}x$. Since either the risk-free rate or the inflation rate are monthly rates, they should be taken as input with Δx_i set to $\frac{1}{12}$. This allows to get any capitalisation and discount factors between two dates.

4.5.3.2 Case 1 : Portfolio of Pure XL treaties, with no clause

Pure Premium Calculations The pure premium are calculated according to the expected value premium principle. The obtained premiums are then summed over the entire portfolio. Considering the ultimate aggregate cumulative payments, this gives :

$$Premium_M = \sum_{m=1}^{M} \mathbb{E}(S^{(m)}_{X^{\Sigma Re}}(t_n)) \tag{4.5.7}$$

Allocated Capital Denote $S^{M}_{X^{\Sigma Re}}(t_n)$ the distribution of the ultimate reinsured aggregate cumulative payments, the initial value of capital need $u(t_1)$ is given as follows:

$$u(t_1) = \rho(S^{M}_{X^{\Sigma Re}}(t_n) - Premium_M) \tag{4.5.8}$$

This initial capital need is then amortised at the speed of claims payment, that is, for times of payment occurrence $t_1, t_2, \ldots, t_n$:

$$u(t_j) = u(t_1) \left[1 - c^{\Sigma}(t_j)\right] \tag{4.5.9}$$

where $c^{\Sigma}(t_j)$ is the j^{th} cumulative payment pattern $c(t_j)$.

4.5.3.3 Case 2 : Portfolio of XL treaties with annual aggregate liability clause

Annual aggregate liability This clause has the same definition as under the single period framework, except again we deal with the ultimate aggregate cumulative payments $S_{X^{\Sigma Re}}(t_n)$. Note that $S_{X^{\Sigma Re}}(t_n)$, as defined in Walhin et al (2001), already captures the annual aggregate liability clause. If this clause is on, denote the ultimate aggregate cumulative payments $S^{(m)}_{Aadl,X^{\Sigma Re}}(t_n) = S^{(m)}_{Aadl}$ for each m^{th} treaty. In total, the reinsurer covering all treaties of its portfolio including this clause is liable for the following claims :

$$S_{M,Aadl} = \sum_{m=1}^{M} S^{(m)}_{Aadl} \tag{4.5.10}$$

Pure Premium Calculations The pure premium per treaty is derived using the expected value premium principle for each treaty :

$$Premium^{(m)}_{Aadl} = \mathbb{E}(S^{(m)}_{Aadl}) \tag{4.5.11}$$

Then these premium are summed over the entire portfolio to end up with the total pure premium of the portfolio :

$$Premium_{Aadl,M} = \sum_{m=1}^{M} Premium^{(m)}_{Aadl} \tag{4.5.12}$$

Allocated Capital Denote $S_{M,Aadl}$ the distribution of the ultimate reinsured aggregate cumulative payments with the Aad/Aal clauses, the initial value of capital need $u(t_1)$ is given as follows:

$$u(t_1) = \rho(S_{M,Aadl} - Premium_M) \tag{4.5.13}$$

Then this initial capital need is again amortised at the speed of claims payment throughout the lifetime of claims.

4.5.3.4 Case 3 : Portfolio of XL treaties with paid reinstatements

Paid reinstatements The clause is defined as under the Single Period Framework, except we deal with the ultimate aggregate cumulative payments $S_{X^{\Sigma Re}}(t_n)$. The random reinstatement premiums depend on $S_{X^{\Sigma Re}}(t_n)$ and are given by:

$$P_{reins,rand} = \frac{P_{reins,init}}{l} \sum_{k=1}^{K} c_k min(l, max(0, S_{X^{\Sigma Re}}(t_n) - (k-1)l)) \tag{4.5.14}$$

with c_k the fraction of the initial premium $P_{reins,init}$ for the K^{th} reinstatements. The retained risk of the reinsurer is :

$$S_{reins} = min(max(S_{X^{\Sigma Re}}(t_n) - P, 0), L) - P_{reins,rand} \tag{4.5.15}$$

In what follows again we consider K equal to 1 (thus L = 2l for comparability purpose with the annual aggregate liability) and c_1 equal to 100%. In other words we consider a single reinstatement @100%, which is a commonly observed clause in the market.

Pure Premium Calculations The methodology based on pure premium principle developed by Sundt (1993) and discussed by Walhin et al (2001) is again considered, for each treaty.

Allocated Capital Denote $S_{M,reins}$ the distribution of the ultimate reinsured aggregate cumulative payments with the reinstatement clauses, the initial value of capital need $u(t_1)$ is given as follows:

$$u(t_1) = \rho(S_{M,reins} - Premium_{reins,M}) \tag{4.5.16}$$

Then this initial capital need is again amortised at the speed of claims payment throughout the lifetime of claims.

4.5.3.5 Case 4 : Portfolio of XL treaties with sliding scale premium

Sliding Scale Premium The methodology based on pure premium principle discussed by Walhin et al. (2001) and by Campana et al. (2022) is used for each treaty, except we consider the value of the ultimate aggregate cumulative payments $S_{X^{\Sigma Re}}(t_n)$. In such case, the reinsurance premium becomes :

$$P_{ss} = \begin{cases} P_{ss,min} & if \quad S_{X^{\Sigma Re}}(t_n) \leq \frac{P_{ss,min}}{f} \\ f\, S_{X^{\Sigma Re}}(t_n) & if \quad \frac{P_{ss,min}}{f} < S_{X^{\Sigma Re}}(t_n) < \frac{P_{ss,max}}{f} \\ P_{ss,max} & if \quad S_{X^{\Sigma Re}}(t_n) \geq \frac{P_{ss,max}}{f} \end{cases} \tag{4.5.17}$$

with f a loading coefficient representing the reinsurer (admin and claims) expenses and cost-of-capital. In our study we again set $f = \frac{100}{80}$. The random part of this reinsurance premium is then $P_{ss,rand} = P_{ss} - P_{ss,min}$. The retained risk of the reinsurer is :

$$S_{ss} = S_{X^{\Sigma Re}}(t_n) - P_{ss,rand} \tag{4.5.18}$$

Again, due to its random part, the initial pure premium per treaty cannot be directly derived. The following methodology is followed : $P_{ss,min}$ is fixed (we set $P^{(m)}_{ss,min} = \frac{Premium^{(m)}}{2}$) and $P_{ss,max}$ is found such as the average premium for a treaty with the sliding scale equals the premium in a pure treaty.

Allocated Capital Denote $S_{M,ss}$ the distribution of the ultimate reinsured aggregate cumulative payments with the sliding scale clauses, the initial value of capital need $u(t_1)$ is given as follows:

$$u(t_1) = \rho(S_{M,ss} - Premium_{ss,init,M}) \tag{4.5.19}$$

Then this initial capital need is again amortised at the speed of claims payment throughout the lifetime of the claims.

4.5.3.6 Intermediary findings

Section D.7 of Appendix D shows the outcome of our study. The results show again that certain clauses have a greater impact on capital needs than others, and that the diversification effect obtained by increasing the number of treaties reduces capital needs. The study also finds that stochastic inflation structures, risk measures, and dependence between risk severities and frequencies affect capital needs. The use of the ES risk measure increases the sensitivity to the presence of stochastic inflation structure. Finally, the study shows that dependence between risks due to exogenous factors captured by copulas may significantly impact capital needs.

4.6 BLOCK 5 : Building Block Pricing Model

4.6.1 Approach

Block 5 is the final block, which consists of a transformation of the initial block (Block 1) by including Blocks 2 to 4 - See again Scheme 4.1.1. Such exercise requires some preliminary steps before being ready for analyses. The following section 4.6.2 details such steps. The next ones exploit the capabilities of the model. Section 4.6.3 deals with the sensibility of the pricer with respect to both stochastic interest and inflation rates. Section 4.6.4 deals with the sensibility of the pricer with respect to stochastic CPP. In section 4.6.5, we search for the 'market' parameter to which the reinsurance commercial price is the most sensitive, according to the pricer. Finally, in section 4.6.6, we look on how the pricing model could lead or not to 'arbitrage' opportunities in prices of XL cover with market-used clauses, and we offer some food for thought to avoid these pricing pitfalls. In each section, all our findings are thoroughly analysed.

4.6.2 Preliminary steps and first insights

4.6.2.1 Integration of rates and CPP stochastic scenarios

Integration of macroeconomic rates Both stochastic risk-free interest and inflation rates are introduced: a number of scenarios of these rates, denoted $N_{scenarios}$, is considered. In practice, $Horizon \times N_{scenarios}$ data frames of forecast monthly risk-free interest rates and inflation rates, which are the output data from previous Block 2, are imported to Block 5. In this section the Horizon is now set to 30 years. Recall from section 4.4 it is the Horizon H found from Block 3. Capitalisation and discount factors between two dates are obtained by using the same technique as the one described in section 4.5.3.1.

Integration of CPP In practice, a CPP $Horizon \times N_{scenarios}$ data frame, computed from Block 3, is directly 'ready for use'. Thus it is directly imported from Block 3 to Block 5.

4.6.2.2 Data update

Complementary to the aforementioned data integration, some data are changed compared to the ones originally set in Block 1. Table E.1.1 of Appendix E details the 'dummy' dataset set in Block 5. Note that, in what follows, 3 layers will be studied : we consider a 3 XS 3 MEUR (denoted layer 1), a 3 XS 6 MEUR (denoted layer 2) and finally an INF XS 6 MEUR (denoted layer 3). Also note the cost-of-capital is not arbitrary set anymore. It is derived according to the Capital Asset Pricing Model (CAPM), and more specifically thanks the formula $r + \beta \times P_R$, where r is the risk-free rate from Block 3, β the unlevered beta corrected for cash, i.e. the average reinsurance company Belgian market sensitivity without the effects of company's debt and cash factors, and P_R the equity risk premium of the Belgian market. According to Prof. Damodaran, these parameters are worth $\beta = 1.042$ and $P_R = 6.97\%$ on

January 1st 2023 (Damodaran Home Page, 2023). We will assume these parameters to be constant in the future years, which is a crude approximation.

In the following sections, we will consider 7 reinsurance portfolios of M identical treaties :

- A portfolio of 'pure' treaties, without any clause (pure treaties with no stability clause);
- A portfolio of treaties with stability clause, but without any other clauses (pure treaties);
- A portfolio of treaties with stability clause, but also with an Annual Aggregate Deductible (AAD). AAD is set to the priority;
- A portfolio of treaties with stability clause, but also with an Annual Aggregate Limit (AAL). AAL set to twice the limit;
- A portfolio of treaties with stability clause, but also with both an Annual Aggregate Deductible (AAD) and Limit (AAL). AAD is set to the priority and AAL set to twice the limit;
- A portfolio of treaties with stability clause, but also with 1 paid reinstatement @100%;
- A portfolio of treaties with stability clause, but also with a sliding scale with $f = \frac{100}{80}$ and $P_{ss,min} = P_{ss,init}$ set to 2%, active from 4 years onwards.

M will be initially set to 100. Note that dependence in exogenous factors others that macroeconomic rates, modelled by copulas, is hard to capture in practice when pricing a treaty. It is out-of-scope of our study in this section.

4.6.2.3 Multivariate Monte-Carlo simulations

Amongst the input parameters of the model, we now have 3 stochastic parameters : the nominal interest risk-free rate, the inflation rate and the CPP. In the Block 5 model, the idea is still to apply resampling techniques, and more specifically Monte-Carlo simulations, on the 3 stochastic parameters.

For doing so, we firstly set a number of simulations, denoted $Nsim$, and then create 3 random numbers b_{rfr}, b_{infl}, b_{pp}, each of them following an univariate distribution $U(1, N_{sim})$. By rounding down the obtained values, a random rank for each of the stochastic parameter scenarios is obtained. Then, for a given set of scenarios, we simulate the values of $S_{X^{\Sigma Re}}$ and $S_{I^{Re}}$, then we deduce Paid and Reserve (recall section 4.2), for each layer. Note that, for sake of comparison between layers, the R code is implemented such that random sampling of the claims frequency and severity is drawn once for all layers. From these distributions, we get the best estimates of the related pure premium, technico-financial premium, and commercial premium.

4.6.2.4 First insights

In this section, we provide a first illustrative example of the rates obtained thanks to the pricer. Recall we consider a portfolio of 100 pure treaties and the input data from Table E.1.1

of Appendix E. The considered risk measure is the Value-at-Risk at 99.5%. Table 4.6.1 below provides an overview of the obtained quotes for layer 1 :

	TR [A] [%]	TR_{disc} [%]	TFR [%]	CR [B] [%]	Multiple [B]/[A] [%]
Quotes	5.71	5.51	6.16	11.52	2.01

Table 4.6.1: Quotes obtained for Layer 1, thanks to our prudent capital allocation

Clearly the obtained multiple for this layer is too high, as the market generally agrees on multiples between 1 and 1.5. We identified that this was due to the capital allocation method, which is too prudent. On one hand, the capital allocation is calculated based on undiscounted ultimate reinsured aggregate cumulative payments. This initial capital need is initially requested to shareholders and then is amortised at the speed of claims payments over 30 years. However the speed of payments only quickens from Year 15 onwards. Therefore, considering the amount of cost-of-capital, the amortised capital release is too low compared to what was initially required. Thus this costly capital need has to be balanced by a higher commercial premium. On the other hand, recall the used risk measures are inherited from ruin theory, the capital obtained from these risk measures is a buffer used to cover both the premiums and reserves risks (we refer to Trufin et al., 2011, for more details). Experience gained from actuaries working in reinsurance industry has enabled us to learn that, in pricing, the reserve risk is withdrawn from capital need as the claims manager estimates improve over time. Since we set the overstatement pattern is out-of-scope of this study (as we set it to 100% for all time t_j, assuming the claims manager is correct from year 1), we will have to use a proxy for considering such issue. Consequently, as a crude approximation, we will consider a capital release of 90% at time 1. We will discuss this crude approximation later in chapter 5. Table 4.6.2 shows the new quotes after correction.

	TR [A] [%]	TR_{disc} [%]	TFR [%]	CR [B] [%]	Multiple [B]/[A] [%]
Quotes	5.71	5.51	6.16	8.39	1.47

Table 4.6.2: Quotes with 90% capital release after 1 year

The updated commercial rate falls now within marketable ranges. From this, we can already conclude that our pricing model both validates and quantifies how the XL reinsurance commercial price for 'long-tail' business is sensitive to the reinsurer capital allocation method. This also confirms such type of business is very expensive in reinsurer solvency capital: a very conservative methodology, like the one we previously applied, may lead to off market prices, triggering the underwriting risk. We will keep this percentage of capital release after 1 year to perform the remaining studies.

4.6.3 Step 1 : Sensitivity to the stochastic rates approach

Question For each layer considered, is the pricer sensitive to the presence of stochastic rates ? What is the impact of clauses on this matter ?

