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Torsion theories in normal categories

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Introduction

Torsion theories were first introduced in the context of abelian categories by Spencer Ernst Dickson in 1963 [9]. Later on, many authors introduced and investigated torsion theories in various non-abelian contexts, including Michael Barr, Walter Tholen, Maria Manuel Clementino, Dikran Dikranjan, Dominique Bourn and Marino Gran, among others.

The aim of this master thesis is to investigate the properties of torsion theories in the context of normal categories, by following the approach developed in [5] and by extending it to more general categories.

Normal categories were introduced by Zurab Janelidze in [13], as regular pointed categories with the property that any regular epimorphism is a normal epimorphism, i.e a cokernel.

Most of the familiar algebraic categories such as groups, rings and algebras are normal. More recently, it was also observed that the category of preordered groups is a normal category that is not homological [8]. This motivated further investigations in torsion theories in this more general context [12]. The present master thesis is structured in three chapters.

The first one is going to introduce some basic definitions and properties that will be useful throughout this work. We are going to define abelian categories, homological categories and normal categories and give some examples. We are also going to show that abelian categories are homological. We will end this section by stating and proving some properties of normal categories. These will allow us to adapt the proofs of [5] to the more general context of normal categories.

The second chapter will focus on the concept of a homological closure operator on kernels in normal categories. This closure operator was defined on homological categories in [5], and we are going to extend it to normal categories. The main aim of the second chapter is to prove Theorem 18 that states the existence of a bijection between the regular epireflective subcategories of a normal category and the homological closure operator on kernels. We are first going to define and give some examples of regular epireflective subcategories. Then we are going to define and state some properties of homological closure operators, and prove Theorem 18.

The third chapter will be devoted to torsion theory. In [5] the authors extended the axioms of torsion theories from abelian categories to homological categories. We will take the generalization to a further level by working in the context of normal categories. After this, we will define torsion-free subcategories and fibered reflections. We are going to show the equivalence between torsion-free subcategories and regular epireflective subcategories with fibered reflection.

We shall conclude the master thesis by explaining the connection between fibered reflection and the concept of semi-left-exact functor introduced by Cassidy, Hébert and Kelly in [6].

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Chapter 1

Introductory concepts

We first introduce some fundamental notions that will be useful throughout this work. In the following, definition and properties of abelian categories are essentially taken from [10] and [2], those about regular categories are mostly taken from [11] and [2].

Definition 1. *A category $\mathcal{C}$ is **abelian** when it satisfies the following properties:*

1. *$\mathcal{C}$ has a zero object.*
2. *every pair of objects has a product and a coproduct.*
3. *every arrow has a kernel and a cokernel.*
4. *every monomorphism is a kernel and every epimorphism is a cokernel.*

We can find some examples of abelian categories in [2]. For instance, if R is a ring, the category R-Mod of R-modules is abelian.
Recall that a category $\mathcal{C}$ is finitely complete when every functor $F : \mathcal{D} \to \mathcal{C}$ with $\mathcal{D}$ finite diagram, has a limit.
For instance, the category Ab of abelian groups and the category Top of topological spaces are complete (hence, finitely complete).
It is well known that a category is finitely complete if:

1. finite products exist,
2. each pair of parallel arrows has an equalizer.

The proof can be found for instance in [1], theorem 2.8.1.

Lemma 2. *In an abelian category, any monomorphism is the kernel of its cokernel.*

Proof. Let q be the cokernel of a monomorphism f in an abelian category $\mathcal{C}$ and k be the kernel of this morphism q. Those two morphisms exist since we are in an abelian category. We have to prove that f is equal to k up to isomorphism.

$$\begin{tikzcd}
K \arrow[r, "k"] & Y \arrow[r, "q"] & Q \\
X \arrow[u, dashed, "\phi"] \arrow[ur, "f"'] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

Since $q = coker(f)$, $q \circ f = 0$ which implies that there exists a unique ϕ with $k \circ \phi = f$. $\mathcal{C}$ is abelian, therefore there exists a morphism g such that f is the kernel of g.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr] & Y \arrow[rr, "q"] \arrow[dr, "g"'] & & Q \arrow[dl, dashed, "\psi"] \\
& 0 \arrow[r] & Z &
\end{tikzcd}$$

Therefore, $g \circ f = 0$ and there exists a unique ψ, such that $\psi \circ q = g$.

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \arrow[r, "g"] & Z \\
& K \arrow[ul, dashed, "i"] \arrow[ur, "k"'] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

Since f is the kernel of g and $g \circ k = \psi \circ q \circ k = \psi \circ 0 = 0$, there exists a unique i such that $f \circ i = k$.
We now have to prove that $i \circ \phi = 1_X$ and $\phi \circ i = 1_K$, but $f \circ i \circ \phi = k \circ \phi = f = f \circ 1_X$ which leads to $i \circ \phi = 1_X$ because f is a monomorphism.
The proof is similar for $\phi \circ i = 1_K$, recall that since k is a kernel, it is a monomorphism, therefore $k \circ \phi \circ i = f \circ i = k = k \circ 1_K$ implies $\phi \circ i = 1_K$. □

Lemma 3. *In an abelian category $\mathcal{C}$, if $g : A \to B$ is a monomorphism, and if we have a morphism $f : C \to B$, then the pullback of f and g exists.*

Proof. Since $\mathcal{C}$ is abelian, we consider $q = coker(g)$ and $k = ker(q \circ f)$.
By Lemma 2, g is the kernel of q.

$$\begin{tikzcd}
A \arrow[rr, "g"] & & B \arrow[r, "q"] & Q \\
& K \arrow[ul, dashed, "l"] \arrow[ur, "f \circ k"'] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

Since k is the kernel of $q \circ f$,we know that $q \circ f \circ k = 0$ and therefore, there exists a unique l such that $g \circ l = f \circ k$.
Let us prove that (K, k, l) is the pullback of f and g.
We already proved that $g \circ l = f \circ k$.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\lambda"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& K \arrow[r, "l"] \arrow[d, "k"'] & A \arrow[d, "g"] \\
& C \arrow[r, "f"'] & B
\end{tikzcd}$$

Let us consider α and β such that $g \circ \alpha = f \circ \beta$. Then, $q \circ g \circ \alpha = q \circ f \circ \beta = 0$.

$$\begin{tikzcd}
K \arrow[r, "k"] & C \arrow[r, "q\circ f"] & Q \\
Z \arrow[u, dashed, "\lambda"] \arrow[ur, "\beta"'] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

Then, since $k = ker(q \circ f)$, there exists a unique λ such that $k \circ \lambda = \beta$.
But $g \circ \alpha = f \circ \beta = f \circ k \circ \lambda = g \circ l \circ \lambda$ which implies $\alpha = l \circ \lambda$ because g is a monomorphism. □

Proposition 4. *Abelian categories are finitely complete.*

Proof. We follow the proof given in [10].
Finite products exist (since $\mathcal{C}$ has binary products and a terminal object).
It suffices to show that the equalizer of any pair of parallel morphisms $f, g : A \to B$ exists.
Let us consider $(1_A, g)$ and $(1_A, f)$

$$\begin{tikzcd}
& A \arrow[dl, "1_A"'] \arrow[d, dashed, "(1_A{,}f)"] \arrow[dr, "f"] & \\
A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[dl, "1_A"'] \arrow[d, dashed, "(1_A{,}g)"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B
\end{tikzcd}$$

which are split monomorphisms; in particular, they are monomorphisms.

Therefore, by Lemma 3, the pullback (P, p_1, p_2) of $(1_A, f)$ and $(1_A, g)$ exists.

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, tail, "{(1_A,g)}"] \\ A \arrow[r, tail, "{(1_A,f)}"'] & A \times B \end{tikzcd}$$

We are going to prove $p := p_1 = p_A \circ (1_A, f) \circ p_1 = p_A \circ (1_A, g) \circ p_2 = p_2$ is the equalizer of f and g.
We take a morphism $e : E \to A$ such that $f \circ e = g \circ e$. The commutativity of the following diagram and the universal property of the pullback (1), gives a unique ϕ such that $p \circ \phi = e$.

$$\begin{tikzcd} E \arrow[drr, bend left, "e"] \arrow[dr, dashed, "\exists!\phi"] \arrow[ddr, bend right, "e"'] & & \\ & P \arrow[r, "p=p_2"] \arrow[d, "p=p_1"'] & A \arrow[d, "{(1_A,g)}"] \\ & A \arrow[r, "{(1_A,f)}"'] & A \times B \end{tikzcd}$$

$\square$

By duality, we also have the finite cocompleteness of abelian categories.

Recall that a kernel pair $(Eq(f), p_1, p_2)$ of an arrow $f : A \to B$ is defined as the following pullback:

$$\begin{tikzcd} Eq(f) \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Note that in the category Set of sets, $Eq(f) = \{(a, a') \in A \times A | f(a) = f(a')\}$.
The notion of regular category is more general than the one of abelian category. Let us recall its definition.

Definition 5. *A finitely complete category is* ***regular*** *if:*

1. *Any kernel pair* $(Eq(f), p_1, p_2)$ *has a coequalizer.*
2. *Regular epimorphism are pullback stable: given a regular epimorphism* $p : E \to B$ *and any morphism* $f : A \to B$*, the morphism* p_2 *in the pullback*

$$\begin{tikzcd} E \times_B A \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "f"] \\ E \arrow[r, two heads, "p"'] & B \end{tikzcd}$$

   is a regular epimorphism.

Examples of regular categories

To illustrate the concept of regular category, let us consider some examples taken from [11].