Strategy All other parameters being set and the average CPP taken as a deterministic input, we aim to compare the best estimates of technical, technico-financial and commercial rates obtained thanks to stochastic rates (denoted scenario S1) with the pure, technico-financial and commercial rates obtained thanks to deterministic average rates scenarios (denoted scenario D). The considered risk measure is the Value-at-Risk at 99.5%.

Findings Tables E.2.1, E.2.2 and E.2.3 of Appendix E show the obtained results for each layer. Firstly note that, regardless of the layer considered, the values of the obtained rates (pure, technico-financial or commercial rates), for each type of treaty, are in line with was we expect. The presence of clauses limiting the liability of reinsurer indeed decreases the price of reinsurance since the reinsured expected loss decreases. Specifically, for layer 3 (INF XS 6 MEUR), which may be considered as an 'extended' case of layer 2 (3 XS 6 MEUR), the obtained rates are deemed logical : the treaty with AAL indeed asks the same premiums as the pure treaty, and the treaty with AAD asks the same premiums as the ones with both AAD and AAL. This is due to the fact that there is no limit set thus the AAL clause is off. Note that, for layer 3, a treaty with a paid reinstatement cannot be priced since no reinstatement is allowed in this case.
We observe that macroeconomic rates stochasticity increases the pure, technico-financial or commercial rates, whatever the layer under consideration. Such finding could be interpreted as an illustrative example of Jensen's inequality, recall from equation 4.5.5 that both capitalisation and discount factors are expressed with an exponential - and thus convex - function. However, the analytical proof of such an inequality in a general case is not straightforward and would deserve a thorough study.
The impacts of stochastic macroeconomic rates on both pure and technico-financial rates are very different depending on the layer considered : there is a slight impact for layer 1 (below 1%), whereas a more significant one for the layer 2 (up to around 2%) and even more significant one for layer 3 (up to around 3%). The impacts tend to increase as both the priority and limit increase. Similarly, for a given layer, a treaty with a clause limiting the liability of the reinsurer is less sensitive to stochastic rates. All these observations are due to the presence of a limit in the first two layers, which by definition caps the value of inflated reinsured aggregate amount. Following the same logic, the absence of index clause seems to reduce the impact of inflation rates stochasticity : index clause indeed inflates both priority and limit each future development year. Note the trend is less clear-cut in this respect for layer 1, due to the aforementioned low impact of stochastic macroeconomic rates (below 1%), which may be blurred by an 'imperfect' convergence (recall from Block 1 - section 4.2 - that our convergence criteria is met when the TR relative convergence gap is below 1%).
The commercial rate is more impacted than the pure or technico-financial ones. Recall, from section 4.5, that a stochastic inflation rate significantly affects the reinsurer capital need. Regardless of the layer considered, the sliding scale clause is the one of the most sensitive to rates stochasticity. This is again in line with the findings from section 4.5 but also from Walhin (Walhin, 2003). The impact of macroeconomic rates stochasticity on commercial premium is also more significant for layer 3 than for layer 2, and more significant for layer 2 than for layer 1. For layer 1, the difference between a stochastic and a deterministic approach is quite negligible (below 1%). This is in line with Kladivko et al. (Kladivko, K. and Zimmermann, P., 2014). However, when we stand in higher layers, thus we look at higher quantiles of the Pareto distribution, the relative impact of stochastic rates has an higher absolute impact on claims severity, and may explain why layer 2 shows higher gaps between scenario S1 and scenario D. As no limit is set in layer 3, some severe inflation scenarios hit the highest quantile

of Pareto distribution. Consequently, the tail of the severity distribution is amplified without being capped. This is also in line with Fackler (Fackler, 2011). The initial reinsurer capital need, calculated on the basis of the ultimate aggregate cumulative payments, is thus higher for layer 3 than for layer 2, and higher for layer 2 than for layer 1. Finally note that the treaty with a paid reinstatement is less sensitive to scenario S1 than its 'equivalent' treaty with AAL, since this clause reduces volatility in the reinsurer net loss cost by returning volatility to the ceding company.

4.6.4 Step 2 : Sensitivity to the stochastic CPP approach

Question For each layer considered, is the pricer sensitive to the presence of stochastic CPP ? What is the impact of clauses on this matter ?

Strategy All other parameters being set, and the average interest and inflation rates scenarios taken as input, we aim to compare the best estimates of pure, technico-financial and commercial rates obtained thanks to stochastic CPP (denoted S2) with the pure, technico-financial and commercial rates obtained thanks to a deterministic average CPP scenario (denoted D). The considered risk measure is still the Value-at-Risk at 99.5%.

Findings Tables E.3.1, E.3.2 and E.3.3 of Appendix E show the obtained results for each layer. The first comments made in Step 1 still apply here. The values of the obtained rates for each type of treaty are still in line with our expectation. Limit cases from layer 3 are still as expected.
On top of this, we also observe that CPP stochasticity increases the pure, technico-financial or commercial rates, regardless of the layer considered. Such outcome could also be interpreted as an illustrative example of a multivariate Jensen's inequality, although this inequality is less intuitive than in step 1, and would also deserve a rigorous proof.
The impact of CPP stochasticity on both pure and technico-financial rates is also different depending on the considered layer : there is a negligible impact for layer 1 (below 1%), whereas a more significant one for the layer 2 (up to around 4%) and even more significant one for layer 3 (up to around 8%). This is also due to the presence of a limit in the two first layers. Stochastic CPP indeed leads to some severe scenario in term of payments, i.e. some cases of very late payments, which impact the inflated claims severity. This increases the total inflated reinsured amount. Layer 1 or 2 have limits which cap this total amount whereas layer 3 does not. For similar reasons, treaties with limits are less impacted by scenario S2 than others. The difference in impact between layers is much greater for this step than for step 1 but it is difficult to give the main reason for this observation. This is indeed due to many factors : the used CPP model, the horizon considered...
The commercial rate is also more impacted than pure or technico-financial ones, recall from section 4.5 that the initial reinsurer capital need is amortised at the speed of payment pattern. On top of what has already been analysed for both pure and technico-financial rates, the stochastic CPP model captures late payments scenarios which then leads to slower capital amortisation. Mechanically this increases the commercial rates. Layer 3 is more impacted than layer 2, which is more impact than layer 1, since the initial reinsurer capital need is calculated on the basis of the reinsured ultimate aggregate cumulative payments. Consequently, the initial reinsurer capital need is again higher for layer 3 than for layer 2, and higher for layer 2 than for layer 1.

4.6.5 Step 3 : Search for the 'market' parameter to which the price is most sensitive

Question What is the 'market' parameter to which the pricer is the most sensitive ? What is the impact of clauses on this matter ?

Strategy All other parameters are set as described in Table E.1.1 of Appendix E, and the average rates and CPP scenarios are taken as input. The scenario with Value-at-Risk at 99.5% and M = 100 is denoted scenario A. We will consider the following sensitivity analyses :

1. The risk measure will be changed to from Value-at-Risk at 99.5% to Expected Shortfall at 99% (scenario B).

2. The size of the portfolio : M initially taken at 100 will be set to 200 (scenario C)

Findings Tables E.4.1, E.4.2 and E.4.3 of Appendix E show the obtained results for each layer.
Again, the first comments made in Step 1 still apply here. The values of the obtained rates, for each type of treaty, are still in line with our expectation. Besides, we observe both scenarios B and C do not impact the technical rate, and the technico-financial rate. This is logical since our scenarios only impact the reinsurer capital need in our model, and thus, since all these rates do not depend on the reinsurer allocated capital (recall section 4.2), they are not affected.
For all layers, the commercial rate is well impacted by up to round 3%, when the reinsurer portfolio size doubles. This is a logical result since the size of reinsurer portfolio directly impacts the reinsurer capital need and its allocation through years. As discussed in section 4.5, the higher the reinsurer portfolio size is, the less initial capital buffer is needed. This is simply due to mutualisation benefits. This is also consistent with the intuitive idea that a reinsurer will be more market competitive as a large-size undertaking. While remaining consistent with section 4.5, the mutualisation effect is on overall improved by the presence of any clauses, despite this trend is less clear for treaties with limit such as these with AAL or paid reinstatements. Thus the related commercial quotes offered by a large-size reinsurer would be even more attractive than the ones offered by smaller companies.
Note that, for all kind of treaty and for layer 3, the impact of reinsurer size on commercial quotes is higher than or similar to the ones of stochastic macroeconomic rates or CPP from previous steps. This emphasizes the need for a reinsurer to carefully look at rates and CPP stochasticity, because this seems to have a higher or similar impact on quotes than the benefit it could get by having a large portfolio.

4.6.6 Step 4 : Search for arbitrage opportunities in the prices of treaties with random premium-making clauses

4.6.6.1 Paid reinstatements

Background and questions In his book "La réassurance" (2012), J.F. Walhin considered an example of arbitrage opportunity in the context of XL cover per event, with paid reinstatements clause. Assume the following treaty in 4 layers (denoted Option 1) and suppose we ask reinsurers to price the same cover but in a single layer (denoted Option 2). Table

E.5.1 (Appendix E) shows the quotes obtained according to Walhin (Walhin, 2012). First option quotes a total sum of 6.7%, which is the same quote as the second option. J.F. Walhin reported that he received from reinsurers commercial rates which were sometimes higher, sometimes lower than this quote. However, it seems logical to think that the option 2 is more favorable to cedant, because the cedant reported claims will have well more chance to hit the limits of the first layers in the first option, and thus pay reinstatement premium than in the second option, in which the limit is very high. Therefore option 2 commercial rate should theoretically quotes higher. Questions are then the following ones: thanks to our stochastic pricing model, do we also come to this conclusion ? How to feed the stochastic model to create an absence of arbitrage opportunity ?

Strategy We simulate a similar situation as the one described above. For achieving this, we consider a portfolio of 100 treaties with a paid reinstatement @100%. The same dataset from Table E.1.1 of Appendix E is used, except the Pareto amplitude (A) is set to 1.5 MEUR to suit the exercise needs. Stochastic rates and CPP scenarios are taken as input. The considered reinsurance portfolio size is still $M = 100$ and the risk measure is still the Value-at-Risk @99.5%.

Findings Table E.5.2 of Appendix E below shows the obtained results. For the treaty with 4 layers, the total sum of technical rate is worth 3.30 % and is lower than the technical rate of the 'equivalent' treaty (3.78 %). This makes sense since the expected loss for a reinsurer is higher in the second option that in the first one, and this is due to the higher reinstated limit in this case. On the contrary, the sum of commercial quotes for the treaty with 4 layers is higher (5.26 %) than the quote for the treaty with a single layer equivalent to the 4 ones (4.87 %). This is illogical because one expects cedant reported claims will have well more chance to hit the limits of the first layer in the first option than in the second one, and thus pay higher reinstatement premium than in the second option, in which the limit and thus reinstatement trigger is very high. This clearly shows our pricing model would lead to an 'arbitrage' opportunity. In other words, in such situation, we end up with a pricing pitfall: a ceding company would be well advised to choose the second option. An efficient way to avoid this pitfall may be to adapt the fixed expenses amount - recall from section 4.2, it is a type of the administrative expenses AE - to each considered layer, since the commercial rates is sensitive to this parameter. It seems indeed logical that the amount of fixed administrative expenses will not be the same depending on the layer considered.

4.6.6.2 Sliding scale

Background and question As stated during the literature review in chapter 3, for treaties with a sliding scale clause, Levi and Walhin agreed that ignoring the fact that the reinsurance premium is not fully paid at the beginning of the contract may lead to erroneous commercial rates (see Levi, 1988 and Walhin, 2003). According to Walhin et al., such XL premium rates are also very sensitive to the contractually agreed first year of premium adjustment of the sliding scale clause (Walhin et al., 2001). Thanks to our stochastic pricer, do we agree with these conclusions ?

Strategy We will follow the methodology from Walhin (2003), but by taking the data set exhibited in Appendix E. The considered reinsurance portfolio size is again $M = 100$ and the risk measure is the Value-at-Risk at 99.5%. The first year for which a premium adjustment occurs - pa_{start} - is set to 3, $P_{min} = P_{init} = 5\%$ and $f = \frac{100}{80}$, as in Walhin (Walhin, 2003).

Findings As previously performed, the optimal maximum rate is obtained by iterations, such that the NPV of the cash-flow model is equal to 0. We obtain an optimum of 7.22%. Total commercial premium ignoring the financial discount on this premium is worth 1 311 501 EUR. If we develop a 'wrong' cash-flow model, which ignores the financial impact, we obtain a positive NPV for this model, equal to 7 556 EUR. In other words, ignoring the financial discount on the commercial premium would lead to incorrect underwriting decision, and in such situation, we also end up with a pricing pitfall. This is in line with the conclusions from the aforementioned papers (Levi, 1988 and Walhin, 2003).

Chapter 5

Conclusions

Wrap-up In this study, we developed a comprehensive cash flow-based simulation model in order to price excess-of-loss reinsurance contracts, respectively without and with the presence of the main marketable clauses. We were able to study and discuss the findings of papers about the importance of macroeconomic factors and their volatility on long-tail XL reinsurance pricing. According to our model, we found that a volatile interest and inflation rate environment seems to justify the growth in quotes often negotiated by reinsurers during recent renewal times, especially in higher layers. In such environment, some marketable clauses leverages the reinsurance price, and this could explain why certain clauses have recently been severely reviewed. Despite a lack of literature on the matter, we proposed a methodology for calibrating and modeling stochastic cedant claims payment pattern, and we conclude that the volatility of such factor has a more significant impact than the one of stochastic rates. The scarcity of literature with respect to reinsurer capital allocation modelling led us to propose a prudent method on this matter. We observed that reinsurance prices were very sensitive to the method used. We conclude that 'long-tail' LoB are very expensive in reinsurer solvency capital and that over prudent capital allocation methodology leads to off market prices. There is no doubt that, in the current European prudential context such as Solvency II, this is one of the key factors in long-tail XL reinsurance pricing.

Closing remarks The model developed in this study could be improved by several ways. A first simple idea could be to test the sensitivity of the insurance portfolio risk model, by proposing some other distributions for both cedant claims frequency and severity. Then other short-term rate models could be tried, a finer dependence structure than simple correlation between risk-free rate and inflation could be taken into account, and even projections of beta and risk premium in the *coc* model could be performed. The proposed methodology for CPP should be challenged. It would be worth to analyse the impact of such changes on reinsurance premiums, and the influence of clauses on these. Some new studies could be conducted based on this model. Amongst them, the research of an optimal reinsurer capital allocation method based on the incurred losses of the ceding company should be conducted on a priority basis. The influence of the overstatement pattern of the ceding company on the reinsurer capital buffer should be carefully analysed. An optimal and regulatory compliant capital allocation method would lead to competitive quotes while dampening the reinsurer underwriting risk.

Appendices

Appendix A

BLOCK 1 - Original pricing model

A.1 Overview of dummy data used

Elements for the pricing	**Parameters**	**Value**
Distribution of Claim Amounts (Pareto)	A (amplitude)	400
Distribution of Claim Amounts (Pareto)	Alpha (shape)	1.50
Distribution of Claim Numbers (Poisson)	Lamdba (mean frequency)	2.50
Claims payment pattern	c (completely developed after 7 years)	(5%,10%,10%,10%,25%, 25%,25%,10%,5%)
Overstatement pattern of the ceding company	d (completely developed after 7 years)	(125%,125%,125%,125%, 105%,105%,100%,100%)
Future inflation	infl (geometric growth)	3%
Superimposed inflation	supinfl (geometric growth)	4.5%
Interest rate obtained on the loss reserve	r	5%
Return obtained on the allocated capital	l	7%
Cost-of-capital	coc	11%
Allocated capital	C(j) (allocated for 3 years)	1.25*standard deviation of the ultimate aggregate claims
Priority of the treaty	P	500
Limit of the treaty	L	2500
Margin on the date of payment stability clause	Applied on the priority and on the limit - based on incurred losses	10%
Interests sharing clause	Delta (portion of interests in the losses)	15%
Estimated premium income	Cedant's EPI	50 000
Share of the reinsurer in the treaty	Share	20%
Brokerage	B	10%
Minimum and deposit premium	MDP (at time 0)	80%
Retrocession costs	Percentage of the commercial premium	3%
Average Paid Claims by the retrocession	Percentage of the claims	2%
Administrative expenses	Fixed part	5
Administrative expenses	Variable part (percentage of the paid losses each year)	4%
Tax rate	Average tax rate	30%

Table A.1.1: Block 1 - Overview of dummy data used

Appendix B

BLOCK 2 - Vasiceck Maximum Likelihood Estimators and Variance

The following demonstrations are based on Gerebrink et al. (Gerebrink et al., 2018). To derive the maximum likelihood estimators for the Vasicek model, we start by simplifying the model and we express the parameters as follows:

$$\begin{aligned} A &= e^{-\lambda t} \\ B &= \mu\left(1 - e^{-\lambda t}\right) \\ C &= \frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda t}\right). \end{aligned} \tag{B.0.1}$$

Thus the log-likelihood function $L(\theta)$ from equation 4.3.2 can be expressed as follows:

$$L(\theta) = L(A, B, C) = -\frac{n}{2}\log C - \frac{n}{2}\log 2\pi - \frac{1}{2C}\sum_{i=1}^{n}\left(i_{t_{i+1}} - Ai_{t_i} - B\right)^2 \tag{B.0.2}$$

Then partial derivations are used to determine the Vasicek parameters λ, μ and σ^2 :

$$\begin{aligned} \frac{\partial L(\theta)}{\partial \lambda} &= \frac{\partial L(\theta)}{\partial A}\frac{\partial A}{\partial \lambda} + \frac{\partial L(\theta)}{\partial B}\frac{\partial B}{\partial \lambda} + \frac{\partial L(\theta)}{\partial C}\frac{\partial C}{\partial \lambda} \\ \frac{\partial L(\theta)}{\partial \mu} &= \frac{\partial L(\theta)}{\partial B}\frac{\partial B}{\partial \mu} \\ \frac{\partial L(\theta)}{\partial \sigma^2} &= \frac{\partial L(\theta)}{\partial C}\frac{\partial C}{\partial \sigma^2}. \end{aligned} \tag{B.0.3}$$

B.1 Proof for σ^2

$$\frac{\partial L(\theta)}{\partial \sigma^2} = \left(-\frac{n}{2C} + \frac{1}{2C^2}\sum_{i=1}^{n}\left(i_{t_{i+1}} - Ai_{t_i} - B\right)^2\right)\frac{1}{2\lambda}\left(1 - e^{-2\lambda t}\right) = 0 \tag{B.1.1}$$

Since $\lambda \neq 0$ by definition of the Vasicek model, we end up with :

$$C = \frac{1}{n}\sum_{i=1}^{n}\left(i_{t_{i+1}} - Ai_{t_i} - B\right)^2 \Rightarrow \hat{\sigma}^2 = \frac{2\lambda}{n\left(1-e^{-2\lambda t}\right)}\sum_{i=1}^{n}\left(i_{t_{i+1}} - Ai_{t_i} - B\right)^2. \tag{B.1.2}$$

B.2 Proof for λ and μ

$$\frac{\partial L(\theta)}{\partial \mu} = \frac{\left(1-e^{-\lambda t}\right)}{C}\sum_{i=1}^{n}\left(i_{t_{i+1}} - Ai_{t_i} - B\right) = 0. \tag{B.2.1}$$

Since $\left(1-e^{-\lambda t}\right) \neq 0$ and $C \neq 0$, this provides :

$$\sum_{i=1}^{n}\left(i_{t_{i+1}} - Ai_{t_i} - B\right) = 0 \tag{B.2.2}$$

It follows from the previous equation :

$$B = \frac{1}{n}\sum_{i=1}^{n}\left(i_{t_{i+1}} - A\, i_{t_i}\right) \tag{B.2.3}$$

Finally for λ:

$$\begin{aligned}\frac{\partial L(\theta)}{\partial \lambda} &= \left(-\frac{te^{-\lambda t}}{C}\right)\sum_{i=1}^{n}\left(i_{t_i}\left(i_{t_{i+1}} - B\right) - Ai_{t_i}^2\right)\\ &+ \left(\frac{1}{C}\sum_{i=1}^{n}\left(i_{t_{i+1}} - Ai_{t_i} - B\right)\right)\frac{\partial B}{\partial \lambda}\\ &+ \left(-\frac{n}{2C} + \frac{1}{2C^2}\sum_{i=1}^{n}\left(i_{t_{i+1}} - Ai_{t_i} - B\right)^2\right)\frac{\partial C}{\partial \lambda}\\ &= 0\end{aligned} \tag{B.2.4}$$

Since the term $-\frac{te^{-\lambda t}}{C} \neq 0$ and considering equation B.2.2 :

$$\sum_{i=1}^{n}\left(i_{t_i}\left(i_{t_{i+1}} - B\right) - Ai_{t_i}^2\right) = 0 \tag{B.2.5}$$

Elaborating the above expression, one is provided with :

$$\sum_{i=1}^{n} i_{t_i} i_{t_{i+1}} = B\sum_{i=1}^{n} i_{t_i} + A\sum_{i=1}^{n} i_{t_i}^2 \tag{B.2.6}$$

Substituting B in previous equation with B from equation B.2.3 :

$$\begin{aligned}
\sum_{i=1}^{n} i_{t_i} i_{t_{i+1}} &= \left(\frac{1}{n}\sum_{i=1}^{n}\left(i_{t_{i+1}} - A i_{t_i}\right)\right)\sum_{i=1}^{n} i_{t_i} + A\sum_{i=1}^{n} i_{t_i}^2 \\
\Rightarrow \sum_{i=1}^{n} i_{t_i} i_{t_{i+1}} - \frac{1}{n}\sum_{i=1}^{n} i_{t_{i+1}} \sum_{i=1}^{n} i_{t_i} &= A\left(\sum_{i=1}^{n} i_{t_i}^2 - \frac{1}{n}\left(\sum_{i=1}^{n} i_{t_i}\right)^2\right) \qquad \text{(B.2.7)} \\
\Rightarrow A &= \frac{n\sum_{i=1}^{n} i_{t_i} i_{t_{i+1}} - \sum_{i=1}^{n} i_{t_i}\cdot\sum_{i=1}^{n} i_{t_{i+1}}}{n\sum_{i=1}^{n} i_{t_i}^2 - \left(\sum_{i=1}^{n} i_{t_i}\right)^2}.
\end{aligned}$$

Recall $A = e^{-\lambda t}$. Thus, the previous equation can be used to express the estimator for λ:

$$\hat{\lambda} = -\frac{1}{t}\log A = -\frac{1}{t}\log\left(\frac{n\sum_{i=1}^{n} i_{t_i} i_{t_{i+1}} - \sum_{i=1}^{n} i_{t_i}\cdot\sum_{i=1}^{n} i_{t_{i+1}}}{n\sum_{i=1}^{n} i_{t_i}^2 - \left(\sum_{i=1}^{n} i_{t_i}\right)^2}\right). \qquad \text{(B.2.8)}$$

Furthermore, since $B = \mu\left(1 - e^{-\lambda t}\right)$ then equation B.2.3 can be used to express μ as a function of λ, namely:

$$\hat{\mu} = \frac{B}{\left(1 - e^{-\lambda t}\right)} = \frac{1}{n\left(1 - e^{-\lambda t}\right)}\left(\sum_{i=1}^{n} i_{t_{i+1}} - e^{-\lambda t}\sum_{i=1}^{n} i_{t_i}\right). \qquad \text{(B.2.9)}$$

B.3 Variance of estimators : negative expected information matrix

By definition of the variance for the maximum likelihood estimators, it follows that the variance of each estimator is the negative expected value of the inverse of the second derivative. The inverse of the variance matrix is commonly called the information matrix (see Denuit et al., 2019, for more details). In the Vasicek model, the negative expected information matrix has the following expression: $-I_{\mathbb{E}}(\lambda, \mu, \sigma) =$

$$\begin{bmatrix}
\frac{2\lambda(\Delta t)^2 e^{-2\lambda\Delta t}\sum_{i=1}^{n}(i_{t_{i-1}}-\mu)^2}{\sigma^2\left(1-e^{-2\lambda\Delta t}\right)} + \frac{n e^{-4\lambda\Delta t}\left(e^{2\lambda\Delta t}-2\lambda\Delta t-1\right)^2}{2\lambda\left(1-e^{-2\lambda\Delta t}\right)^2} & -\frac{2\lambda\Delta t\sum_{i=1}^{n}\left(i_{t_{i-1}}-\mu\right)}{\sigma^2\left(1+e^{\lambda\Delta t}\right)} & -\frac{n\left(1-e^{-2\lambda\Delta t}(2\lambda\Delta t+1)\right)}{\sigma\lambda\left(1-e^{-2\lambda\Delta t}\right)} \\
-\frac{2\lambda\Delta t\sum_{i=1}^{n}\left(i_{t_{i-1}}-\mu\right)}{\sigma^2\left(1+1+e^{\lambda\Delta t}\right)} & \frac{2n\lambda\left(1-e^{-\lambda\Delta t}\right)}{\sigma^2\left(1+e^{-\lambda\Delta t}\right)} & 0 \\
-\frac{n\left(1-e^{-2\lambda\Delta t}(2\lambda\Delta t+1)\right)}{\sigma\lambda\left(1-e^{-2\lambda\Delta t}\right)} & 0 & \frac{2n}{\sigma^2}
\end{bmatrix}^{-1} \qquad \text{(B.3.1)}$$

Proof for $\frac{\partial^2 L(\theta)}{\partial\sigma^2}$

Proof.

$$\begin{aligned}\frac{\partial^2 L(\theta)}{\partial \sigma^2} &= \frac{\partial^2 L(\theta)}{\partial C^2}\left(\frac{\partial C}{\partial \sigma}\right)^2 \\ &= \frac{\partial}{\partial C}\left[-\frac{n}{2C}+\frac{1}{2C^2}\sum_{i=1}^{n}\left(i_{t_{i+1}}-Ai_{t_i}-B\right)^2\right]\left(\frac{\sigma}{\lambda}\left(1-e^{-2\lambda t}\right)\right)^2 \\ &= \left(\frac{\sigma}{\lambda}\left(1-e^{-2\lambda t}\right)\right)^2\left(\frac{n}{2C^2}-\frac{1}{C^3}\sum_{i=1}^{n}\left(i_{t_{i+1}}-Ai_{t_i}-B\right)^2\right).\end{aligned} \tag{B.3.2}$$

The expected value of previous equation then needs to be calculated. Since the only stochastic term in the sum is $i_{t_{i+1}}$, this provides the following:

$$\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial \sigma^2}\right] = \left(\frac{\sigma}{\lambda}\left(1-e^{-2\lambda t}\right)\right)^2\left(\frac{n}{2C^2}-\frac{1}{C^3}\sum_{i=1}^{n}\mathbb{E}\left[\left(i_{t_{i+1}}-Ai_{t_i}-B\right)^2 \mid i_{t_i}\right]\right). \tag{B.3.3}$$

Moreover, at any time t_{t_i} the interest rate i_{t_i} is given and the expected interest rate at time $t_{t_{i+1}}$ depends on the known interest rate i_{t_i}. Hence

$$\begin{aligned}\mathbb{E}\left[\left(i_{t_{i+1}}-Ai_{t_i}-B\right)^2 \mid i_{t_i}\right] &= \mathbb{E}\left[\left(i_{t_{i+1}}-(Ai_{t_i}+B)\right)^2 \mid i_{t_i}\right] \\ &= \mathbb{E}\left[i^2_{t_{i+1}} \mid i_{t_i}\right]-2\left(Ai_{t_i}+B\right)\mathbb{E}\left[i_{t_{i+1}} \mid i_{t_i}\right]+\left(Ai_{t_i}+B\right)^2\end{aligned} \tag{B.3.4}$$

Note that $\mathbb{E}\left[i^2_{t_{i+1}} \mid i_{t_i}\right]$ is the same as the squared mean plus the variance of $i_{t_{i+1}}$. Whilst $\mathbb{E}\left[i_{t_{i+1}} \mid i_{t_i}\right]$ is the mean of $i_{t_{i+1}}$. Hence

$$\mathbb{E}\left[i^2_{t_{i+1}} \mid i_{t_i}\right] = \frac{\sigma^2}{2\lambda}\left(1-e^{-2\lambda t}\right)+\left(i_{t_i}e^{-\lambda t}+r\left(1-e^{-\lambda t}\right)\right)^2 = C+\left(Ai_{t_i}+B\right)^2 \tag{B.3.5}$$

and

$$\mathbb{E}\left[i_{t_{i+1}} \mid i_{t_i}\right] = i_{t_i}e^{-\lambda t}+r\left(1-e^{-\lambda t}\right) = Ai_{t_i}+B \tag{B.3.6}$$

Hence, equation B.3.3 gives us that $\mathbb{E}\left[\left(i_{t_{i+1}}-Ai_{t_i}-B\right)^2 \mid i_{t_i}\right] = C$. Thus, this can be replaced in equation B.3.4, which provides us with:

$$\begin{aligned}\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial \sigma^2}\right] &= \left(\frac{\sigma}{\lambda}\left(1-e^{-2\lambda t}\right)\right)^2\left(\frac{n}{2C^2}-\frac{1}{C^3}\sum_{i=1}^{n}C\right) = -\frac{n}{2C^2}\left(\frac{\sigma}{\lambda}\left(1-e^{-2\lambda t}\right)\right)^2 \\ &= -\frac{n}{2\left(\frac{\sigma^2}{2\lambda}\left(1-e^{-2\lambda t}\right)\right)^2}\left(\frac{\sigma}{\lambda}\left(1-e^{-2\lambda t}\right)\right)^2 = -\frac{2n}{\sigma^2}\end{aligned} \tag{B.3.7}$$