The categories *Set* of sets, *Grp* of groups and *Ab* of abelian groups are regular

Let us prove that the category *Grp* of groups is regular.
In the category *Grp*, the kernel pair $(Eq(f), p_1, p_2)$ of a morphism $f : X \to Y$ is given by

$$Eq(f) = \{(x, y) \in X \times X | f(x) = f(y)\}$$

together with the projections $p_1 : Eq(f) \to X$ and $p_2 : Eq(f) \to X$, where

$$p_1(x, y) = x \quad \text{and} \quad p_2(x, y) = y.$$

This kernel pair always has a coequalizer $\pi : X \to X/Eq(f)$, which is the quotient of X by the equivalence relation $Eq(f)$ on X.

$$\begin{tikzcd} Eq(f) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \\ & & X/Eq(f) \arrow[ur, dashed, "\exists\, !\, i"'] & \end{tikzcd}$$

To check the second property of a regular category, recall that the pullback of two morphisms $f : X \to Y$ and $g : Z \to Y$ is defined as $(X \times_Y Z, p_1, p_2)$
with $X \times_Y Z = \{(x, z) \in X \times Z | f(x) = g(z)\}$.
If g is a regular epimorphism (i.e. a surjective homomorphism in the category Grp),

$$\begin{tikzcd}
X \times_Y Z \arrow[r, "p_2"] \arrow[d, "p_1"'] & Z \arrow[d, "f"] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}$$

we have to check that p_2 is also a surjective homomorphism. For this, let us take $x \in X$. Since g is surjective, there exists $z \in Z$ such that $g(z) = f(x)$. By definition of the pullback $X \times_Y Z$, we have that $(x, z) \in X \times_Y Z$ and $p_2(x, z) = z$.
The proofs that the category *Set* of sets and the category *Ab* of abelian groups are regular are identical.

The category $Grp(Top)$ of topological groups is regular

Recall that in the category $Grp(Top)$, the objects are the topological groups that are topological spaces with a group structure $(G, ., \tau)$ in a way that the group operation and the inversion map are continuous.
The morphisms of $Grp(Top)$ are continuous homomorphisms.
The proof that every kernel pair $(Eq(f), p_1, p_2)$ of a morphism $f : X \to Y$ has a coequalizer is exactly the same as in the category Grp of groups.
The coequalizer of a kernel pair $(Eq(f), p_1, p_2)$ is the canonical quotient
$\pi : X \to X/Eq(f)$ where the quotient group $X/Eq(f)$ is equipped with the quotient topology.

$$\begin{tikzcd}
Eq(f) \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \\
& & X/Eq(f) \arrow[ur, dashed, "\exists\,!\,i"'] &
\end{tikzcd}$$

Let us check that in $Grp(Top)$, this canonical quotient is an open surjective homomorphism.
We consider $K = ker(\pi)$, we are going to prove that for any open $V \in \tau_X$, $\pi(V)$ is open.
Observe that if $V \in \tau_X$, $\pi^{-1}(\pi(V)) = V \cdot K$. Indeed, if $v.k \in V \cdot K$,

$$\pi(v.k) = \pi(v).\pi(k) = \pi(v) \in \pi(V)$$

and this implies that $v.k \in \pi^{-1}(\pi(V))$. Conversely, if $z \in \pi^{-1}(\pi(V))$, there exists $v \in V$ such that $\pi(z) = \pi(v)$ which implies that $v^{-1}.z \in K$. Hence, $z \in V \cdot K$.

Therefore,

$$\pi^{-1}(\pi(V)) = \bigcup_{k \in K} V.k$$

which is in τ_X as a union of the open subsets $V.k$. Hence, π is an open surjective homomorphism.

Now that we have characterized the regular epimorphisms in $Grp(Top)$ we can check the regularity of the category $Grp(Top)$.
We have already seen that every kernel pair $(Eq(f), ., \tau_i)$ of a morphism $f : (X, ., \tau_X) \to (Y, ., \tau_Y)$ exists and is the canonical quotient $\pi : (X, ., \tau_X) \to (X/Eq(f), ., \tau_q)$.
We have to prove that open surjective homomorphisms are pullback stable.
We consider a pullback:

$$\begin{tikzcd}
(E \times_B A, ., \tau) \arrow[rr, "\pi_2"] \arrow[dd, "\pi_1"'] \arrow[dr, tail] & & (A, ., \tau_A) \arrow[dd, "f"] \\
& (E \times A, ., \tau_{E\times A}) \arrow[ur] \arrow[dl] & \\
E \arrow[rr, "p"'] & & B
\end{tikzcd}$$

with p an open surjective homomorphism. We have to check that π_2 is an open surjective homomorphism.
$(E \times_B A, \tau)$ is a topological subspace of $(E \times A, \tau_{E\times A})$.
Therefore, we can take an open set in the basis of τ that is the restriction to $E \times_A B$ of an open set $U_{E\times_A B}$ in the product topology $\tau_{E\times A}$ of $E \times A$.
We have that

$$\begin{aligned}
\pi_2\left(\{(e,a) \in U_E \times U_A | p(e) = f(a)\}\right) &= \{a \in U_A | \exists e \in U_E,\ p(e) = f(a)\} \\
&= \{a \in U_A | a \in f^{-1}\left(p(U_E)\right)\} \\
&= U_A \cap f^{-1}\left(p(U_E)\right).
\end{aligned}$$

But p is open, therefore $p(U_E) \in \tau_B$ and $f^{-1}(p(U_E)) \in \tau_A$ since f is continuous.
It follows that the image by π_2 of every open in $E \times_B A$ is open in A which implies that π_2 is open.

Definition 6. *[4] A homological category is a regular pointed category such that, given a commutative diagram:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & B \arrow[r] \arrow[d, two heads, "v"] \arrow[dr, phantom, "(2)"] & C \arrow[d] \\
D \arrow[r] & E \arrow[r] & F
\end{tikzcd}$$

if v is a regular epimorphism and if the left-hand square (1) and the whole rectangle (1)+(2) are pullbacks, then the right-hand square (2) is a pullback.

Lemma 7. *In an abelian category $\mathcal{C}$, consider the following commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & K[f] \arrow[r, "ker(f)"] \arrow[d, "\tau"] & A \arrow[r, "f"] \arrow[d, "g"] \arrow[dr, phantom, "(1)"] & B \arrow[r] \arrow[d, "g'"] & 0 \\
0 \arrow[r] & K[f'] \arrow[r, "ker(f')"'] & A' \arrow[r, "f'"'] & B' &
\end{tikzcd}$$

The square (1) is a pullback $\iff$ τ is an isomorphism.

Proof. If the right-hand square (1) is a pullback,

$$\begin{tikzcd}
K[f] \arrow[rr] \arrow[dd, "\tau"'] \arrow[dr] & & A \arrow[dd] \arrow[dr] & \\
& 0 \arrow[rr] \arrow[dd, equal] & & B \arrow[dd] \\
K[f'] \arrow[rr] \arrow[dr] & & A' \arrow[dr] \arrow[ur, phantom, "(1)"] & \\
& 0 \arrow[rr] & & B'
\end{tikzcd}$$

then the left-hand side of the cube is a pullback.

$$\begin{tikzcd}
K[f] \arrow[r] \arrow[d, "\tau"'] & 0 \arrow[d, equal] \\
K[f'] \arrow[r] & 0
\end{tikzcd}$$

Therefore, τ is an isomorphism.
Conversely, if τ is an isomorphism,

$$\begin{tikzcd}
0 \arrow[r] & K[f] \arrow[rr, "ker(f)"] \arrow[dr, dashed, "\exists!\gamma"] \arrow[dd, "\tau"', "\simeq"] & & A \arrow[rr, "f"] \arrow[dr, dotted, "\exists!\alpha"] \arrow[dd, "g"'] & & B \arrow[r] \arrow[dd, "g'"] & 0 \\
0 \arrow[rr, bend left] & & K[p_2] \arrow[rr, bend left, "ker(p_2)"] \arrow[dl, dashed, "\exists!\beta"] & & A' \underset{B'}{\times} B \arrow[ur, "p_2"'] \arrow[dl, "p_1"] & & \\
0 \arrow[r] & K[f'] \arrow[rr, "ker(f')"'] & & A' \arrow[rr, "f'"'] & & B' &
\end{tikzcd}$$

We consider the pullback $(A' \underset{B'}{\times} B, p_1, p_2)$ of f' and g'.

Because of the commutativity of the diagram, there exists a unique α such that $p_2 \circ \alpha = f$ and $p_1 \circ \alpha = g$.
We consider $ker(p_2) : K[p_2] \to A' \times B$, the kernel of p_2.
Therefore, $f' \circ p_1 \circ ker(p_2) = g' \circ p_2 \circ ker(p_2) = 0$ implies the existence of a unique β such that $ker(f') \circ \beta = p_1 \circ ker(p_2)$.
Similarly, since $p_2 \circ \alpha \circ ker(f) = f \circ ker(f) = 0$, there exists a unique γ such that $ker(p_2) \circ \gamma = \alpha \circ ker(f)$.
Moreover, since $ker(f') \circ \beta \circ \gamma = p_1 \circ ker(p_2) \circ \gamma = p_1 \circ \alpha \circ ker(f) = g \circ ker(f) = ker(f') \circ \tau$, we know that $\tau = \beta \circ \gamma$, since $ker(f')$ is a monomorphism.
By the first part of the proof applied to the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K[p_2] \arrow[r, "ker(p_2)"] \arrow[d, "\beta"] & A' \underset{B'}{\times} B \arrow[r, "p_2"] \arrow[d, "p_1"] & B \arrow[r] \arrow[d, "g'"] & 0 \\
0 \arrow[r] & K[f'] \arrow[r, "ker(f')"'] & A' \arrow[r, "f'"'] & B' &
\end{tikzcd}$$

β is an isomorphism, therefore since $\tau = \beta \circ \gamma$ and τ is an isomorphism, we have that γ is also an isomorphism.
Therefore, we get a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K[f] \arrow[r, "ker(f)"] \arrow[d, "\gamma"] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K[p_2] \arrow[r, "ker(p_2)"'] & A' \times B \arrow[r, "p_2"] & B \arrow[r] & 0
\end{tikzcd}$$

Therefore by the short five lemma [2], that holds in abelian categories, we conclude that α is also an isomorphism, and (1) is a pullback. $\square$

Proposition 8. *Abelian categories are homological categories.*

Proof. We have already observed that abelian categories are regular categories.
Let us take a commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & B \arrow[r] \arrow[d, two heads, "v"] \arrow[dr, phantom, "(2)"] & C \arrow[d] \\
D \arrow[r] & E \arrow[r] & F
\end{tikzcd}$$

with v a regular epimorphism and with the left-hand square (1) and the whole rectangle (1)+(2) pullbacks.
We have to show the right-hand square (2) is a pullback.

Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K[a] \arrow[r, "ker(a)"] \arrow[d, dashed, "\exists! \hat{e}"] & A \arrow[r, "a"] \arrow[d, "e"] \arrow[dr, phantom, "(1)"] & D \arrow[d, "g"] & \\
0 \arrow[r] & K[b] \arrow[r, "ker(b)"] \arrow[d, dashed, "\exists! \hat{f}"] & B \arrow[r, "b"] \arrow[d, "f"] \arrow[dr, phantom, "(2)"] & E \arrow[r, dashed] \arrow[d, "h"] & 0 \\
0 \arrow[r] & K[c] \arrow[r, "ker(c)"] & C \arrow[r, "c"] & F &
\end{tikzcd}$$

Since $c \circ f \circ ker(b) = h \circ b \circ ker(b) = 0$, there exists a unique $\hat{f}$ such that $ker(c) \circ \hat{f} = f \circ ker(b)$.
Similarly, since $b \circ e \circ ker(a) = g \circ a \circ ker(a) = 0$, there exists a unique $\hat{e}$ such that $e \circ ker(a) = ker(b) \circ \hat{e}$.
Since the squares (1) and (1)+(2) are pullbacks, by Lemma 7, $\hat{e}$ and $\hat{f} \circ \hat{e}$ are isomorphisms.
Therefore, $\hat{f}$ is also an isomorphism and since b is an epimorphism, by Lemma 7, (2) is a pullback. $\square$

Definition 9. *[13] A regular category $\mathcal{C}$ is normal if:*

1. *$\mathcal{C}$ is pointed, i.e. $\mathcal{C}$ has a zero object.*
2. *any regular epimorphism is a normal epimorphism, i.e. a cokernel.*

Examples of normal categories

Let us illustrate the concept of normal categories with some examples.

The categories Grp of groups, Ab of abelian groups and $Grp(Top)$ of topological groups are normal

We have already shown that these categories are regular.
These categories are pointed, the zero object is the trivial group.
We have to show that every surjective homomorphism is the cokernel of a morphism (its kernel). Let us take a surjective homomorphism $f : X \to Y$, its kernel $k : K[f] \to A$ and $\pi : A \to A/K[f]$ the canonical quotient.

$$\begin{tikzcd}
K[f] \arrow[r, "k"] & A \arrow[rr, "f"] \arrow[dr, two heads, "\pi"'] & & B \\
& & A/K[f] \arrow[ur, tail, "i"'] &
\end{tikzcd}$$

Since f is surjective, by the first isomorphism theorem, the morphism i provides an isomorphism $A/K[f] \cong B$. But π is the cokernel of k, hence f is the cokernel of k.

The category $PreOrdGrp$ of preordered groups is normal

The proof that this group is normal is made in detail in [8]. Here, we are only going to describe the category and some of its properties.
Let us describe what the category of preordered groups is. The objects of $PreOrdGrp$ $(G, \leq)$ are groups $(G, +)$ endowed with a preorder relation $\leq$ on G such that

$$\forall a, b, c, d \in G,\ a \leq c \text{ and } b \leq d \text{ implies } a + b \leq c + d.$$

The morphisms of $PreOrdGrp$ are monotone maps that preserve the group structure.
An essential property to describe this category is that giving a preorder relation $\leq$ on G such that $(G, \leq) \in PreOrdGrp$ is equivalent to giving a submonoid P_G of G such that $\forall a \in P_G$ and $b \in G$, $b + a - b \in P_G$.
With the help of this property, we can describe regular epimorphisms. Indeed, a morphism $f : X \to Y$ is a regular epimorphism in $PreOrdGrp$ if and only if f is surjective and $f(P_X) = P_Y$.

$$\begin{tikzcd} P_X \arrow[r, "f", -{Triangle[open]}] \arrow[d, tail] & P_Y \arrow[d, tail] \\ X \arrow[r, "f"', -{Triangle[open]}] & Y \end{tikzcd}$$

Therefore, regular epimorphisms are easily seen to be pullback stable.

Proposition 10. *[2] If $\mathcal{C}$ is a normal category, every regular epimorphism $f : X \to Y$ is the cokernel of its kernel.*

Proof. If f is a regular epimorphism, the normality of $\mathcal{C}$ implies that there is a morphism g such that f is the cokernel of g.
Let us consider the morphisms k defined as the kernel of f and q defined as the cokernel of k.

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, dashed, "\exists\,!\,\psi"'] & & X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\ & K \arrow[ur, "k"'] & & Q \arrow[ur, dashed, "\exists\,!\,\phi"'] & \end{tikzcd}$$

Since f is the cokernel of g, $f \circ g = 0$.

$$\begin{tikzcd}
Z \arrow[drr, bend left] \arrow[dr, dashed, "\exists\,!\,\psi"] \arrow[ddr, bend right, "g"'] & & \\
& K \arrow[r] \arrow[d, "k"] & 0 \arrow[d] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Therefore, since k is the kernel of f, there exists a unique ψ such that $k \circ \psi = g$.
Similarly, since k is the kernel of f, $f \circ k = 0$, and there exists a unique ϕ such that $\phi \circ q = f$:

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "k"] & 0 \arrow[d] \arrow[ddr, bend left] & \\
X \arrow[r, "q"] \arrow[drr, bend right, "f"'] & Q \arrow[dr, dashed, "\exists\,!\,\phi"] & \\
& & Y
\end{tikzcd}$$

We have that $q \circ g = q \circ k \circ \psi = 0$.

$$\begin{tikzcd}
Z \arrow[r] \arrow[d, "g"] & 0 \arrow[d] \arrow[ddr, bend left] & \\
X \arrow[r, "f"] \arrow[drr, bend right, "q"'] & Y \arrow[dr, dashed, "\exists\,!\,i"] & \\
& & Q
\end{tikzcd}$$

Therefore, since f is the cokernel of g, there exists a unique i such that $i \circ f = q$.
Observe that $\phi \circ i \circ f = \phi \circ q = f = 1_Y \circ f$ and since f is an epimorphism, we have that $\phi \circ i = 1_Y$.
Similarly, $i \circ \phi \circ q = i \circ f = q = 1_Q \circ q$ and since q is an epimorphism, we have that $i \circ \phi = 1_Q$.
Therefore, f is the cokernel of its kernel. $\square$

Proposition 11. *If $\mathcal{C}$ is a regular category, any morphism $f : X \to Y$ has a factorization $f = i \circ p$ with p a regular epimorphism and i a monomorphism. This factorization is unique.*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & I \arrow[ur, dashed, "\exists\,!\,i"'] & \end{tikzcd} \tag{1}$$

Proof. Let us first prove the existence of the factorization.
We consider the kernel pair $(Eq(f), p_1, p_2)$ of f and the coequalizer p of p_1 and p_2:

$$\begin{tikzcd} Eq(f) \arrow[r, shift left, "p_2"] \arrow[r, shift right, "p_1"'] & X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & & I \arrow[ur, dashed, "\exists\,!\,i"'] & \end{tikzcd}$$

Since $f \circ p_2 = f \circ p_2$ there exists a unique i such that $i \circ p = f$.
By definition, p is a regular epimorphism, we still have to prove that i is a monomorphism.
Let us consider the kernel pair $(Eq(i), i_1, i_2)$ of i. To prove that i is a monomorphism, we have to prove that the projections i_1 and i_2 are equal.
We consider the following diagram:

$$\begin{tikzcd} Eq(f) \arrow[dr, equal] \arrow[drrr, bend left, "p_2"] \arrow[dddr, bend right, "p_1"'] & & & \\ & P \arrow[r, "\alpha_2"] \arrow[d, "\alpha_1"'] & P_2 \arrow[r, "p_2'"] \arrow[d, "p_1'"] & X \arrow[d, "p"] \arrow[dd, bend left, "f"] \\ & P_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Eq(i) \arrow[r, "i_2"] \arrow[d, "i_1"] & I \arrow[d, "i"] \\ & X \arrow[r, "p"'] \arrow[rr, bend right, "f"'] & I \arrow[r, "i"'] & Y \end{tikzcd}$$

The four squares are pullbacks, therefore, $(P, p_2' \circ \alpha_2, \pi_1 \circ \alpha_1)$ is the kernel pair of f, $P = Eq(f)$, $p_1' \circ \alpha_1 = p_1$ and $\pi_2 \circ \alpha_2 = p_2$.
By definition of regular category, and since p is a regular epimorphism, both p_1' and α_1 are regular epimorphisms.
Similarly, π_2 and α_2 are regular epimorphisms.
This implies that $\pi_2 \circ \alpha_1$ is an epimorphism.
But we have that
$i_1 \circ \pi_2 \circ \alpha_1 = p \circ \pi_1 \circ \alpha_1 = p \circ p_1 = p \circ p_2 = p \circ p_2' \circ \alpha_2 = i_2 \circ p_1' \circ \alpha_2 = i_2 \circ \pi_2 \circ \alpha_1.$

Therefore, $i_1 = i_2$ and i is a monomorphism as desired.
Let us then prove the uniqueness of this factorization. We consider two factorizations $f = j \circ q = i \circ p$ with i and j monomorphisms and p and q regular epimorphisms.

$$\begin{tikzcd}
& I \arrow[dr, tail, "i"] & \\
X \arrow[ur, -{Triangle[open]}, "p"] \arrow[rr, "f"] \arrow[dr, -{Triangle[open]}, "q"'] & & Y \\
& J \arrow[ur, tail, "j"'] &
\end{tikzcd}$$