Since the variance is the inverse of the negative expected value of the second derivative, the previous equation then gives:

$$\frac{\partial^2 L(\theta)}{\partial \sigma^2} = -\left(-\frac{1}{\frac{2n}{\sigma^2}}\right) = \frac{\sigma^2}{2n}. \tag{B.3.8}$$

□

Proof for $\frac{\partial^2 L(\theta)}{\partial \mu^2}$

Proof. The variance of $\hat{\mu}$ is calculated in the same manner, namely:

$$\begin{aligned}\frac{\partial^2 L(\theta)}{\partial \mu^2} &= \frac{\partial^2 L(\theta)}{\partial B^2}\left(\frac{\partial B}{\partial \mu}\right)^2 \\ \frac{\partial^2 L(\theta)}{\partial \mu^2} &= \frac{\partial}{\partial B}\left[\frac{1}{C}\sum_{i=1}^{n}\left(i_{t_{i+1}} - Ai_{t_i} - B\right)\right]\left(1-e^{-\lambda t}\right)^2 = -\frac{n}{C}\left(1-e^{-\lambda t}\right)^2.\end{aligned} \tag{B.3.9}$$

Substituting C in previous equation then gives us :

$$\frac{\partial^2 L(\theta)}{\partial \mu^2} = -\frac{2n\lambda\left(1-e^{-\lambda t}\right)^2}{\sigma^2\left(1-e^{-2\lambda t}\right)} \tag{B.3.10}$$

As stated before, the variance is the inverse of the negative expected value of the second derivative and therefore previous equation can be used to finally express the variance as :

$$\frac{\partial^2 L(\theta)}{\partial \mu^2} = -\left(-\frac{1}{\frac{2n\lambda\left(1-e^{-\lambda t}\right)^2}{\sigma^2\left(1-e^{-2\lambda}\right)}}\right) = \frac{\sigma^2\left(1-e^{-2\lambda t}\right)}{2n\lambda\left(1-e^{-\lambda t}\right)^2}. \tag{B.3.11}$$

□

Proof for $\frac{\partial^2 L(\theta)}{\partial \lambda^2}$

Proof. The differentiation for $\frac{\partial^2 L(\theta)}{\partial \lambda^2}$ is more complex and will therefore be done in several steps. First of all observe:

$$\begin{aligned}\frac{\partial^2 L(\theta)}{\partial \lambda^2} &= \frac{\partial^2 L(\theta)}{\partial A^2}\left(\frac{\partial A}{\partial \lambda}\right)^2 + \frac{\partial^2 L(\theta)}{\partial B^2}\left(\frac{\partial B}{\partial \lambda}\right)^2 + \frac{\partial^2 L(\theta)}{\partial C^2}\left(\frac{\partial C}{\partial \lambda}\right)^2 + \\ &2\frac{\partial^2 L(\theta)}{\partial A \partial B}\left(\frac{\partial A}{\partial \lambda}\right)\left(\frac{\partial B}{\partial \lambda}\right) + 2\frac{\partial^2 L(\theta)}{\partial A \partial C}\left(\frac{\partial A}{\partial \lambda}\right)\left(\frac{\partial C}{\partial \lambda}\right) + 2\frac{\partial^2 L(\theta)}{\partial B \partial C}\left(\frac{\partial B}{\partial \lambda}\right)\left(\frac{\partial C}{\partial \lambda}\right).\end{aligned} \tag{B.3.12}$$

Then the expected value of each of these derivatives can be calculated as follows

$$\begin{aligned}\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial A^2}\left(\frac{\partial A}{\partial \lambda}\right)^2\right] &= -\frac{1}{C}\sum_{i=1}^{n}\left(i_{t_i}^2\right)(-tA)^2 \\ \mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial B^2}\left(\frac{\partial B}{\partial \lambda}\right)^2\right] &= -\frac{(\mu t A)^2 n}{C} \\ \mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial C^2}\left(\frac{\partial C}{\partial \lambda}\right)^2\right] &= -\frac{n}{2C^2}\Psi^2\end{aligned} \tag{B.3.13}$$

where $\Psi = \frac{\partial C}{\partial \lambda} = -\frac{\sigma^2 A^2\left(-2\lambda t + A^{-2} - 1\right)}{2\lambda^2}$

$$\begin{aligned}\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial A \partial B}\left(\frac{\partial A}{\partial \lambda}\right)\left(\frac{\partial B}{\partial \lambda}\right)\right] &= \frac{\mu}{C}(tA)^2 \sum_{i=1}^{n} i_{t_i} \\ \mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial A \partial C}\left(\frac{\partial A}{\partial \lambda}\right)\left(\frac{\partial C}{\partial \lambda}\right)\right] = \mathbb{E}&\left[2(tA)\Psi \sum_{i=1}^{n}\left(A i_{t_i}^2 - i_{t_i}\left(i_{t_{i+1}} - B\right)\right)\right] = 0.\end{aligned} \tag{B.3.14}$$

As previously shown, $\mathbb{E}\left[i_{t_{i+1}} \mid i_{t_i}\right] = A i_{t_i} + B$. Thus, the expected value of previous equation will just be zero since :

$$\mathbb{E}\left[\sum_{i=1}^{n}\left(A i_{t_i}^2 - i_{t_i}\left(i_{t_{i+1}} - B\right)\right)\right] = \sum_{i=1}^{n}\left(A i_{t_i}^2 - i_{t_i}\left(A i_{t_i} + B - B\right) = 0\right. \tag{B.3.15}$$

Similarly:

$$\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial B \partial C}\left(\frac{\partial B}{\partial \lambda}\right)\left(\frac{\partial C}{\partial \lambda}\right)\right] = \mathbb{E}\left[(\mu t A)\Psi\left(-\frac{1}{C^2}\right)\sum_{i=1}^{n}\left(i_{t_{i+1}} - A i_{t_i} - B\right)\right] = 0. \tag{B.3.16}$$

Again using the properties of the expected value of $i_{t_{i+1}}$, the summation turns out as follows

$$\mathbb{E}\left[\sum_{i=1}^{n}\left(i_{t_{i+1}} - A i_{t_i} - B\right)\right] = \sum_{i=1}^{n}\left(A i_{t_i} + B - A i_{t_i} - B\right) = 0 \tag{B.3.17}$$

Hence

$$\begin{aligned}\frac{\partial^2 L(\theta)}{\partial \lambda^2} &= \left(\frac{(tA)^2}{C}\sum_{i=1}^{n} i_{t_i}^2 + \frac{n(\mu t A)^2}{C} + \frac{n\Psi^2}{2C^2} - 2\frac{\mu(tA)^2}{C}\sum_{i=1}^{n} i_{t_i}\right)^{-1} \\ &= \left(\frac{(tA)^2}{C}\left(\sum_{i=1}^{n}\left(i_{t_i} - \mu\right)^2 + \frac{n\Psi^2}{2C(tA)^2}\right)\right)^{-1} \\ &= \left(\frac{2\lambda t^2 e^{-2\lambda t}}{\sigma^2\left(1 - e^{-2\lambda t}\right)}\sum_{i=1}^{n}\left(i_{t_i} - \mu\right)^2 + \frac{n e^{-4\lambda t}\left(e^{2\lambda t} - 2\lambda t - 1\right)^2}{2\lambda\left(1 - e^{-2\lambda t}\right)^2}\right)^{-1}.\end{aligned} \tag{B.3.18}$$

□

Proof for $\frac{\partial^2 L(\theta)}{\partial \mu \partial \sigma}$

Proof.

$$\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial \mu \partial \sigma}\right] = \mathbb{E}\left[\frac{\partial}{\partial \mu}\left[\frac{\partial L}{\partial C}\frac{\partial C}{\partial \sigma}\right]\right] = \mathbb{E}\left[\left(\frac{\partial^2 L}{\partial A \partial L}\frac{\partial A}{\partial \mu} + \frac{\partial^2 L}{\partial B \partial C}\frac{\partial B}{\partial \mu} + \frac{\partial^2 L}{\partial C^2}\frac{\partial C}{\partial \mu}\right)\frac{\partial C}{\partial \sigma}\right] \tag{B.3.19}$$

Since neither A nor C contains μ, both $\frac{\partial A}{\partial \mu}$ and $\frac{\partial C}{\partial \mu}$ becomes zero. Furthermore, $\mathbb{E}\left[\frac{\partial^2 L}{\partial B \partial C}\right]$ also equals zero due to equation B.3.16, hence

$$\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial\mu\partial\sigma}\right] = 0. \tag{B.3.20}$$

□

Proof for $\frac{\partial^2 L(\theta)}{\partial\mu\partial\lambda}$

Proof. First, note that $\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial B\partial C}\right] = 0$ from previous sections. Moreover,

$$\begin{aligned}
\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial\mu\partial\lambda}\right] &= \mathbb{E}\left[\frac{\partial}{\partial\lambda}\left[\frac{\partial L(\theta)}{\partial B}\frac{\partial B}{\partial r}\right]\right] \\
&= \mathbb{E}\left[\left(\frac{\partial^2 L(\theta)}{\partial A\partial B}\frac{\partial A}{\partial\lambda} + \frac{\partial^2 L(\theta)}{\partial B^2}\frac{\partial B}{\partial\lambda} + \frac{\partial^2 L(\theta)}{\partial B\partial C}\frac{\partial C}{\partial\lambda}\right)\frac{\partial B}{\partial r}\right] \\
&= \left(\frac{tA}{C}\sum_{i=1}^{n} i_{t_i} - \frac{\mu ntA}{C}\right)\left(1 - e^{-\lambda t}\right) = \frac{tA\left(1 - e^{-\lambda t}\right)}{C}\sum_{i=1}^{n}\left(i_{t_i} - \mu\right) \\
&= \frac{2\lambda te^{-\lambda t}\left(1 - e^{-\lambda t}\right)}{\sigma^2\left(1 - e^{-2\lambda t}\right)}\sum_{i=1}^{n}\left(i_{t_i} - \mu\right) = \frac{2\lambda t}{\sigma^2\left(1 + e^{\lambda t}\right)}\sum_{i=1}^{n}\left(i_{t_i} - \mu\right).
\end{aligned} \tag{B.3.21}$$

Since $I(\hat{\lambda}, \hat{\mu}, \hat{\sigma})$ contains of the negative seconds derivatives, we only need to take the negative value of previous equation and this ends the proof.

□

Proof for $\frac{\partial^2 L(\theta)}{\partial\lambda\partial\sigma}$

Proof. Note from previous sections that $\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial B\partial C}\right] = 0$ and that $\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial A\partial C}\right] = 0$, which will be used later. Also note that this is the derivative with respect to σ and not σ^2. Hence

$$\begin{aligned}
\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial\lambda\partial\sigma}\right] &= \mathbb{E}\left[\frac{\partial}{\partial\lambda}\left[\frac{\partial L(\theta)}{\partial C}\frac{\partial C}{\partial\sigma}\right]\right] \\
&= \mathbb{E}\left[\left(\frac{\partial^2 L(\theta)}{\partial A\partial C}\frac{\partial A}{\partial\lambda} + \frac{\partial^2 L(\theta)}{\partial B\partial C}\frac{\partial B}{\partial\lambda} + \frac{\partial^2 L(\theta)}{\partial C^2}\frac{\partial C}{\partial\lambda}\right)\frac{\partial C}{\partial\sigma}\right].
\end{aligned} \tag{B.3.22}$$

Also note from previous sections that $\mathbb{E}\left[\frac{\partial^2 L(\theta)}{\partial C^2}\right] = -\frac{n}{2C^2}$. Hence :

$$\begin{aligned}
-\frac{n}{2C^2}\Psi\frac{\sigma\left(1 - e^{-2\lambda t}\right)}{\lambda} &= -\frac{n}{2\left(\frac{\sigma^2}{2\lambda}\left(1 - e^{-2\lambda t}\right)\right)^2}\left(-\frac{\sigma^2 e^{-2\lambda t}\left(e^{2\lambda t} - 2\lambda t - 1\right)}{2\lambda^2}\right)\frac{\sigma\left(1 - e^{-2\lambda t}\right)}{\lambda} \\
&= \frac{n}{\sigma\lambda}\frac{1 - e^{-2\lambda t}(2\lambda t + 1)}{1 - e^{-2\lambda t}}
\end{aligned} \tag{B.3.23}$$

where $\Psi = \frac{\partial C}{\partial\lambda} = -\frac{\sigma^2 A^2\left(-2\lambda t + A^{-2} - 1\right)}{2\lambda^2}$ (see previously). This completes the final part of the proof.