Recall that regular epimorphisms are strong epimorphisms.

$$\begin{tikzcd}
X \arrow[r, -{Triangle[open]}, "p"] \arrow[d, "q"'] & I \arrow[d, "i"] \arrow[dl, dashed, "\exists\,!\,t"] \\
J \arrow[r, tail, "j"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, -{Triangle[open]}, "q"] \arrow[d, "p"'] & J \arrow[d, "j"] \arrow[dl, dashed, "\exists\,!\,\bar{t}"] \\
I \arrow[r, tail, "i"'] & Y
\end{tikzcd}$$

We have therefore a morphism t such that $t \circ p = q$ and $j \circ t = i$ and another morphism $\bar{t}$ such that $\bar{t} \circ q = p$ and $i \circ \bar{t} = j$.
Hence, $i \circ \bar{t} \circ t = j \circ t = i = i \circ 1_I$ which implies that $\bar{t} \circ t = 1_I$.
Similarly, $j \circ t \circ \bar{t} = i \circ \bar{t} = j = j \circ 1_J$ which implies that $t \circ \bar{t} = 1_J$.
We can therefore conclude that the factorization is unique. $\square$

Additionally, we can remark that in a normal category, given any $f : A \to B$, the normal epi/mono factorization can be obtained as follows:

$$\begin{tikzcd}
K[f] \arrow[r, "ker(f)"] \arrow[dr] & A \arrow[r, "f"] \arrow[dr, -{Triangle[open]}, "q"'] & B \\
& 0 \arrow[r] & Q \arrow[u, tail, "m"']
\end{tikzcd}$$

where $q = coker(ker(f))$.
This follows easily from Proposition 11 and the fact that any regular epimorphism is the cokernel of its kernel by Proposition 10.

Proposition 12. *In a normal category, f is a monomorphism if and only if its kernel is zero: $K[f] \cong 0$.*

Proof. If f is a monomorphism, then one has $f \circ ker(f) = 0 = f \circ 0$, hence $ker(f) = 0$ since f is a monomorphism. Conversely, let us assume that the kernel of f is the zero object.

$$\begin{tikzcd} K[f] \arrow[rr, tail, "ker(f)"] \arrow[dr, "\simeq"'] & & A \\ & 0 \arrow[ur] & \end{tikzcd}$$

Since we are in a normal category, by Proposition 11 we can write f as the composite of a normal epimorphism followed by a monomorphism.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "q"'] & & B \\ & Q \arrow[ur, tail, "m"'] & \end{tikzcd}$$

Let us show the normal epimorphism q is an isomorphism.
Since m is a monomorphism, we have that the kernel of m is zero.

$$\begin{tikzcd} 0 = K[f] \arrow[r, "k"] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "q"] & B \\ 0 = K[m] \arrow[r, "k'"'] & Q \arrow[ur, tail, "m"] & \end{tikzcd}$$

Since $q = coker(k)$ and $K[f] = 0$, the left-hand square in the diagram is a pushout. Therefore, q is an isomorphism. Since f is the composite of a monomorphism and an isomorphism, we conclude that f is a monomorphism. $\square$

Proposition 13. *[4]*
In a normal category $\mathcal{C}$, pullbacks reflect monomorphisms:

$$\begin{tikzcd}
C \underset{B}{\times} A \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "f"] \\
C \arrow[r, "g"'] & B
\end{tikzcd}$$

i.e if p_1 is a monomorphism then f is a monomorphism.

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
K[p_1] \arrow[rr, "\hat{p_2}"] \arrow[dr] \arrow[dd] & & K[f] \arrow[dr] \arrow[dd] & \\
& C \underset{B}{\times} A \arrow[rr, "p_2"] \arrow[dd, "p_1"'] & & A \arrow[dd, "f"] \\
0 \arrow[rr, equal] \arrow[dr] & & 0 \arrow[dr] & \\
& C \arrow[rr, "g"'] & & B
\end{tikzcd}$$

Since the rectangle:

$$\begin{tikzcd}
K[p_1] \arrow[r, "\hat{p_2}"] \arrow[d] \arrow[dr, phantom, "(1)"] & K[f] \arrow[r, "ker(f)"] \arrow[d] \arrow[dr, phantom, "(2)"] & A \arrow[d, "f"] \\
0 \arrow[r, equal] & 0 \arrow[r] & B
\end{tikzcd}$$

is a pullback and (2) is a pullback, (1) is a pullback. This implies that $\hat{p_2}$ is an isomorphism.
Since p_1 is a monomorphism, $K[p_1] = 0$. Therefore $K[f] = 0$ and f is a monomorphism by Proposition 12. □

Chapter 2

Homological closure operator

In this section, we are going to introduce and characterize a particular type of closure operator that will be called homological. To do this, we are going to adapt the proof from [5] which is the main source of this chapter.
Although we are currently working in normal categories, we choose to keep the name "homological" as in [5] to avoid any confusion with another closure operator that is called "normal" in [7].
Let us first recall what a regular epireflective subcategory is.

Definition 14. *A* ***regular epireflective subcategory*** $\mathcal{A}$ *of a category* $\mathcal{B}$

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend right, "F"'] \end{tikzcd}$$

is a full replete subcategory of $\mathcal{B}$ *such that for every* $B \in \mathcal{B}$ *the* B*-component* $\eta_B : B \to UF(B)$ *of the unit* η *of the adjunction is a regular epimorphism.*

Examples of regular epireflective subcategories

The category *Ab* of abelian categories is regular epireflective in *Grp*

$$\begin{tikzcd} Ab \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & Grp \arrow[l, bend right, "ab"'] \end{tikzcd}$$

Given a group B in Grp, the B-component η_B of the unit η of the adjunction is the projection morphism $B \longmapsto \frac{B}{[B,B]}$

where $[B,B] =< [x,y] \mid x \in G \text{ and } y \in G >_G$ is the derived subgroup of B.

$$\begin{tikzcd}
K[f] \arrow[dr] & & & \\
& B \arrow[rr, "\eta_B"] \arrow[dr, "\forall f"'] & & B/[B,B] \arrow[dl, dashed, "\exists\,!\,\hat{f}"] \\
{[B,B]} \arrow[uu] \arrow[ur] & & A &
\end{tikzcd}$$

Given any morphism $f : B \to A$, with A abelian, one easily checks that $[B,B] \subseteq K[f]$. The universal property of the quotient $B/[B,B]$ gives a unique morphism $\hat{f}$ such that $\hat{f} \circ \eta_B = f$.
Moreover, since $[B,B]$ is a normal subgroup of B, this projection is surjective.
This shows that the category Ab of abelian groups is regular epireflective in the category Grp of groups.

The category $Ab_{t.f}$ of torsion-free abelian groups is regular epireflective in the category Ab of abelian groups

$$\begin{tikzcd}
Ab_{t.f} \arrow[r, bend right, "V"'] \arrow[r, phantom, "\perp"] & Ab \arrow[l, bend right, "F"']
\end{tikzcd}$$

Given an abelian group B, the B-component η_B of the unit η of the adjunction is the quotient morphism $B \longmapsto \frac{B}{T(B)}$, where $T(B) = \{b \in B | \exists n \in \mathbb{N}^*,\ n.b = 0\}$ is the "torsion subgroup" of B.
Let us show that any morphism $f : B \to A$ where $A \in Ab_{t.f}$, uniquely factors through η_B.

$$\begin{tikzcd}
K[f] \arrow[dr] & & & \\
& B \arrow[rr, "\eta_B"] \arrow[dr, "f"'] & & B/T(B) \arrow[dl, dashed, "\exists\,!\hat{f}"] \\
T(B) \arrow[uu] \arrow[ur] & & A &
\end{tikzcd}$$

Indeed, if $b \in T(B)$, there exists $n \in \mathbb{N}^*$ such that $n.b = 0$, therefore $0 = f(n.b) = n.f(b)$ which implies that $f(b) = 0$, and $b \in K[f]$.
The universal property of the quotient $B/T(B)$ gives a unique morphism $\hat{f} : B/T(B) \to A$ such that $\hat{f} \circ \eta_B = f$.
Since $T(B)$ is a normal subgroup of B, the projection η_B is surjective.
This shows that the category $Ab_{t.f}$ of torsion-free abelian groups is regular epireflective in the category Ab of abelian groups.

Note that when a category $\mathcal{A}$ is a regular epireflective subcategory of a regular category $\mathcal{B}$ which is also a regular epireflective subcategory of a regular category $\mathcal{C}$, then $\mathcal{A}$ is a regular epireflective subcategory of $\mathcal{C}$.
This implies that $Ab_{t.f}$ is regular epireflective in Grp.

$$\begin{tikzcd} Ab_{t.f} \arrow[r, bend right, "V"'] \arrow[r, phantom, "\perp"] & Ab \arrow[l, bend right, "F"'] \arrow[r, bend right, "U"'] \arrow[r, phantom, "\perp"] & Grp \arrow[l, bend right, "ab"'] \end{tikzcd}$$

The following definition was introduced in [5]:

Definition 15 (**Idempotent closure operator** $\overline{()}$)**.** *An idempotent closure operator $\overline{()}$ on kernels consists in giving, for every kernel $S \to X$, another kernel $\overline{S} \to X$, called the closure of S in X. This assignment has to satisfy the following properties, where $S \to X$ and $T \to X$ are kernels and $f \in \mathcal{C}(Y, X)$:*

1. $S \subseteq \overline{S}$;
2. $S \subseteq T \implies \overline{S} \subseteq \overline{T}$;
3. $\overline{f^{-1}(S)} \subseteq f^{-1}(\overline{S})$;
4. $\overline{\overline{S}} = \overline{S}$.

A kernel is said to be closed when $S = \overline{S}$, and dense when $\overline{S} = X$.

Definition 16 (**Homological closure operator**)**.** *[5] An idempotent closure operator on kernels is a homological closure operator if it also satisfies the following axiom:*
For any regular epimorphism f in $\mathcal{C}(Y, X)$, one has that $\overline{f^{-1}(S)} = f^{-1}(\overline{S})$ for any kernel $S \to X$.

Lemma 17. *Let $\overline{()}$ be a homological closure operator, let $s : S \to Y$ be a kernel and $f : X \to Y$ a regular epimorphism. Then in the pullback*

$$\begin{tikzcd} f^{-1}(S) \arrow[r, tail, "f^{-1}(s)"] \arrow[d, "g"] & X \arrow[d, "f"] \\ S \arrow[r, tail, "s"'] & Y \end{tikzcd}$$

s is a closed (respectively, dense) kernel if and only if $f^{-1}(s)$ is a closed (dense) kernel.

Proof. Assume that s is dense. One has to prove that $\overline{f^{-1}(S)} = X$. Since S is a regular epimorphism, $\overline{f^{-1}(S)} = f^{-1}(\overline{S})$.
But we have $\overline{S} = Y$ as s is dense. Therefore, $\overline{f^{-1}(S)} = f^{-1}(Y) = X$.
For the other case, let us take s, a closed kernel. Similarly, $\overline{f^{-1}(S)} = f^{-1}(\overline{S}) = f^{-1}(S)$.
This leads to conclude $f^{-1}(s)$ is a closed kernel.

Let us consider the other implication.
In [5], the proof is given in the closed case. Here, we consider the dense case.
Let us assume that $f^{-1}(S)$ is dense, i.e $\overline{f^{-1}(S)} \cong X$.
Since f is a regular epimorphism, $\overline{f^{-1}(S)} \cong f^{-1}(\overline{S}) \cong X$.

$$\begin{tikzcd}
f^{-1}(\overline{S}) \arrow[r, "\cong"] \arrow[d] & X \arrow[d, two heads, "f"] \\
\overline{S} \arrow[r, tail, "\overline{s}"'] & Y
\end{tikzcd}$$

The commutativity of the previous diagram shows that $\overline{s}$ is a regular epimorphism. But $\overline{s}$ is the composite of an isomorphism and a regular epimorphism. Therefore, $\overline{s}$ is also a monomorphism. It is then an isomorphism which proves that s is dense. $\square$

Theorem 18. *Let $\mathcal{C}$ be a normal category. There is a bijection between the regular epireflective subcategories of $\mathcal{C}$ and the homological closure operators on kernels.*

This theorem and its proof for the case of homological categories can be found in [5]. Here, we observe that the same proof still holds for normal categories.

Proof. To prove the first implication, one has to take a homological closure operator on kernels $\overline{()}$ and prove that the full replete subcategory $\mathcal{F}$ of $\mathcal{C}$ defined by:

$$F \in \mathcal{F} \text{ if and only if } 0 \to F \text{ is closed}$$

is regular epireflective in $\mathcal{C}$.
Given X in $\mathcal{C}$, let us define η_X as the cokernel of k_X the closure $\overline{0}_X$ of 0 in X.

$$\begin{tikzcd}
\overline{0}_X \arrow[r, "k_X"] \arrow[d] & X \arrow[d, "\eta_X"] \\
0 \arrow[r] & X/\overline{0}_X
\end{tikzcd}$$

By construction, in the previous pullback, one has that η_X is a regular epimorphism. Then, Lemma 17 tells us that $0 \to X/\overline{0}_X$ is a closed kernel. Therefore, $F(X) := X/\overline{0}_X$ is in $\mathcal{F}$.
It is well known that proving that $F : \mathcal{C} \to \mathcal{F}$ is the left adjoint of $U : \mathcal{C} \to \mathcal{F}$ is equivalent to proving the existence of a natural transformation $\eta : Id_{\mathcal{B}} \to UF$ with each X-component $\eta_X : X \to UF(X)$ universal from X to U.

We are going to prove the universality of η_X. Let us take $f : X \to F$ with $F \in \mathcal{F}$.

$$\begin{tikzcd}
X \arrow[rr, "\eta_X"] \arrow[dr, "f"'] & & F(X) \arrow[dl, "\exists! \overline{f}"] \\
& F &
\end{tikzcd}$$

One has to prove the existence of a unique $\overline{f} : F(X) \to F$ such that $\overline{f} \circ \eta_X = f$.
Since $\overline{()}$ is a homological closure operator, $0 \subseteq Ker(f) = f^{-1}(0)$ implies that

$$\overline{0_X} \subseteq \overline{f^{-1}(0_F)} \subseteq f^{-1}(\overline{0_F}) = f^{-1}(0_F) = Ker(f)$$

Therefore, we have a factorization of $f \circ k_X$ through the 0-object.

$$\begin{tikzcd}
K(X) \arrow[r, "k_X"] \arrow[dr] & X \arrow[rr, "\eta_X"] \arrow[dr, "f"] & & F(X) \arrow[dl, dashed, "\overline{f}"] \\
& 0 \arrow[r] & F &
\end{tikzcd}$$

Recall that η_X is defined as the cokernel of k_X, therefore $f \circ k_X = 0$ implies the existence of a unique $\overline{f} : F(X) \to F$ such that $\overline{f} \circ \eta_X = f$, as desired.

Conversely, assume that we have a regular epireflective subcategory $\mathcal{F}$ of $\mathcal{C}$ with reflector $F : \mathcal{C} \to \mathcal{F}$.
For a kernel $s : S \to X$, we will define its closure $\overline{S}$ as the pullback of the kernel $k_{X/S}$ of $\eta_{X/S}$ along the quotient q of X by S:

$$\begin{tikzcd}
& \overline{S} \arrow[r] \arrow[d, "\overline{s}"] & K[X/S] \arrow[d, "k_{X/S}"] \\
S \arrow[ur, "i_S"] \arrow[r, "s"'] & X \arrow[r, "q"'] \arrow[dr, "\eta_{X/S} \circ q"'] & X/S \arrow[d, "\eta_{X/S}"] \\
& & F(X/S)
\end{tikzcd}$$

We must show that our closure respects the axioms of a homological closure operator on kernel.
(1)$S \subseteq \overline{S}$ since $\overline{s}$ is the kernel of $\eta_{X/S} \circ q$.
(2) $S \subseteq T \implies \overline{S} \subseteq \overline{T}$. For this, consider the following commutative diagram,

$$\begin{tikzcd}
 & \overline{S} \arrow[rr, "\overline{q}_S"] \arrow[dr, tail, "\overline{s}"] \arrow[dd, dashed, "3)\exists!\overline{\phi}"'] & & K[X/S] \arrow[dr, "k_{X/S}"] \arrow[dd, dashed, "2)\exists!\tilde{\phi}"] & \\
S \arrow[ur, tail, "i_S"] \arrow[rr, tail, "s"'] \arrow[dd, "i"'] & & X \arrow[rr, "q_S"'] \arrow[dd, equal] & & X/S \arrow[dd, dashed, "1)\exists!\phi"] \\
 & \overline{T} \arrow[rr, "\overline{q}_T"] \arrow[dr, tail, "\overline{t}"] & & K[X/T] \arrow[dr, "k_{X/T}"] & \\
T \arrow[ur, tail, "i_T"] \arrow[rr, tail, "t"'] & & X \arrow[rr, "q_T"'] & & X/T
\end{tikzcd}$$

that is built as follows.

1. **There exists a unique ϕ such that $\phi \circ q_S = q_T$.**
   Since q_S is the quotient of X by S, the following diagram is a pushout, and we have to prove that $q_T \circ s$ is a zero-morphism to prove the existence of such a ϕ.

$$\begin{tikzcd}
S \arrow[r, "s"] \arrow[d] & X \arrow[d, "q_S"] \arrow[ddr, bend left, "q_T"] & \\
0 \arrow[r] \arrow[drr, bend right] & X/S \arrow[dr, dashed, "\phi"] & \\
 & & X/T
\end{tikzcd}$$

   Since $S \subseteq T$, we have $q_T \circ s = q_T \circ t \circ i = 0$, hence we have the existence of a unique ϕ such that $\phi \circ q_S = q_T$.

2. **There exists a unique $\tilde{\phi}$ such that $\phi \circ k_{X/S} = k_{X/T} \circ \tilde{\phi}$.**
   Since F is a functor, we have that $F(\phi) \circ \eta_{X/S} = \eta_{X/T} \circ \phi$.

Hence, $\eta_{X/T} \circ \phi \circ k_{X/S} = F(\phi) \circ \eta_{X/S} \circ k_{X/S} = 0$.

$$\begin{tikzcd}
K[X/S] \arrow[dr, dashed, "\exists!\tilde{\phi}"] \arrow[drr, bend left, "\phi\circ k_{X/S}"] \arrow[ddr, bend right] & & \\
& K[X/T] \arrow[r, "k_{X/T}"] \arrow[d] & X/T \arrow[d, "\eta_{X/T}"] \\
& 0 \arrow[r] & UF(X/T)
\end{tikzcd}$$

Therefore, since $k_{X/T}$ is the kernel of $\eta_{X/T}$, we have the existence of $\tilde{\phi}$ such that $\phi \circ k_{X/S} = k_{X/T} \circ \tilde{\phi}$.