□

Appendix C

BLOCK 3 - Run-Off Triangle of As if Belgian MTPL Inflated Claims Paid from 2005-2018

C.1 Deterministic Chain-Ladder Model

Paid inflated

Year	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2005	506,644	3,107,700	7,214,802	9,501,190	13,498,872	16,534,232	20,346,002	28,250,575	30,426,469	33,992,023	38,628,296	43,530,592	44,206,992	47,304,475
2006	94,169	2,053,246	3,960,737	6,075,237	9,316,154	13,578,571	19,438,443	23,243,159	27,317,921	29,422,200	34,052,962	34,664,720	35,726,144	38,229,394
2007	829,538	3,898,961	7,208,932	9,861,732	12,996,215	15,117,859	16,907,188	18,411,612	22,730,529	26,112,214	29,730,932	32,183,353	32,898,601	35,203,731
2008	146,696	1,045,372	2,892,815	3,831,203	6,122,617	10,108,389	11,503,717	14,881,138	16,903,215	20,168,729	21,537,152	23,212,492	23,728,370	25,390,963
2009	488,291	2,120,868	6,163,772	7,908,835	10,653,393	19,138,866	26,449,901	29,270,746	33,073,699	34,284,088	38,739,082	41,752,532	42,680,446	45,670,969
2010	498,896	2,799,265	4,424,833	5,561,538	7,263,692	12,084,055	14,573,029	18,897,212	21,631,951	23,875,112	26,977,528	29,076,066	29,722,256	31,804,827
2011	459,654	2,918,629	4,924,875	6,633,850	8,646,147	9,999,900	18,233,885	19,055,108	21,796,736	24,056,985	27,183,034	29,297,558	29,948,671	32,047,106
2012	125,132	3,723,002	6,231,242	7,932,766	9,929,138	11,612,591	13,189,130	15,730,402	17,993,675	19,859,559	22,440,180	24,185,765	24,723,273	26,455,576
2013	378,530	2,498,948	4,920,957	7,279,408	8,310,401	9,943,084	12,927,341	15,418,172	17,636,521	19,465,370	21,994,768	23,705,706	24,232,545	25,930,464
2014	71,733	1,959,296	4,020,243	6,405,270	6,970,852	9,492,874	12,342,008	14,720,057	16,837,963	18,584,004	20,998,874	22,632,342	23,135,327	24,756,366
2015	537,040	3,930,128	5,175,530	5,919,114	7,813,174	10,639,945	13,833,354	16,498,754	18,872,577	20,829,601	23,536,272	25,367,120	25,930,883	27,747,801
2016	474,476	2,322,265	3,327,149	4,478,424	5,911,477	8,050,223	10,466,369	12,483,020	14,279,064	15,759,755	17,807,633	19,192,860	19,619,405	20,994,092
2017	563,219	1,735,717	3,241,482	4,363,114	5,759,269	7,842,947	10,196,882	12,161,609	13,911,408	15,353,975	17,349,124	18,698,684	19,114,246	20,453,538
2018	111,583	735,691	1,373,917	1,849,326	2,441,092	3,324,268	4,321,994	5,154,752	5,896,412	6,507,851	7,353,504	7,925,521	8,101,659	8,669,324
link ratios		**6.593**	**1.868**	**1.346**	**1.320**	**1.362**	**1.300**	**1.193**	**1.144**	**1.104**	**1.130**	**1.078**	**1.022**	**1.070**

Payment Patterns

Year	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2005	0.00	0.03	0.06	0.08	0.12	0.15	0.18	0.25	0.27	0.30	0.34	0.39	0.39	0.42
2006	0.00	0.02	0.04	0.07	0.10	0.15	0.22	0.26	0.31	0.33	0.38	0.39	0.40	0.43
2007	0.01	0.03	0.06	0.09	0.12	0.13	0.15	0.16	0.20	0.23	0.26	0.29	0.29	0.31
2008	0.00	0.01	0.04	0.05	0.08	0.13	0.15	0.19	0.22	0.26	0.28	0.30	0.30	0.33
2009	0.01	0.02	0.07	0.09	0.12	0.21	0.29	0.32	0.36	0.37	0.42	0.45	0.46	0.50
2010	0.01	0.03	0.05	0.06	0.08	0.14	0.16	0.21	0.24	0.27	0.30	0.33	0.34	0.36
2011	0.00	0.03	0.05	0.07	0.09	0.10	0.18	0.19	0.22	0.24	0.27	0.30	0.30	0.32
2012	0.00	0.03	0.05	0.07	0.09	0.10	0.11	0.14	0.15	0.17	0.19	0.21	0.21	0.23
2013	0.00	0.03	0.06	0.09	0.10	0.12	0.16	0.19	0.22	0.24	0.27	0.29	0.30	0.32
2014	0.00	0.02	0.04	0.07	0.07	0.10	0.13	0.16	0.18	0.20	0.22	0.24	0.25	0.26
2015	0.00	0.03	0.04	0.05	0.07	0.09	0.12	0.14	0.16	0.18	0.20	0.22	0.22	0.24
2016	0.01	0.03	0.04	0.06	0.07	0.10	0.13	0.16	0.18	0.20	0.22	0.24	0.25	0.26
2017	0.01	0.02	0.03	0.04	0.06	0.08	0.10	0.12	0.14	0.15	0.17	0.18	0.19	0.20
2018	0.00	0.02	0.03	0.04	0.05	0.07	0.09	0.11	0.13	0.14	0.16	0.17	0.18	0.19

Figure C.1.1: Deterministic Chain-Ladder Model

Appendix D

BLOCK 4 - Capital Allocation Modelling

D.1 Single Period Framework - Data used

		Value
Loss Distribution Parameters	λ (Poisson)	2.50
	α (Pareto)	1.50
	A (Pareto)	400
Inflation states	(i_1, p_1)	(1.02,0.20)
	(i_2, p_2)	(1.03,0.20)
	(i_3, p_3)	(1.10,0.60)
XL cover	priority (p)	500
	limit (l)	2500

Table D.1.1: Capital Allocation Modelling : single period framework - data used

D.2 Single Period Framework - Monte Carlo simulations

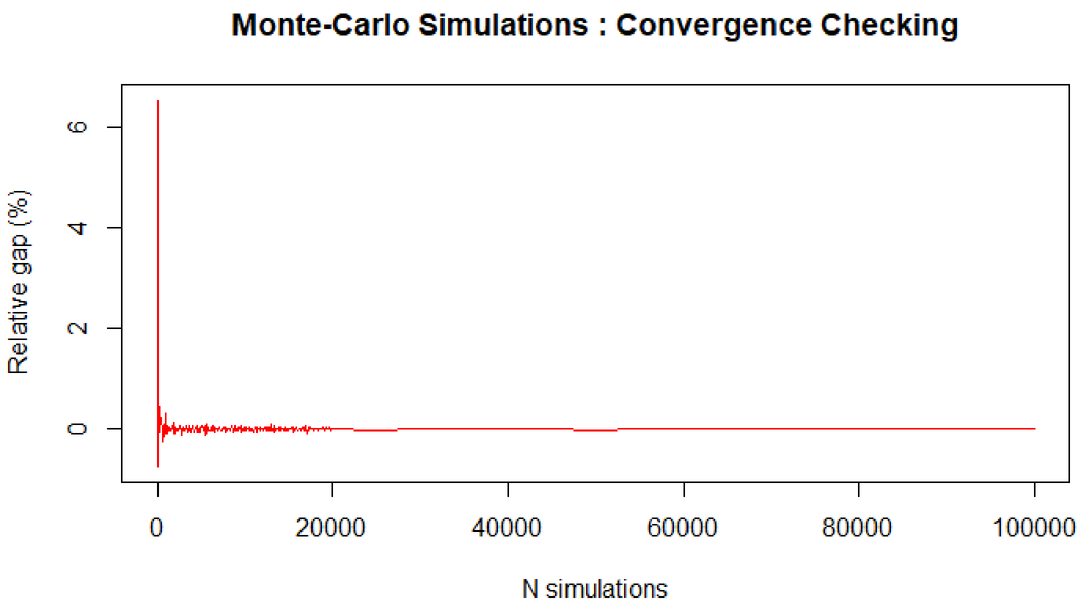

Figure D.2.1: Monte Carlo simulations : convergence checking - case of Pure Treaty, M=1, VaR 99.5%

D.3 Single Period Framework - Case 2 : Annual aggregate liability

D.3.1 Reinsurer total loss

Aad is set to P and Aal is set to 2L. The resulting reinsurer total loss is then :

$$S_{Aadl} = min(Aal, max(0, S - Aad)) \tag{D.3.1}$$

This equation is applied to each reinsurance treaty. The reinsurer covering all its own portfolio with such clauses is liable for the following total claims :

$$S_{M,Aadl} = \sum_{m=1}^{M} S_{Aadl}^{(m)} \tag{D.3.2}$$

D.3.2 Pure Premium Calculations

The pure premium per treaty is derived using the expected value premium principle for each treaty :

$$Premium_{Aadl}^{(m)} = \mathbb{E}(S_{Aadl}^{(m)}) \tag{D.3.3}$$

Then these premium are summed over the entire portfolio to end up with the total pure premium for a given line of business:

$$Premium_{Aadl,M} = \sum_{m=1}^{M} Premium_{Aadl}^{(m)} \tag{D.3.4}$$

D.3.3 Allocated Capital

Denote $S_{Aadl,M}$ the distribution of the total reinsured aggregate claims :

$$u = \rho(S_{Aadl,M} - Premium_{Aadl,M}) \tag{D.3.5}$$

D.3.4 Limit Case Checks

Without any clause (Pure Treaty)		**Aggregate Annual D/L**		**Check - Gap**	
Priority	500	AAD	0		
Limit	2 500	AAL	Inf		
Pure Premium	1 065	Pure Premium	1 065	0.00	OK
Capital Needs ($VaR99.5\%$)	4 865	Capital Needs ($VaR99.5\%$)	4,865	0.00	OK

Table D.3.1: AAD/AAL limit case checks

D.4 Single Period Framework - Case 3 : Paid reinstatements

D.4.1 Reinsurer total loss

The corresponding reinsurer total loss is given by:

$$P_{reins,rand} = \frac{P_{reins,init}}{L} \sum_{k=1}^{K} c_k min(L, max(0, S - (k-1)L)) \tag{D.4.1}$$

with c_k the fraction of the initial premium $P_{reins,init}$ for the K^{th} reinstatements. The retained risk of the reinsurer is then :

$$S_{reins} = min(max(S - P, 0), L) - P_{reins,rand} \tag{D.4.2}$$

In this section we consider K equal to 1 and c_1 equal to 100%. In other words we consider one paid reinstatement @100%.

D.4.2 Pure Premium Calculations

The pure premium per treaty with such a clause now embeds a random part and cannot be directly derived. We use the methodology based on pure premium principle studied by Sundt for each treaty (Sundt, 1993). In brief, we find the initial pure premium as the premium for which the expected premium income is equal to the expected claim payments for each reinstatement. Then these obtained premium are summed over the entire reinsurance portfolio to end up with the total pure premium for a given line of business:

$$Premium_{reins,M} = \sum_{m=1}^{M} Premium_{reins}^{(m)} \tag{D.4.3}$$

D.4.3 Allocated Capital

Denote $S_{reins,M}$ the total reinsured aggregate claims with the presence of single paid reinstatements:

$$u = \rho(S_{reins,M} - Premium_{reins,M}) \tag{D.4.4}$$

D.4.4 Limit Case Checks

This clause is challenged the following ways:

- Consider a limit case where the aggregate priority is set to 0 and aggregate limit to infinity (infinite number of free reinstatements);

- Consider a limit case where the aggregate priority is set to p and aggregate limit to 2l (1 free reinstatement);

Without any clause (Pure Treaty)		**Paid reinstatement**		**Check - Gap**	
Priority	500	NÂ° of reinst	Inf		
Limit	2 500	Reinstatement %	0 %		
Pure Premium	1 065	Pure Premium	1 065	0.00	OK
Capital Needs (VaR99.5%)	4 865	Capital Needs (VaR99.5%)	4 865	0.00	OK

Aggregate Annual D/L		**Paid reinstatement**		**Check - Gap**	
AAD	500	NÂ° of reinst	1		
AAL	5 000	Reinstatement %	0 %		
Pure Premium	727	Pure Premium	727	0.00	OK
Capital Needs (VaR99.5%)	4 273	Capital Needs (VaR99.5%)	4 273	0.00	OK

Table D.4.1: Paid reinstatements limit case checks

- Consider model and XL cover parameters used in Sundt (Sundt, 1993) :

Model parameters	idem Sundt,1993
lambda Poisson	0.5
a Pareto	100
Alpha Pareto	1.2
Paid reinstatement	idem Sundt,1993
AAD	100
AAL	100

Nb of reinstatements	**0**	**1**	**1**	**2**	**2**
Reinstatement %		0%	100%	0%	100%
Pure Premium - Monte-Carlo Simulations [nSim = 100K]	4.12	4.52	4.34	4.55	4.35
Pure Premium - Sundt, 1993	4.09	4.49	4.31	4.52	4.32
Diff.	-0.03	-0.04	-0.03	-0.03	-0.03
Diff. (%)	−0.81%	−0.82%	−0.78%	−0.75%	−0.74%
CHECK	OK NS	OK NS	OK NS	OK NS	OK NS

Table D.4.2: Paid reinstatements limit case checks - Continued

D.5 Single Period Framework - Case 4 : Sliding Scale Premium

D.5.1 Reinsurer total loss

Based on pure premium principle discussed by Walhin et al. (Walhin et al., 2001) but also by Campana and Ferretti (Campana and Ferretti, 2022), the reinsurance premium per treaty is given by :

$$P_{ss} = \begin{cases} P_{ss,min} & \text{if} \quad S \leq \frac{P_{ss,min}}{f} \\ fS & \text{if} \quad \frac{P_{ss,min}}{f} < S < \frac{P_{ss,max}}{f} \\ P_{ss,max} & \text{if} \quad S \geq \frac{P_{ss,max}}{f} \end{cases} \tag{D.5.1}$$

with f a loading coefficient which represents the reinsurer expenses and cost-of-capital. In this section we set a typical value for such coefficient: $f = \frac{100}{80}$. The random part of this reinsurance premium is then $P_{ss,rand} = P_{ss} - P_{ss,min}$.The retained risk of the reinsurer is :

$$S_{ss} = S - P_{ss,rand} \tag{D.5.2}$$

Again, due to its random part, the initial pure premium per treaty cannot be directly derived. Thus the following methodology is followed : $P_{ss,min}$ is fixed (we set $P^{(m)}_{ss,min} = \frac{Premium^{(m)}}{2}$) and then $P_{ss,max}$ is found such as the average premium for a treaty with the sliding scale equals the premium of a pure treaty, i.e. a treaty without any clause.

D.5.2 Allocated Capital

Denote $S_{ss,M}$ total reinsured aggregate claims with the presence of sliding scale clauses :

$$u = \rho(S_{ss,M} - Premium_{ss,init,M}) \tag{D.5.3}$$

D.5.3 Limit Case Checks

Without any clause (Pure Treaty)		**Sliding scale**		**Check- Gap**	
Priority	500	Loading coefficient	1		
Limit	2 500	"Flat" Scale	=Pure Premium (Min == Max == Pure)		
Pure Premium	1 065	Pure Premium	1 065	0.00	OK
Capital Needs ($VaR99.5\%$)	4 865	Capital Needs ($VaR99.5\%$)	4 865	0.00	OK

Table D.5.1: Sliding scale limit case checks

D.6 Single Period Framework - Intermediary findings

D.6.1 Influence of portfolio size on 1-year reinsurer capital need

All other parameters being fixed or deterministic.

Risk Measure	VaR @ 99.5%				ES @ 99%			
Clause	Pure Treaty - No Clause	AAD/ AAL Clause	1 Reins @100%	Sliding Scale	Pure Treaty - No Clause	AAD/ AAL Clause	1 Reins @100%	Sliding Scale
M = 1	4 865	4 160	3 704	4 021	5 174	4 158	3 701	4330
M = 10	1 274	743	449	307	1 307	783	481	345
M = 100	360	318	311	239	374	330	322	248
M = 1000	109	97	119	69	112	99	122	72

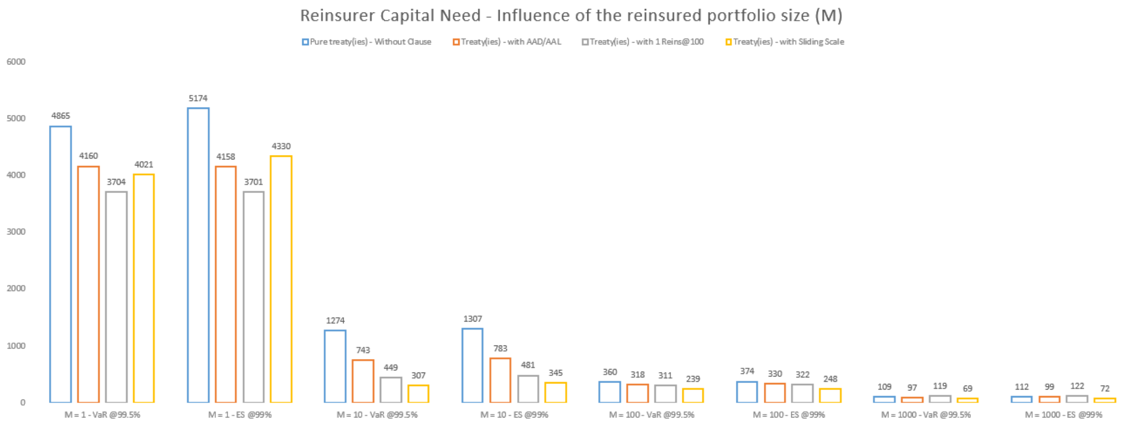

Figure D.6.1: Influence of portfolio size on 1-year reinsurer capital need

D.6.2 Influence of portfolio dependence in inflation on initial reinsurer capital need

All other parameters being fixed or deterministic. M = 100 treaties and treaties being independent of other exogenous factors.

Risk Measure	VaR @ 99.5%				ES @ 99%			
Clause	Pure Treaty - No Clause	AAD/ AAL Clause	1 Reins @100%	Sliding Scale	Pure Treaty - No Clause	AAD/ AAL Clause	1 Reins @100%	Sliding Scale
iid severities	360	318	311	239	374	330	322	248
infl dpd severities	368	325	316	239	385	337	327	252

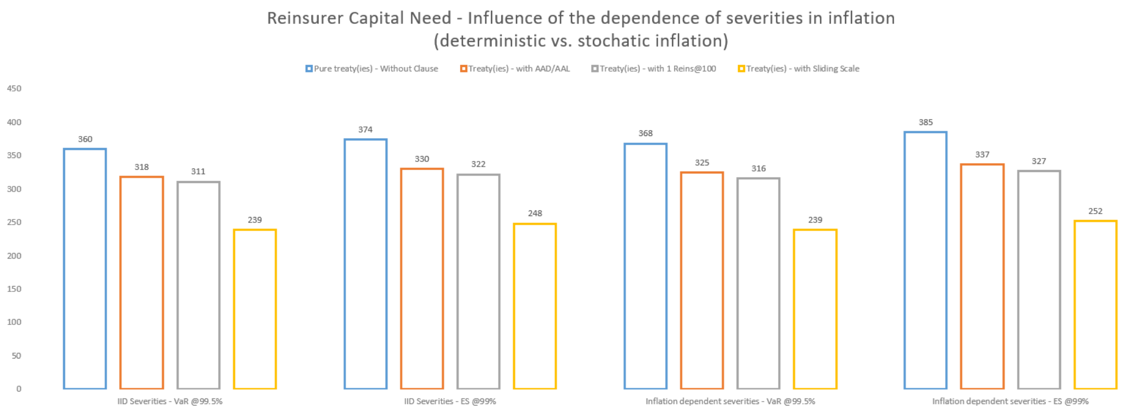

Figure D.6.2: Influence of portfolio dependence in inflation on 1-year reinsurer capital need

D.6.3 Influence of portfolio dependence in other exogenous factors on 1-year reinsurer capital need

All other parameters being fixed or deterministic. M = 100 treaties and treaties being independent of inflation - inflation is set deterministic.

Risk Measure	VaR @ 99.5%				ES @ 99%			
Clause	Pure Treaty - No Clause	AAD/ AAL Clause	1 Reins @100%	Sliding Scale	Pure Treaty - No Clause	AAD/ AAL Clause	1 Reins @100%	Sliding Scale
iid severities	360	318	311	239	374	330	322	248
Normal copula ($\rho = 0.5$) on frequency	1 586	1 371	1 218	998	1 687	1 448	1 288	1 090
Normal copula ($\rho = 0.5$) on severity	1 888	1 703	1 508	1 271	1 976	1 765	1 568	1 359
Normal copula ($\rho = 0.5$) on severityxfrequency	2 757	2 419	2 140	1 957	2 930	2 528	2 236	2 143
t copula ($\rho = 0.5$,df=3) on frequency	1 723	1 499	1 330	1 131	1 842	1 587	1 411	1 231
t copula ($\rho = 0.5$,df=3) on severity	2 002	1 818	1 617	1 395	2 099	1 883	1 672	1 479
t copula ($\rho = 0.5$,df=3) on severity x frequency	3 006	2 598	2 299	2 234	3 144	2 695	2 387	2 382

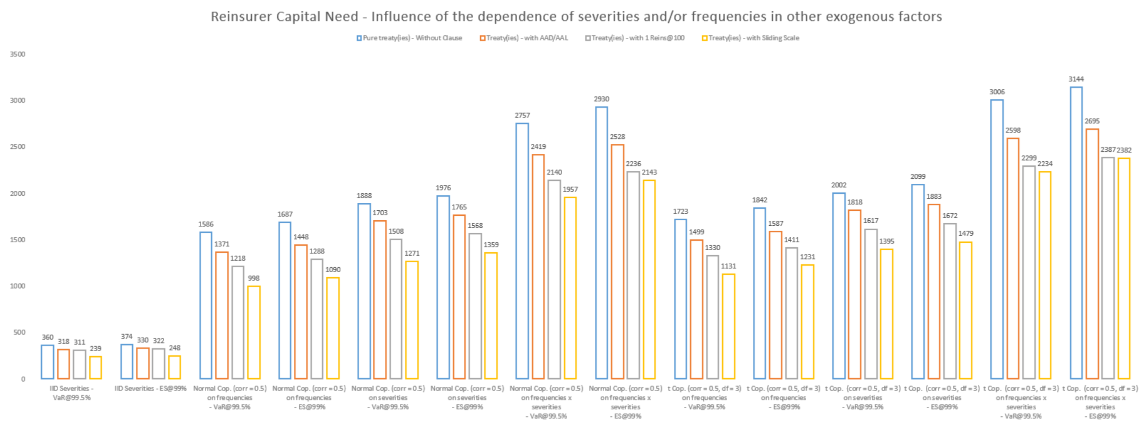

Figure D.6.3: Influence of portfolio dependence in other exogenous factors on 1-year reinsurer capital need

D.7 Multiple Period Framework - Intermediary findings

D.7.1 Influence of portfolio size on initial reinsurer capital need

All other parameters being fixed or deterministic.

Risk Measure	VaR @ 99.5%				ES @ 99%			
Clause	Pure Treaty - No Clause	AAD/ AAL Clause	1 Reins @100%	Sliding Scale	Pure Treaty - No Clause	AAD/ AAL Clause	1 Reins @100%	Sliding Scale
M = 1	14 404	4 063	3 604	3 580	14 641	4 063	3 604	3 639
M = 10	1 452	1 174	1 064	1 034	1 517	1 218	1 106	1 086
M = 100	412	339	326	273	425	351	337	284
M = 1000	123	105	119	80	128	109	123	84

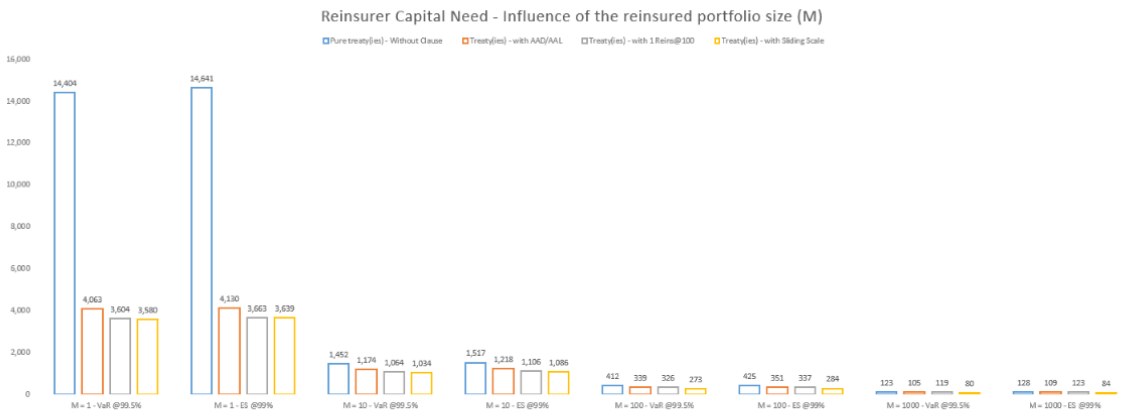

Figure D.7.1: Influence of portfolio size on initial reinsurer capital need

D.7.2 Influence of portfolio dependence in inflation on initial reinsurer capital need

All other parameters being fixed or deterministic. M = 100 treaties and treaties being independent of other exogenous factors.

Risk Measure	VaR @ 99.5%				ES @ 99%			
Clause	Pure Treaty - No Clause	AAD/ AAL Clause	1 Reins @100%	Sliding Scale	Pure Treaty - No Clause	AAD/ AAL Clause	1 Reins @100%	Sliding Scale
iid severities	412	339	326	273	425	351	337	284
infl dpd severities	413	339	326	278	431	352	342	292

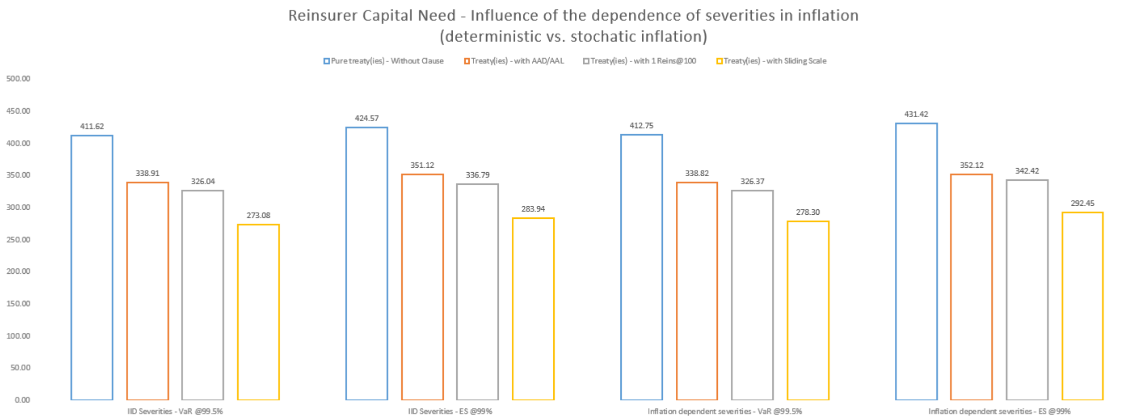

Figure D.7.2: Influence of portfolio dependence in inflation on initial reinsurer capital need

D.7.3 Influence of portfolio dependence in other exogenous factors on initial reinsurer capital need

All other parameters being fixed or deterministic. M = 100 treaties and treaties being independent of inflation - inflation is set deterministic.

Risk Measure	VaR @ 99.5%				ES @ 99%			
Clause	Pure Treaty - No Clause	AAD/ AAL Clause	1 Reins @100%	Sliding Scale	Pure Treaty - No Clause	AAD/ AAL Clause	1 Reins @100%	Sliding Scale
iid severities	412	339	326	273	425	351	337	284
Normal copula ($\rho = 0.5$) on frequency	1 566	1 261	1 113	971	1 613	1 308	1 134	1 015
Normal copula ($\rho = 0.5$) on severity	2 263	1 877	1 671	1 533	2,372	1 959	1 717	1 644
Normal copula ($\rho = 0.5$) on severityxfrequency	3 199	2 555	2 228	2 172	3 351	2 637	2 295	2 244
t copula ($\rho = 0.5$,df=3) on frequency	1 727	1 363	1 226	1 097	1 852	1 426	1 289	1 104
t copula ($\rho = 0.5$,df=3) on severity	2 512	2 115	1 842	1 772	2 620	2 173	1 900	1 873
t copula ($\rho = 0.5$,df=3) on severityxfrequency	3 640	2 790	2 458	2 315	2 844	2 521	2 889	2 289

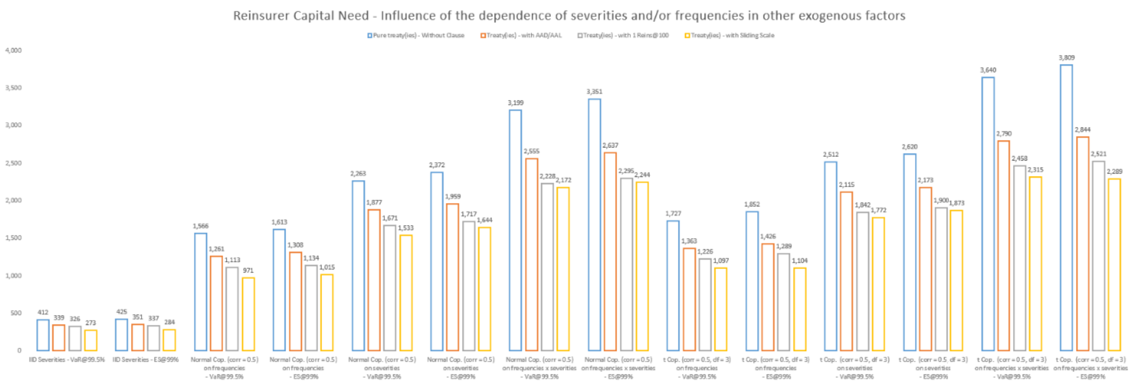

Figure D.7.3: Influence of portfolio dependence in other exogenous factors on initial reinsurer capital need