3. **There exists a unique $\overline{\phi}$ such that $\overline{t} \circ \overline{\phi} = \overline{s}$.**
   Observe that

$$k_{X/T} \circ \tilde{\phi} \circ \overline{q}_S = \phi \circ k_{X/S} \circ q_S = \phi \circ q_S \circ \overline{s} = q_T \circ \overline{s}.$$

   Therefore, the following diagram is a pullback by construction and there exists a unique $\overline{\phi}$ such that $q_T \circ \overline{\phi} = \tilde{\phi} \circ \overline{q}_S$ and $\overline{t} \circ \overline{\phi} = \overline{s}$.

$$\begin{tikzcd}
\overline{S} \arrow[dr, dashed, "\exists!\overline{\phi}"] \arrow[drr, bend left, "\tilde{\phi}\circ\overline{q}_S"] \arrow[ddr, bend right, "\overline{s}"'] & & \\
& \overline{T} \arrow[r, "\overline{q}_T"] \arrow[d, "\overline{t}"'] & K[X/T] \arrow[d, "k_{X/T}"] \\
& X \arrow[r, "q_T"'] & X/T
\end{tikzcd}$$

   The last equality implies that $\overline{\phi}$ is a monomorphism since $\overline{s}$ is a monomorphism. Therefore we have the desired inclusion $\overline{S} \subseteq \overline{T}$.

(3) $\overline{f^{-1}(S)} \subseteq f^{-1}(\overline{S})$

Consider the following commutative diagram:

$$
\begin{tikzcd}
\overline{f^{-1}(S)} \arrow[rr, "\overline{q_{f^{-1}(S)}}"] \arrow[dr, "\overline{f^{-1}(s)}"] \arrow[dd, dashed, "\overline{\phi}"'] & & K[Y/f^{-1}(S)] \arrow[dr, "k_{Y/f^{-1}(S)}"] \arrow[dd, dashed, "\tilde{\phi}" near start] & \\
& Y \arrow[rr, "q_{f^{-1}(S)}"' near start] \arrow[dd, "f" near end] & & Y/f^{-1}(S) \arrow[dd, dashed, "\phi"] \\
\overline{S} \arrow[rr, "\overline{q_S}" near start] \arrow[dr, "\overline{s}"'] & & K[X/S] \arrow[dr, "k_{X/S}"] & \\
& X \arrow[rr, "q_S"'] & & X/S
\end{tikzcd}
$$

that is built as follows.

1. **There exists a unique ϕ such that $\phi \circ q_{f^{-1}(S)} = q_S \circ f$.**
   Let us consider the following pullback:

$$
\begin{tikzcd}
f^{-1}(S) \arrow[r, "f^{-1}(s)"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\
S \arrow[r, "s"'] & X
\end{tikzcd}
$$

since $q_{f^{-1}(S)}$ is the quotient of Y by $f^{-1}(S)$, the following diagram is a pushout.

$$
\begin{tikzcd}
f^{-1}(S) \arrow[r, "f^{-1}(s)"] \arrow[d] & Y \arrow[d, "q_{f^{-1}(S)}"] \arrow[ddr, bend left, "q_s \circ f"] & \\
0 \arrow[r] \arrow[drr, bend right] & Y/f^{-1}(S) \arrow[dr, dashed, "\exists\,!\,\phi"] & \\
& & X/S
\end{tikzcd}
$$

By the definition of q_S we have that $q_S \circ s = 0$ and therefore, $q_S \circ f \circ f^{-1}(s) = q_S \circ s \circ g = 0$ and by the property of the previous pushout, we have the existence of a unique ϕ such that $\phi \circ q_{f^{-1}(S)} = q_S \circ f$.

2. **There exists a unique $\tilde{\phi}$ such that $\phi \circ k_{Y/f^{-1}(S)} = k_{X/S} \circ \tilde{\phi}$.**
The functoriality of F implies that $F(\phi) \circ \eta_{Y/f^{-1}(S)} = \eta_{X/S} \circ \phi$.
Hence $\eta_{X/S} \circ \phi \circ k_{Y/f^{-1}(S)} = F(\phi) \circ \eta_{Y/f^{-1}(S)} \circ k_{Y/f^{-1}(S)} = 0$.

$$\begin{tikzcd}
K[Y/f^{-1}(S)] \arrow[dr, dashed, "\exists\,!\,\tilde{\phi}"] \arrow[drr, bend left, "\phi \circ k_{Y/f^{-1}(S)}"] \arrow[ddr, bend right] & & \\
& K[X/S] \arrow[r, "k_{X/S}"] \arrow[d] & X/S \arrow[d, "\eta_{X/S}"] \\
& 0 \arrow[r] & UF(X/S)
\end{tikzcd}$$

Hence $\eta_{X/S} \circ \phi \circ k_{Y/f^{-1}(S)} = F(\phi) \circ \eta_{Y/f^{-1}(S)} \circ k_{Y/f^{-1}(S)} = 0$ and by the previous pushout we have the existence of the desired $\tilde{\phi}$.

3. **There exists a unique $\overline{\phi}$ such that $\bar{s} \circ \bar{\phi} = f \circ \overline{f^{-1}(s)}$.**
We have that

$$k_{X/S} \circ \tilde{\phi} \circ \overline{q_{f^{-1}(S)}} = \phi \circ k_{Y/f^{-1}(S)} \circ \overline{q_{f^{-1}(S)}} = \phi \circ q_{f^{-1}(S)} \circ \overline{f^{-1}(s)} = q_S \circ f \circ \overline{f^{-1}(s)}.$$

$$\begin{tikzcd}
\overline{f^{-1}(S)} \arrow[dr, dashed, "\exists\,!\,\overline{\phi}"] \arrow[drr, bend left, "\tilde{\phi} \circ \overline{q_{f^{-1}(S)}}"] \arrow[ddr, bend right, "f \circ \overline{f^{-1}(s)}"'] & & \\
& \overline{S} \arrow[r, "\overline{q_S}"] \arrow[d, "\bar{s}"] & K[X/S] \arrow[d, "k_{X/S}"] \\
& X \arrow[r, "q_S"'] & X/S
\end{tikzcd}$$

By definition of the closure operator, the previous diagram is a pullback and there exists a unique $\overline{\phi}$ such that $\bar{s} \circ \bar{\phi} = f \circ \overline{f^{-1}(s)}$.

We consider the following diagram which is a pullback by construction,

$$\begin{tikzcd}
\overline{f^{-1}(S)} \arrow[dr, dashed, "\exists\,!\,t"] \arrow[drr, bend left, "\overline{f^{-1}(s)}"] \arrow[ddr, bend right, "\overline{\phi}"'] & & \\
& f^{-1}(\overline{S}) \arrow[r, "f^{-1}(\overline{s})"] \arrow[d, "g"] & Y \arrow[d, "f"] \\
& \overline{S} \arrow[r, "\bar{s}"'] & X
\end{tikzcd}$$

since $\overline{f^{-1}(s)}$ is a monomorphism, we have the desired inclusion $\overline{f^{-1}(S)} \subseteq f^{-1}(\overline{S})$.

(4) $\overline{\overline{S}} = \overline{S}$
The quotient of X by $\overline{S}$ is $\eta_{X/S} \circ q$, because this latter is the cokernel of its kernel $\overline{s}$. But $F(X/S) \in \mathcal{F}$ which implies that $\overline{\overline{s}}$ is also defined as the kernel of $\eta_{X/S} \circ q$. Therefore, $\overline{\overline{S}} = \overline{S}$.

(5) For any regular epimorphism $f : X \to Y$, $\overline{f^{-1}(S)} = f^{-1}(\overline{S})$.
The proof is similar as for axiom (2), therefore we are going to give less details.
Consider the following commutative diagram:

$$\begin{tikzcd}
 & & & & \overline{f^{-1}(S)} \arrow[rr, "\overline{q'}"] \arrow[dd, "\overline{f^{-1}(s)}"] \arrow[dl, dashed, "t"'] & & K[Y/f^{-1}(S)] \arrow[dd, "k_{Y/f^{-1}(S)}"] \arrow[dl, dashed, "\tilde{\phi}"] & & \\
 & & & \overline{S} \arrow[rr, "\bar{q}"'] \arrow[dd, "\bar{s}"'] & & K[X/S] \arrow[dd, "k_{X/S}"] & & & \\
 & 0 \arrow[r] & f^{-1}(S) \arrow[rr, "f^{-1}(s)"] \arrow[dl, "g"'] \arrow[uurr, bend left, "i_{f^{-1}(S)}"] & & Y \arrow[rr, "q'"] \arrow[dl, "f"] & & Y/f^{-1}(S) \arrow[rr] \arrow[dl, dashed, "\phi"] & & 0 \\
0 \arrow[r] & S \arrow[rr, "s"'] \arrow[uurr, bend left, "i_S"] & & X \arrow[rr, "q"'] & & X/S \arrow[rr] & & 0 &
\end{tikzcd}$$

Since q' is the quotient of Y by $f^{-1}(S)$ and $q \circ f \circ f^{-1}(s) = q \circ s \circ g = 0$, there exists a unique ϕ such that $\phi \circ q' = q \circ f$. $q \circ f$ is a regular epimorphism, hence the morphism ϕ is a regular epimorphism.
Since F is a functor, we have that $F(\phi) \circ \eta_{Y/f^{-1}(S)} = \eta_{X/S} \circ \phi$.
Therefore, $\eta_{X/S} \circ \phi \circ k_{Y/f^{-1}(S)} = F(\phi) \circ \eta_{Y/f^{-1}(S)} \circ k_{Y/f^{-1}(S)} = 0$.
But we have that $k_{X/S}$ is the kernel of $\eta_{X/S}$, hence there exists a unique $\tilde{\phi}$ such that $k_{X/S} \circ \tilde{\phi} = \phi \circ k_{Y/f^{-1}(S)}$.
We have that $q \circ f \circ \overline{f^{-1}(s)} = \phi \circ q' \circ \overline{f^{-1}(S)} = \phi \circ k_{Y/f^{-1}(S)} \circ \overline{q'} = k_{X/S} \circ \tilde{\phi} = \overline{q'}$.
Therefore, by definition of $\overline{S}$, there exists a unique t such that $\overline{s} \circ t = f \circ \overline{f^{-1}(s)}$ and $\overline{q} \circ t = \tilde{\phi} \circ \overline{q'}$.
We already know that ϕ is a regular epimorphism, therefore since it is also a monomorphism, ϕ is an isomorphism.