Appendix E

BLOCK 5 - Final pricing model

E.1 Overview of data used

Elements for the pricing	Parameters	Value
Distribution of Claim Amounts (Pareto)	A (amplitude)	2 500 000
Distribution of Claim Amounts (Pareto)	Alpha (shape)	1.50
Distribution of Claim Numbers (Poisson)	Lamdba (mean frequency)	2.50
Claims payment pattern	c (completely developed after 30 years)	Stochastic Variable Set in BLOCK 4
Overstatement pattern of the ceding company	d (completely developed after 30 years)	100% for t [0,30]
Future inflation	infl (geometric growth)	Stochastic Variable Set in BLOCK 3
Superimposed inflation	supinfl (geometric growth)	$infl + 1\%$
Interest rate obtained on the loss reserve	r	Stochastic Variable Set in BLOCK 3
Return obtained on the allocated capital	l	Equal to r
Cost-of-capital	coc	$r + \beta \times P_r$
Allocated capital	C(j) (allocated for 3 years)	Stochastic Variable Set in BLOCK 2
Priority 1 of the treaty	P1	3 000 000
Limit 1 of the treaty	L1	6 000 000
Priority 2 of the treaty	P2	6 000 000
Limit 2 of the treaty	L2	9 000 000
Priority 3 of the treaty	P3	6 000 000
Limit 3 of the treaty	L3	Inf
Margin on the date of payment stability clause	Applied on the priority and on the limit - based on incurred losses	10%
Interests sharing clause	Delta (portion of interests in the losses)	0%
Estimated premium income	Cedant's EPI	100 000 000
Share of the reinsurer in the treaty	Share	20%
Brokerage	B	10%
Minimum and deposit premium	MDP	80%
Retrocession costs	Percentage of the commercial premium	3%
Average Paid Claims by the retrocession	Percentage of the claims	2%
Administrative expenses	Fixed part	25 000
Administrative expenses	Variable part (percentage of the paid losses each year)	4%
Tax rate	Average tax rate	0%

Table E.1.1: Block 5 - Overview of dummy data used

E.2 Step 1 : Sensitivity to the stochastic rates approach

LAYER 1	**3 xs 3 MEUR**	Scenario S1	Scenario D	Gap	Relative Gap [%]
Pure Treaty	TR [%] [A]	5.72	5.71	0.01	0.16%
	TR_{disc} [%]	5.52	5.51	0.01	0.18%
	TFR [%]	6.17	6.16	0.01	0.14%
	CR [%] [B]	8.41	8.39	0.02	0.26%
	Multiple [B]/[A]	1.47	1.47	0.00	0.10%
Pure Treaty No stability clause	TR [%] [A]	8.37	8.35	0.02	0.24%
	TR_{disc} [%]	8.18	8.17	0.01	0.14%
	TFR [%]	7.50	7.49	0.01	0.15%
	CR [%] [B]	10.12	10.11	0.01	0.09%
	Multiple [B]/[A]	1.21	1.21	0.00	-0.16%
AAD Clause (Aad = p)	TR [%] [A]	3.50	3.49	0.01	0.39%
	TR_{disc} [%]	3.34	3.32	0.01	0.42%
	TFR [%]	3.98	3.97	0.01	0.30%
	CR [%] [B]	5.59	5.58	0.01	0.22%
	Multiple [B]/[A]	1.60	1.60	0.00	-0.17%
AAL Clause (Aal = 2l)	TR [%] [A]	5.17	5.16	0.01	0.11%
	TR_{disc} [%]	5.00	5.00	0.00	0.04%
	TFR [%]	5.48	5.47	0.01	0.11%
	CR [%] [B]	7.40	7.40	0.00	0.00%
	Multiple [B]/[A]	1.43	1.43	0.00	-0.12%
AAD/AAL Clause (Aad = p, Aal = 2l)	TR [%] [A]	3.22	3.22	0.01	0.20%
	TR_{disc} [%]	3.08	3.08	0.01	0.25%
	TFR [%]	3.63	3.63	0.00	0.08%
	CR [%] [B]	5.07	5.07	0.00	0.09%
	Multiple [B]/[A]	1.57	1.58	0.00	-0.11%
Paid Reinstatements Clause (K = 1 @100%)	TR [%] [A]	5.17	5.16	0.01	0.11%
	TR_{disc} [%]	5.00	5.00	0.00	0.02%
	TFR [%]	5.48	5.47	0.01	0.11%
	CR [%] [B]	6.27	6.27	0.00	0.08%
	Multiple [B]/[A]	1.21	1.21	0.00	-0.03%
Sliding Scale Clause (f=100/80, $P_{min} = 2\%$)	P_{max} [%]	10.42	10.42	0.00	0.00%
	Average Premium [%]	8.69	8.67	0.02	0.18%

Table E.2.1: Step 1 Outcome - Layer 1

LAYER 2	**3 xs 6 MEUR**	Scenario S1	Scenario D	Gap	Relative Gap [%]
Pure Treaty	TR [%] [A]	1.88	1.86	0.02	0.91%
	TR_{disc} [%]	1.81	1.80	0.02	0.95%
	TFR [%]	2.17	2.15	0.02	0.99%
	CR [%] [B]	3.28	3.25	0.04	1.10%
	Multiple [B]/[A]	1.74	1.74	0.00	0.18%
Pure Treaty No stability clause	TR [%] [A]	3.46	3.45	0.01	0.34%
	TR_{disc} [%]	3.33	3.32	0.01	0.35%
	TFR [%]	3.10	3.09	0.01	0.34%
	CR [%] [B]	4.48	4.46	0.02	0.33%
	Multiple [B]/[A]	1.30	1.30	0.00	0.00%
AAD Clause (Aad = p)	TR [%] [A]	0.67	0.65	0.02	2.37%
	TR_{disc} [%]	0.64	0.62	0.02	2.50%
	TFR [%]	0.70	0.68	0.01	2.11%
	CR [%] [B]	1.27	1.25	0.02	1.95%
	Multiple [B]/[A]	1.92	1.92	-0.01	-0.43%
AAL Clause (Aal = 2l)	TR [%] [A]	1.85	1.83	0.01	0.68%
	TR_{disc} [%]	1.78	1.77	0.01	0.72%
	TFR [%]	2.14	2.12	0.02	0.89%
	CR [%] [B]	3.22	3.20	0.02	0.74%
	Multiple [B]/[A]	1.74	1.74	0.00	0.06%
AAD/AAL Clause (Aad = p, Aal = 2l)	TR [%] [A]	0.66	0.64	0.01	2.21%
	TR_{disc} [%]	0.63	0.62	0.01	2.34%
	TFR [%]	0.69	0.68	0.01	2.05%
	CR [%] [B]	1.26	1.24	0.02	1.27%
	Multiple [B]/[A]	1.91	1.93	-0.02	-0.96%
Paid Reinstatements Clause (K = 1 @100%)	TR [%] [A]	1.85	1.83	0.01	0.68%
	TR_{disc} [%]	1.78	1.77	0.01	0.72%
	TFR [%]	2.14	2.12	0.02	0.89%
	CR [%] [B]	2.91	2.88	0.02	0.77%
	Multiple [B]/[A]	1.57	1.57	0.00	0.08%
Sliding Scale Clause (f=100/80, P_{min} = 2%)	P_{max} [%]	5.06	5.06	0.00	0.00%
	Average Premium [%]	3.00	2.92	0.08	2.59%

Table E.2.2: Step 1 Outcome - Layer 2

LAYER 3	**Inf xs 6 MEUR**	Scenario S1	Scenario D	Gap	Relative Gap [%]
Pure Treaty	TR [%] [A]	2.49	2.49	0.00	0.02%
	TR_{disc} [%]	2.40	2.40	0.00	0.01%
	TFR [%]	2.89	2.87	0.01	0.48%
	CR [%] [B]	4.95	4.92	0.03	0.61%
	Multiple [B]/[A]	1.99	1.98	0.01	0.59%
Pure Treaty No stability clause	TR [%] [A]	4.61	4.60	0.01	0.17%
	TR_{disc} [%]	4.44	4.43	0.01	0.17%
	TFR [%]	4.13	4.12	0.01	0.17%
	CR [%] [B]	6.59	6.54	0.05	0.80%
	Multiple [B]/[A]	1.43	1.42	0.01	0.62%
AAD Clause (Aad = p)	TR [%] [A]	1.32	1.27	0.05	3.71%
	TR_{disc} [%]	1.26	1.22	0.04	3.09%
	TFR [%]	1.45	1.41	0.05	3.24%
	CR [%] [B]	3.07	3.01	0.06	2.02%
	Multiple [B]/[A]	2.32	2.36	-0.04	-1.76%
AAL Clause (Aal = 2l)	TR [%] [A]	2.49	2.49	0.00	0.02%
	TR_{disc} [%]	2.40	2.40	0.00	0.01%
	TFR [%]	2.89	2.87	0.01	0.48%
	CR [%] [B]	4.95	4.92	0.03	0.61%
	Multiple [B]/[A]	1.99	1.98	0.01	0.59%
AAD/AAL Clause (Aad = p, Aal = 2l)	TR [%] [A]	1.32	1.27	0.05	3.71%
	TR_{disc} [%]	1.26	1.22	0.04	3.09%
	TFR [%]	1.45	1.41	0.05	3.24%
	CR [%] [B]	3.07	3.01	0.06	2.02%
	Multiple [B]/[A]	2.32	2.36	-0.04	-1.76%
Paid Reinstatements Clause (K = 1 @100%)	TR [%] [A]	2.49	2.49	0.00	0.02%
	TR_{disc} [%]	2.40	2.40	0.00	0.01%
	TFR [%]	2.89	2.87	0.01	0.48%
	CR [%] [B]	NA	NA		
	Multiple [B]/[A]	NA	NA		
Sliding Scale Clause	P_{max} [%]	11.19	11.95	-0.77	-6.84%
(f=100/80, $P_{min} = 2\%$)	Average Premium [%]	4.71	4.70	0.02	0.33%

Table E.2.3: Step 1 Outcome - Layer 3

E.3 Step 2 : Sensitivity to the stochastic CPP approach

LAYER 1	**3 xs 3 MEUR**	Scenario S2	Scenario D	Gap	Relative Gap [%]
Pure Treaty	TR [%] [A]	5.72	5.71	0.01	0.10%
	TR_{disc} [%]	5.52	5.51	0.01	0.24%
	TFR [%]	6.17	6.16	0.01	0.13%
	CR [%] [B]	8.39	8.39	0.01	0.06%
	Multiple [B]/[A]	1.47	1.47	0.00	-0.04%
Pure Treaty No stability clause	TR [%] [A]	8.36	8.35	0.01	0.12%
	TR_{disc} [%]	8.18	8.17	0.01	0.15%
	TFR [%]	7.49	7.49	0.00	0.01%
	CR [%] [B]	10.11	10.11	0.00	0.02%
	Multiple [B]/[A]	1.21	1.21	0.00	-0.10%
AAD Clause (Aad = p)	TR [%] [A]	3.50	3.49	0.01	0.18%
	TR_{disc} [%]	3.34	3.32	0.01	0.39%
	TFR [%]	3.98	3.97	0.01	0.23%
	CR [%] [B]	5.59	5.58	0.01	0.17%
	Multiple [B]/[A]	1.60	1.60	0.00	0.00%
AAL Clause (Aal = 2l)	TR [%] [A]	5.17	5.16	0.01	0.17%
	TR_{disc} [%]	5.01	5.00	0.01	0.26%
	TFR [%]	5.48	5.47	0.00	0.07%
	CR [%] [B]	7.40	7.40	0.00	0.05%
	Multiple [B]/[A]	1.43	1.43	0.00	-0.12%
AAD/AAL Clause (Aad = p, Aal = 2l)	TR [%] [A]	3.22	3.22	0.00	0.03%
	TR_{disc} [%]	3.08	3.08	0.01	0.21%
	TFR [%]	3.63	3.63	0.00	0.11%
	CR [%] [B]	5.08	5.07	0.01	0.15%
	Multiple [B]/[A]	1.58	1.58	0.00	0.12%
Paid Reinstatements Clause (K = 1 @100%)	TR [%] [A]	5.17	5.16	0.01	0.17%
	TR_{disc} [%]	5.00	5.00	0.00	0.06%
	TFR [%]	5.48	5.47	0.00	0.07%
	CR [%] [B]	6.27	6.27	0.00	0.03%
	Multiple [B]/[A]	1.21	1.21	0.00	-0.13%
Sliding Scale Clause (f=100/80, $P_{min} = 2\%$)	P_{max} [%]	10.43	10.42	0.01	0.08%
	Average Premium [%]	8.69	8.67	0.02	0.26%

Table E.3.1: Step 2 Outcome - Layer 1

LAYER 2	**3 xs 6 MEUR**	Scenario S2	Scenario D	Gap	Relative Gap [%]
Pure Treaty	TR [%] [A]	1.87	1.86	0.01	0.40%
	TR_{disc} [%]	1.81	1.80	0.01	0.50%
	TFR [%]	2.16	2.15	0.01	0.31%
	CR [%] [B]	3.26	3.25	0.01	0.41%
	Multiple [B]/[A]	1.74	1.74	0.00	0.01%
Pure Treaty No stability clause	TR [%] [A]	3.45	3.45	0.01	0.25%
	TR_{disc} [%]	3.33	3.32	0.01	0.35%
	TFR [%]	3.10	3.09	0.01	0.25%
	CR [%] [B]	4.48	4.46	0.02	0.38%
	Multiple [B]/[A]	1.30	1.30	0.00	0.13%
AAD Clause (Aad = p)	TR [%] [A]	0.66	0.65	0.01	1.15%
	TR_{disc} [%]	0.63	0.62	0.01	1.32%
	TFR [%]	0.69	0.68	0.01	0.98%
	CR [%] [B]	1.26	1.25	0.01	1.05%
	Multiple [B]/[A]	1.92	1.92	0.00	-0.10%
AAL Clause (Aal = 2l)	TR [%] [A]	1.84	1.83	0.01	0.29%
	TR_{disc} [%]	1.78	1.77	0.01	0.39%
	TFR [%]	2.13	2.12	0.01	0.26%
	CR [%] [B]	3.21	3.20	0.01	0.32%
	Multiple [B]/[A]	1.74	1.74	0.00	0.04%
AAD/AAL Clause (Aad = p, Aal = 2l)	TR [%] [A]	0.65	0.64	0.01	1.11%
	TR_{disc} [%]	0.62	0.62	0.01	1.30%
	TFR [%]	0.69	0.68	0.01	1.02%
	CR [%] [B]	1.24	1.24	0.00	0.38%
	Multiple [B]/[A]	1.92	1.93	-0.01	-0.74%
Paid Reinstatements Clause (K = 1 @100%)	TR [%] [A]	1.84	1.83	0.01	0.29%
	TR_{disc} [%]	1.78	1.77	0.01	0.39%
	TFR [%]	2.13	2.12	0.01	0.26%
	CR [%] [B]	2.89	2.88	0.01	0.20%
	Multiple [B]/[A]	1.57	1.57	0.00	-0.08%
Sliding Scale Clause (f=100/80, $P_{min} = 2\%$)	P_{max} [%]	5.77	5.06	0.71	12.23%
	Average Premium [%]	3.07	2.92	0.15	4.82%