Consider now the following commutative diagram:

$$\begin{tikzcd}
\overline{f^{-1}(S)} \arrow[r, "\overline{f^{-1}(s)}"] \arrow[d, "t"] & Y \arrow[r, "q'"] \arrow[d, "f"] & Y/f^{-1}(S) \arrow[d, "\phi"] \\
\overline{S} \arrow[r, "\overline{s}"'] & X \arrow[r, "q"'] & X/S
\end{tikzcd}$$

Since ϕ is an isomorphism, (1)+(2) and (2) are pullbacks. This implies that (1) is a pullback. Therefore, $\overline{f^{-1}(S)}$ is $f^{-1}(\bar{S})$.
We still have to prove the existence of a bijection between the previous construction of epireflections and homological closure operators on kernels.
Let us consider $\overline{()}$ a homological closure operator, F the functor defined in the first part

of the proof and $\tilde{()}$ the homological closure operator associated with F as in the second part of the proof.
We have to show that $\overline{()} = \tilde{()}$ that is for every kernel $S \to X$, $\overline{S} = \tilde{S}$.
Consider the following pullback:

$$\begin{tikzcd} & \tilde{S} \arrow[r] \arrow[d, "\overline{s}"] & \overline{0_{X/S}} \arrow[d, "k_{X/S}"] \\ S \arrow[ur, "i_S"] \arrow[r, "s"'] & X \arrow[r, "q"'] & X/S \end{tikzcd}$$

By axiom (5) of the homological closure operator $\overline{()}$,

$$\tilde{S} = q^{-1}\left(\overline{0_{X/S}}\right) = \overline{q^{-1}(0_{X/S})} = \overline{Ker(q)} = \overline{S}.$$

Conversely, let us take an epireflection $F : \mathcal{C} \to \mathcal{F}$, the associated closure operator $\overline{()}$ given by the second part of the proof and $\overline{F} : \mathcal{C} \to \mathcal{F}$ the functor defined in the first part of the proof. We have to show that $F = \overline{F}$.

Recall by the construction of $\overline{\eta}$ that $\overline{\eta_{X/S}}$ is the cokernel of $k_{X/S}$.
But by definition of $\overline{S}$, $k_{X/S}$ is the kernel of $\eta_{X/S}$.
Therefore, since $\eta_{X/S}$ is a regular epimorphism it is also the cokernel of $k_{X/S}$ which implies that $\overline{\eta_{X/S}} = \eta_{X/S}$. $\square$

Chapter 3

Torsion theories

We are now able to introduce the notion of torsion theory. We are first going to define the concept itself. Then, we are going to define what a torsion-free subcategory of a normal category is. Finally, we are going to prove the equivalence between fibered reflectors and semi-left-exact reflectors.
The sources of this chapter are mainly [5] and [3].

Definition 19. *A **torsion theory** in a normal category $\mathcal{C}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathcal{C}$ such that:*

1. *For any $T \in \mathcal{T}$ and $F \in \mathcal{F}$, the only morphism $f : T \to F$ is the 0-morphism.*

2. *For any object $X \in \mathcal{C}$, there exists a short exact sequence*

$$0 \longrightarrow T \xrightarrow{t_X} X \xrightarrow{\eta_X} F \longrightarrow 0$$

   with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Lemma 20. *If $(\mathcal{T}, \mathcal{F})$ is a torsion theory in a normal category $\mathcal{C}$. Then for any X in $\mathcal{C}$, there exists, up to isomorphism, exactly one short exact sequence*

$$0 \longrightarrow T \xrightarrow{t_X} X \xrightarrow{\eta_X} F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Proof. Let us consider two short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r, "t_X"] & X \arrow[r, "\eta_X"] \arrow[d, equal] & F \arrow[r] & 0 \\
0 \arrow[r] & T' \arrow[r, "t'_X"] & X \arrow[r, "\eta'_X"] & F' \arrow[r] & 0
\end{tikzcd}$$

with T, T' in $\mathcal{T}$ and F, F' in $\mathcal{F}$. By definition of torsion theory, the morphisms $\eta'_X \circ t_X$ and $\eta_X \circ t'_X$ are the zero morphisms.

$$\begin{tikzcd}
T \arrow[r, "t_X"] \arrow[dr] & X \arrow[rr, "\eta_X"] \arrow[dr, "\eta'_X"] & & F \arrow[dl, dashed, "\phi"] \\
& 0 \arrow[r] & F' &
\end{tikzcd}$$

Since $\eta_X = Coker(t_X)$ and $\eta'_X \circ t_X = 0$, there exists a unique ϕ such that $\phi \circ \eta_X = \eta'_X$. Similarly, since $\eta'_X = Coker(t'_X)$ and $\eta_X \circ t'_X = 0$, there exists a unique ψ such that $\psi \circ \eta'_X = \eta_X$.
Moreover, $\psi \circ \phi \circ \eta_X = \psi \circ \eta'_X = \eta_X = 1_F \circ \eta_X$ and η_X is an epimorphism, which implies that $\psi \circ \phi = 1_F$.
Similarly, one proves that $\phi \circ \psi = 1_{F'}$. Hence, $F \cong F'$.
The proof that $T \cong T'$ is similar. □

Lemma 21. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in a normal category $\mathcal{C}$. Then:*

1. $\mathcal{T} \bigcap \mathcal{F} = 0$,

2. *$\mathcal{T}$ is closed in $\mathcal{C}$ under regular quotients,*

3. *$\mathcal{F}$ is closed in $\mathcal{C}$ under subobjects.*

Proof. (1) If $X \in \mathcal{T} \bigcap \mathcal{F}$, the identity morphism $1_X : X \to X$ is the zero morphism which implies that $X = 0$.
(2) Let $f : X \to Y$ be a regular epimorphism with $X \in \mathcal{T}$. We have to prove that $Y \in \mathcal{T}$. We consider the exact sequence (unique by previous lemma) associated with Y:

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r, "t_Y"] & Y \arrow[r, "\eta_Y"] & F \arrow[r] & 0 \\ & X \arrow[u, dashed, "g"] \arrow[ur, "f"'] & & & \end{tikzcd}$$

Since $F \in \mathcal{F}$, $\eta_Y \circ f = 0$.
But t_Y is the kernel of η_Y and therefore, there exists a unique morphism g such that $t_Y \circ g = f$.
We can therefore conclude that t_Y is a regular epimorphism and since it is also a monomorphism, $t_Y : T \to Y$ is an isomorphism and $Y \in \mathcal{T}$.
(3) Let $f : X \to Y$ be a monomorphism with $Y \in \mathcal{F}$. We have to prove that $X \in \mathcal{F}$. We consider the exact sequence associated with Y (unique by Lemma 20):

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r, "t_X"] & X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & F \arrow[r] \arrow[d, dashed, "g"] & 0 \\ & & & Y & \end{tikzcd}$$

Since $T \in \mathcal{T}$, one has that $f \circ t_X = 0$.
But η_X is the cokernel of t_X and therefore, there exists a unique morphism g such that $g \circ \eta_X = f$.
We can therefore conclude that η_X is a monomorphism and since it is also a regular epimorphism, $\eta_X : X \to F$ is an isomorphism and $X \in \mathcal{F}$. $\square$

Lemma 22. *Let $(\mathcal{T}, \mathcal{F})$ be a pair of full replete subcategories of a normal category $\mathcal{C}$ with the property that for any X in $\mathcal{C}$ there is an exact sequence*

$$0 \longrightarrow T \xrightarrow{t_X} X \xrightarrow{\eta_X} F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Then the following conditions are equivalent:

(1) $(\mathcal{T}, \mathcal{F})$ is a torsion theory.
(2) $\mathcal{T}$ is closed in $\mathcal{C}$ under regular quotients, $\mathcal{F}$ is closed in $\mathcal{C}$ under subobjects and $\mathcal{T} \cap \mathcal{F} = 0$.

Proof. The previous lemma gives the first implication.
Conversely, let $f : X \to Y$ be any arrow with $X \in \mathcal{T}$ and $Y \in \mathcal{F}$.
We consider the factorization of f,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "r"'] & & Y \\ & Z \arrow[ur, tail, "m"'] & \end{tikzcd}$$

with r a regular epimorphism and m a monomorphism. This factorization exists by Proposition 11.
Therefore, the object Z belongs to $\mathcal{T}$ (since τ is closed in $\mathcal{C}$ under regular quotients) and to $\mathcal{F}$ (since $\mathcal{F}$ is closed in $\mathcal{C}$ under subobjects). It follows that $Z \in \mathcal{T} \cap \mathcal{F} = 0$ and $f = 0$ so that $(\mathcal{T}, \mathcal{F})$ is a torsion theory. □

Definition 23. *A full replete subcategory $\mathcal{F}$ of a normal category $\mathcal{C}$ is a **torsion-free subcategory** if there exists a full replete subcategory $\mathcal{T}$ with the property that $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$.*

Let $\mathcal{F}$ be a regular epireflective subcategory of a normal category $\mathcal{C}$ with reflection $F : \mathcal{C} \to \mathcal{F}$. We define

$$\mathcal{T}_{\mathcal{F}} := \{T \in \mathcal{C} \mid T \simeq K(X) \text{ for some X}\}$$

and

$$\mathcal{F}^{\leftarrow} := \{T \in \mathcal{C} \mid \hom_{\mathcal{C}}(T, F) = \{0\}, \forall F \in \mathcal{F}\}.$$

Lemma 24. *Let $\mathcal{F}$ be a regular epireflective subcategory of a normal category $\mathcal{C}$, with reflection $F : \mathcal{C} \to \mathcal{F}$. Then*

$$\mathcal{F}^{\leftarrow} = \{T \in \mathcal{C} | F(T) = 0\}.$$

Proof. First, let us take $T \in \mathcal{C}$ such that $F(T) = 0$. Then, any morphism $f : T \to F$ with $T \in \mathcal{F}^{\leftarrow}$ factors through 0.