Table E.3.2: Step 2 Outcome - Layer 2

LAYER 3	**Inf xs 6 MEUR**	Scenario S2	Scenario D	Gap	Relative Gap [%]
Pure Treaty	TR [%] [A]	2.52	2.49	0.03	1.31%
	TR_{disc} [%]	2.43	2.40	0.03	1.41%
	TFR [%]	2.91	2.87	0.03	1.09%
	CR [%] [B]	5.32	4.92	0.40	7.46%
	Multiple [B]/[A]	2.11	1.98	0.13	6.24%
Pure Treaty No stability clause	TR [%] [A]	4.64	4.60	0.04	0.83%
	TR_{disc} [%]	4.47	4.43	0.04	0.93%
	TFR [%]	4.16	4.12	0.03	0.83%
	CR [%] [B]	6.96	6.54	0.42	6.06%
	Multiple [B]/[A]	1.50	1.42	0.08	5.27%
AAD Clause (Aad = p)	TR [%] [A]	1.31	1.27	0.03	2.52%
	TR_{disc} [%]	1.26	1.22	0.03	2.66%
	TFR [%]	1.44	1.41	0.03	2.21%
	CR [%] [B]	3.47	3.01	0.46	13.28%
	Multiple [B]/[A]	2.65	2.36	0.29	11.03%
AAL Clause (Aal = 2l)	TR [%] [A]	2.52	2.49	0.03	1.31%
	TR_{disc} [%]	2.43	2.40	0.03	1.41%
	TFR [%]	2.91	2.87	0.03	1.09%
	CR [%] [B]	5.31	4.92	0.39	7.33%
	Multiple [B]/[A]	2.11	1.98	0.13	6.11%
AAD/AAL Clause (Aad = p, Aal = 2l)	TR [%] [A]	1.31	1.27	0.03	2.52%
	TR_{disc} [%]	1.26	1.22	0.03	2.66%
	TFR [%]	1.44	1.41	0.03	2.21%
	CR [%] [B]	3.47	3.01	0.46	13.28%
	Multiple [B]/[A]	2.65	2.36	0.29	11.03%
Paid Reinstatements Clause (K = 1 @100%)	TR [%] [A]	2.52	2.49	0.03	1.31%
	TR_{disc} [%]	2.43	2.40	0.03	1.41%
	TFR [%]	2.91	2.87	0.03	1.09%
	CR [%] [B]	NA	NA		
	Multiple [B]/[A]	NA	NA		
Sliding Scale Clause (f=100/80, $P_{min} = 2\%$)	P_{max} [%]	14.87	11.95	2.92	19.62%
	Average Premium [%]	5.13	4.70	0.43	8.41%

Table E.3.3: Step 2 Outcome - Layer 3

E.4 Step 3 : Search for the market parameter to which the price is most sensitive

LAYER 1	3 xs 3 MEUR	SCENARIO A Scenario M = 100, VaR @99.5%	SCENARIO B Scenario M = 100, ES @99%	SCENARIO C Scenario M = 200, VaR @99.5%	Gap B - A	Gap C - A	Gap B - A [%]	Gap C - A [%]
Pure Treaty	TR [%] [A]	5.71	5.71	5.71	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	5.51	5.51	5.51	0.00	0.00	0.00%	0.00%
	TFR [%]	6.16	6.16	6.16	0.00	0.00	0.00%	0.00%
	CR [%] [B]	8.39	8.41	8.23	0.03	-0.16	0.32%	-1.88%
	Multiple [B]/[A]	1.47	1.47	1.44	0.00	-0.03	0.32%	-1.88%
Pure Treaty No stability clause	TR [%] [A]	8.35	8.35	8.35	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	8.17	8.17	8.17	0.00	0.00	0.00%	0.00%
	TFR [%]	7.49	7.49	7.49	0.00	0.00	0.00%	0.00%
	CR [%] [B]	10.11	10.14	9.94	0.03	-0.16	0.31%	-1.62%
	Multiple [B]/[A]	1.21	1.21	1.19	0.00	-0.02	0.31%	-1.62%
AAD Clause (Aad = p)	TR [%] [A]	3.49	3.49	3.49	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	3.32	3.32	3.32	0.00	0.00	0.00%	0.00%
	TFR [%]	3.97	3.97	3.97	0.00	0.00	0.00%	0.00%
	CR [%] [B]	5.58	5.60	5.44	0.02	-0.15	0.36%	-2.63%
	Multiple [B]/[A]	1.60	1.61	1.56	0.01	-0.04	0.36%	-2.63%
AAL Clause (Aal = 2l)	TR [%] [A]	5.16	5.16	5.16	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	5.00	5.00	5.00	0.00	0.00	0.00%	0.00%
	TFR [%]	5.47	5.47	5.47	0.00	0.00	0.00%	0.00%
	CR [%] [B]	7.40	7.41	7.28	0.01	-0.12	0.18%	-1.61%
	Multiple [B]/[A]	1.43	1.44	1.41	0.00	-0.02	0.18%	-1.61%
AAD/AAL Clause (Aad = p, Aal = 2l)	TR [%] [A]	3.22	3.22	3.22	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	3.08	3.08	3.08	0.00	0.00	0.00%	0.00%
	TFR [%]	3.63	3.63	3.63	0.00	0.00	0.00%	0.00%
	CR [%] [B]	5.07	5.09	4.96	0.02	-0.11	0.34%	-2.22%
	Multiple [B]/[A]	1.58	1.58	1.54	0.01	-0.04	0.34%	-2.22%
Paid Reinstatements Clause (K = 1 @100%)	TR [%] [A]	5.16	5.16	5.16	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	5.00	5.00	5.00	0.00	0.00	0.00%	0.00%
	TFR [%]	5.47	5.47	5.47	0.00	0.00	0.00%	0.00%
	CR [%] [B]	6.27	6.27	6.20	0.01	-0.07	0.11%	-1.13%
	Multiple [B]/[A]	1.21	1.22	1.20	0.00	-0.01	0.11%	-1.13%
Sliding Scale Clause (f=100/80, P_{min} = 2%)	P_{max} [%]	10.42	10.42	10.42	0.00	0.00	0.00%	-0.02%
	Average Premium [%]	8.67	8.90	8.50	0.23	-0.17	2.67%	-1.94%

Table E.4.1: Step 3 Outcome - Layer 1

LAYER 2	**3 xs 6 MEUR**	SCENARIO A Scenario M = 100, VaR @99.5%	SCENARIO B Scenario M = 100, ES @99%	SCENARIO C Scenario M = 200, VaR @99.5%	Gap B - A	Gap C - A	Gap B - A [%]	Gap C - A [%]
Pure Treaty	TR [%] [A]	1.86	1.86	1.86	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	1.80	1.80	1.80	0.00	0.00	0.00%	0.00%
	TFR [%]	2.15	2.15	2.15	0.00	0.00	0.00%	0.00%
	CR [%] [B]	3.25	3.27	3.12	0.02	-0.13	0.76%	-3.96%
	Multiple [B]/[A]	1.74	1.75	1.67	0.01	-0.07	0.76%	-3.96%
Pure Treaty No stability clause	TR [%] [A]	3.45	3.45	3.45	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	3.32	3.32	3.32	0.00	0.00	0.00%	0.00%
	TFR [%]	3.09	3.09	3.09	0.00	0.00	0.00%	0.00%
	CR [%] [B]	4.46	4.48	4.33	0.02	-0.14	0.36%	-3.05%
	Multiple [B]/[A]	1.30	1.30	1.26	0.00	-0.04	0.36%	-3.05%
AAD Clause (Aad = p)	TR [%] [A]	0.65	0.65	0.65	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	0.62	0.62	0.62	0.00	0.00	0.00%	0.00%
	TFR [%]	0.68	0.68	0.68	0.00	0.00	0.00%	0.00%
	CR [%] [B]	1.25	1.26	1.17	0.01	-0.08	1.00%	-6.37%
	Multiple [B]/[A]	1.92	1.94	1.80	0.02	-0.12	1.00%	-6.37%
AAL Clause (Aal = 2l)	TR [%] [A]	1.83	1.83	1.83	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	1.77	1.77	1.77	0.00	0.00	0.00%	0.00%
	TFR [%]	2.12	2.12	2.12	0.00	0.00	0.00%	0.00%
	CR [%] [B]	3.20	3.21	3.07	0.02	-0.12	0.52%	-3.87%
	Multiple [B]/[A]	1.74	1.75	1.67	0.01	-0.07	0.52%	-3.87%
AAD/AAL Clause (Aad = p, Aal = 2l)	TR [%] [A]	0.64	0.64	0.64	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	0.62	0.62	0.62	0.00	0.00	0.00%	0.00%
	TFR [%]	0.68	0.68	0.68	0.00	0.00	0.00%	0.00%
	CR [%] [B]	1.24	1.25	1.16	0.01	-0.08	0.47%	-6.74%
	Multiple [B]/[A]	1.93	1.94	1.80	0.01	-0.13	0.47%	-6.74%
Paid Reinstatements Clause (K = 1 @100%)	TR [%] [A]	1.83	1.83	1.83	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	1.77	1.77	1.77	0.00	0.00	0.00%	0.00%
	TFR [%]	2.12	2.12	2.12	0.00	0.00	0.00%	0.00%
	CR [%] [B]	2.88	2.89	2.82	0.01	-0.06	0.28%	-2.22%
	Multiple [B]/[A]	1.57	1.58	1.54	0.00	-0.03	0.28%	-2.22%
Sliding Scale Clause (f=100/80, $P_{min} = 2\%$)	P_{max} [%]	5.06	5.06	4.83	0.00	-0.23	0.00%	-4.59%
	Average Premium [%]	2.92	3.03	2.82	0.10	-0.10	3.54%	-3.42%

Table E.4.2: Step 3 Outcome - Layer 2

LAYER 3	**Inf xs 6 MEUR**	SCENARIO A Scenario M = 100, VaR @99.5%	SCENARIO B Scenario M = 100, ES @99%	SCENARIO C Scenario M = 200, VaR @99.5%	Gap B - A	Gap C - A	Gap B - A [%]	Gap C - A [%]
Pure Treaty	TR [%] [A]	2.49	2.49	2.49	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	2.40	2.40	2.40	0.00	0.00	0.00%	0.00%
	TFR [%]	2.87	2.87	2.87	0.00	0.00	0.00%	0.00%
	CR [%] [B]	4.92	4.99	4.48	0.06	-0.44	1.28%	-9.03%
	Multiple [B]/[A]	1.98	2.00	1.80	0.03	-0.18	1.28%	-9.03%
Pure Treaty No stability clause	TR [%] [A]	4.60	4.60	4.60	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	4.43	4.43	4.43	0.00	0.00	0.00%	0.00%
	TFR [%]	4.12	4.12	4.12	0.00	0.00	0.00%	0.00%
	CR [%] [B]	6.54	6.63	6.12	0.09	-0.42	1.34%	-6.46%
	Multiple [B]/[A]	1.42	1.44	1.33	0.02	-0.09	1.34%	-6.46%
AAD Clause (Aad = p)	TR [%] [A]	1.27	1.27	1.27	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	1.22	1.22	1.22	0.00	0.00	0.00%	0.00%
	TFR [%]	1.41	1.41	1.41	0.00	0.00	0.00%	0.00%
	CR [%] [B]	3.01	3.08	2.60	0.08	-0.41	2.54%	-13.62%
	Multiple [B]/[A]	2.36	2.42	2.04	0.06	-0.32	2.54%	-13.62%
AAL Clause (Aal = 2l)	TR [%] [A]	2.49	2.49	2.49	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	2.40	2.40	2.40	0.00	0.00	0.00%	0.00%
	TFR [%]	2.87	2.87	2.87	0.00	0.00	0.00%	0.00%
	CR [%] [B]	4.92	4.99	4.48	0.07	-0.45	1.33%	-9.08%
	Multiple [B]/[A]	1.98	2.00	1.80	0.03	-0.18	1.33%	-9.08%
AAD/AAL Clause (Aad = p, Aal = 2l)	TR [%] [A]	1.27	1.27	1.27	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	1.22	1.22	1.22	0.00	0.00	0.00%	0.00%
	TFR [%]	1.41	1.41	1.41	0.00	0.00	0.00%	0.00%
	CR [%] [B]	3.01	3.08	2.59	0.07	-0.42	2.47%	-13.83%
	Multiple [B]/[A]	2.36	2.42	2.03	0.06	-0.33	2.47%	-13.83%
Paid Reinstatements Clause (K = 1 @100%)	TR [%] [A]	2.49	2.49	2.49	0.00	0.00	0.00%	0.00%
	TR_{disc} [%]	2.40	2.40	2.40	0.00	0.00	0.00%	0.00%
	TFR [%]	2.87	2.87	2.87	0.00	0.00	0.00%	0.00%
	CR [%] [B]	NA	NA	NA				
	Multiple [B]/[A]	NA	NA	NA				
Sliding Scale Clause	P_{max} [%]	11.95	11.19	9.66	-0.77	-2.29	-6.41%	-19.18%
(f=100/80, $P_{min} = 2\%$)	Average Premium [%]	4.70	4.83	4.41	0.13	-0.29	2.82%	-6.13%

Table E.4.3: Step 3 Outcome - Layer 3

E.5 Step 4 : Search for arbitrage opportunities in the prices of certain treaties with clauses

Layer	Priority (MEUR)	Limit (MEUR)	Reinstatements	CR (%)
1	2	4	1 @100%	2.3
2	4	8	1 @100%	2.1
3	8	16	1 @100%	1.4
4	16	30	1 @100%	0.9

Layer	Priority (MEUR)	Limit (MEUR)	Reinstatements	CR (%)
1	2	30	1 @100%	6.7

Table E.5.1: Option 1 (upper table) versus Option 2 (lower table) quotes

Layer	Priority (MEUR)	Limit (MEUR)	Reinst.	TR [%]	TR_{disc} [%]	TFR [%]	CR [%]
1	2	4	1 @100%	2.03	1.90	2.21	2.79
2	4	8	1 @100%	0.83	0.77	0.90	1.42
3	8	16	1 @100%	0.31	0.29	0.33	0.67
4	16	30	1 @100%	0.13	0.12	0.14	0.38
			Total	**3.30**	**3.08**	**3.57**	**5.26**

Layer	Priority (MEUR)	Limit (MEUR)	Reinst.	TR [%]	TR_{disc} [%]	TFR [%]	CR [%]
1	2	30	1 @100%	**3.78**	**3.58**	**3.99**	**4.87**

Table E.5.2: Step 4 - Treaty with reinstatements - 4 layers problem - Outcome

Appendix F

List of Abbreviations

- AAD : Annual Aggregate Deductible
- AAL : Annual Aggregate Limit
- CAPM : Capital Allocation Pricing Model
- Cat Nat : Natural Catastrophes
- CPI : Consumer Price Index
- CPP : Claims Payment Pattern
- DYC: Development Year Closing
- ES : Expected Shortfall
- IBNR : Incurred But Not Reported (claims)
- LDY : Last Development Year
- LoB : Line of Business
- MTPL : Motor Third Party Liability
- VaR : Value-at-Risk
- XL : excess-of-loss

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