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr, "\eta_T"'] & & F \\ & 0 = F(T) \arrow[ur] & \end{tikzcd}$$

For the other inclusion, let us take $T \in \mathcal{C}$ such that $\hom_{\mathcal{C}}(T, F) = \{0\}$ for all $F \in \mathcal{F}$. In particular, the T-component η_T of the unit of the adjunction is the zero arrow.

But, since η_T is a regular epimorphism, $F(T) = 0$.

$$\begin{tikzcd} T \arrow[rr, "\eta_T"] \arrow[dr] & & F(T) \\ & 0 \arrow[ur, "\cong"'] & \end{tikzcd}$$

Indeed, $0 \to F(T)$ is also a regular epimorphism but since it is a monomorphism, it is also an isomorphism. $\square$

Theorem 25. *Let $\mathcal{F}$ be a full replete subcategory of a normal category $\mathcal{C}$. The following conditions are equivalent:*
(1) $\mathcal{F}$ is a torsion-free subcategory of $\mathcal{C}$,
(2) $\mathcal{F}$ is regular epireflective in $\mathcal{C}$ and $\mathcal{T}_{\mathcal{F}} = \mathcal{F}^{\leftarrow}$,
(3) $\mathcal{F}$ is regular epireflective in $\mathcal{C}$, and $\hom_{\mathcal{C}}(T, F) = \{0\}$ for any $T \in \mathcal{T}_{\mathcal{F}}$ and $F \in \mathcal{F}$,
(4)$\mathcal{F}$ is regular epireflective in $\mathcal{C}$, $\mathcal{F}$ is closed in $\mathcal{C}$ under extensions, and $\mathcal{T}_{\mathcal{F}}$ is closed under quotients.

Proof. (1) $\implies$ (2)
We are going to prove that if $\mathcal{F}$ is a torsion-free subcategory of $\mathcal{C}$, then $\mathcal{F}$ is regular epireflective in $\mathcal{C}$ and $\mathcal{T}_{\mathcal{F}} = \mathcal{F}^{\leftarrow}$.
We consider $\mathcal{F}$ a torsion-free subcategory of a normal category $\mathcal{C}$.
Let us prove that $\mathcal{F}$ is regular epireflective in $\mathcal{C}$:
Since $\mathcal{F}$ is a torsion-free subcategory of $\mathcal{C}$, there is a full replete subcategory $\mathcal{T}$ such that $(\mathcal{T}, \mathcal{F})$ is a torsion theory in $\mathcal{C}$. Therefore, for any $X \in \mathcal{C}$, there is a unique exact sequence:

$$0 \longrightarrow K \xrightarrow{k_X} X \xrightarrow{\eta_X} F \longrightarrow 0$$

with $K \in \mathcal{T}$ and $F \in \mathcal{F}$. We define the left adjoint F of the inclusion functor U on an object X by $F(X) = F$. Let us check the universal property of $\eta_X : X \to F(X)$.
If $f : X \to F'$ is a morphism with $F' \in \mathcal{F}$, we have $f \circ k_X = 0$.
Since η_X is the cokernel of k_X, there exists a unique morphism $g : F(X) \to F'$ with $g \circ \eta_X = f$. Therefore, $F : \mathcal{C} \to \mathcal{F}$ is the left adjoint of $U : \mathcal{F} \to \mathcal{C}$.
Let us prove that $\mathcal{T}_{\mathcal{F}} \subseteq \mathcal{F}^{\leftarrow}$:
If $X' \in \mathcal{T}_{\mathcal{F}}$, by definition $X' \cong K(X)$ for some X in $\mathcal{C}$.

$$0 \longrightarrow X' \cong K(X) \xrightarrow{k_X} X \xrightarrow{\eta_X} F(X) \longrightarrow 0$$

$\mathcal{T}$ is replete in $\mathcal{C}$ then $X' \in \mathcal{T}$ because $K(X) \in \mathcal{T}$. Therefore, we have that $\mathcal{T}_{\mathcal{F}} \subseteq \mathcal{T}$ but by definition of torsion theory $(\mathcal{T}, \mathcal{F})$, $\mathcal{T} \subseteq \mathcal{F}^{\leftarrow}$.

Let us then prove that $\mathcal{F}^{\leftarrow} \subseteq \mathcal{T}_{\mathcal{F}}$:
If $X \in \mathcal{F}^{\leftarrow}$, consider the diagram

$$\begin{tikzcd} K(X) \arrow[r, "k_X"] & X \arrow[rr, "\eta_X"] \arrow[dr] & & F(X) \arrow[dl] \\ & & 0 & \end{tikzcd}$$

Then, since isomorphisms are stable under pullback, k_X is an isomorphism.

$$\begin{tikzcd} K(X) \arrow[r, "k_X"] \arrow[d] & X \arrow[d] \\ 0 \arrow[r, equal] & 0 \end{tikzcd}$$

Therefore, $X \cong K(X)$ and by definition, $X \in \mathcal{T}_{\mathcal{F}}$.

(2) $\Longrightarrow$ (3)
For any $T \in \mathcal{T}_{\mathcal{F}}$ and $F \in \mathcal{F}$, we know that $T \in \mathcal{F}^{\leftarrow}$ and therefore any morphism from T to F is the zero morphism.

(3) $\Longrightarrow$ (4)
Let $f : T \in \mathcal{T}_{\mathcal{F}} \to Y$ be a regular epimorphism.

$$\begin{tikzcd} K(Y) \arrow[rr, "k_Y"] & & Y \arrow[r, "\eta_Y"] & F(Y) \in \mathcal{F} \\ & T \in \mathcal{T}_{\mathcal{F}} \arrow[ul, dashed, "i"] \arrow[ur, "f"'] & & \end{tikzcd}$$

Since $F(Y) \in \mathcal{F}$ and $T \in \mathcal{T}_{\mathcal{F}}$, the morphism $\eta_Y \circ f$ is the 0-morphism.
Therefore, since $k_Y = ker(\eta_Y)$, there exists a unique i such that $k_Y \circ i = f$.
But since f is a regular epimorphism, k_Y is also a regular epimorphism, which implies that k_Y is an isomorphism and $Y \in \mathcal{T}_{\mathcal{F}}$.
We also have to prove that $\mathcal{F}$ is closed in $\mathcal{C}$ under extensions. Given an exact sequence

$$0 \longrightarrow F_1 \xrightarrow{a} X \xrightarrow{b} F_2 \longrightarrow 0$$

with F_1 and F_2 in $\mathcal{F}$, one has to prove that X is in $\mathcal{F}$.
Let us consider $k_X : K[X] \to X$, the kernel of the component $\eta_X : X \to F(X)$ of the unit η of the adjunction.
We have that $K[X] \in \mathcal{T}_{\mathcal{F}}$ and $F_2 \in \mathcal{F}$. Therefore, $b \circ k_X = 0$. But a is the kernel of b,

hence there is a unique morphism $l : T \to F_1$ making the following diagram commute:

$$\begin{tikzcd} F_1 \arrow[r, "a"] & X \arrow[r, "b"] & F2 \\ T \arrow[u, dashed, "\exists! l"] \arrow[ur, "k_X"'] \arrow[r] & 0 \arrow[ur] & \end{tikzcd}$$

Since the only arrow from T to F_1 is the zero arrow, $k_X = 0$ and since $\eta_X : X \to F(X)$ is the cokernel of k_X it follows that η_X is an isomorphism and $X \in \mathcal{F}$.

(4) $\implies$ (1)
We would like to prove that $(\mathcal{T}_{\mathcal{F}}, \mathcal{F})$ is a torsion theory. By Lemma 22, we only have to prove that $\mathcal{T}_{\mathcal{F}} \cap \mathcal{F} = 0$.
If $X \in \mathcal{T}_{\mathcal{F}} \cap \mathcal{F}$, there exists X_1 such that $X \cong K[X_1]$.
We can consider the short exact sequence:

$$0 \longrightarrow X \longrightarrow X_1 \xrightarrow{\eta_{X_1}} F(X_1) \longrightarrow 0$$

Since $\mathcal{F}$ is closed in $\mathcal{C}$ under extensions, it follows that X_1 is in $\mathcal{F}$. Hence, η_{X_1} is an isomorphism and $X = 0$. $\square$

Definition 26. *An epireflection is called a* ***fibered reflection*** *if the units of the adjunction are pullback stable along arrows in the reflective subcategory $\mathcal{F}$.*
This means that in a pullback

$$\begin{tikzcd} P \arrow[r] \arrow[d, dashed] & Y \arrow[d, "\eta_Y"] \\ X \arrow[r, "f"'] & UF(Y) \end{tikzcd}$$

where X is in $\mathcal{F}$, the vertical dashed arrow is necessarily η_P.

Theorem 27. *Let $\mathcal{F}$ be a full replete subcategory of a normal category $\mathcal{C}$. Then the following conditions are equivalent:*
(1) $\mathcal{F}$ is a torsion-free subcategory of $\mathcal{C}$;
(2) $\mathcal{F}$ is regular epireflective in $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{F}$ is ***normal****, i.e $F(K(X)) = 0 \; \forall X \in \mathcal{C}$;*
(3) $\mathcal{F}$ is regular epireflective in $\mathcal{C}$ and the reflection is fibered.

Proof. The equivalence (1) $\iff$ (2) is given by Theorem 25.
Let us prove that (2) $\implies$ (3).
We consider $\mathcal{F}$ regular epireflective in $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{F}$.

We consider now the following pullback:

$$\begin{tikzcd}
K(Y) \arrow[r, "l"] \arrow[d, "\eta_{K(Y)}"'] \arrow[rr, bend left, "t_Y"] & P \arrow[r, "g"] \arrow[d, "h"] & Y \arrow[d, "\eta_Y"] \\
0 \arrow[r] & X \arrow[r, "f"'] & F(Y)
\end{tikzcd}$$

Since η_Y is a regular epimorphism, h is also a regular epimorphism.
One has to prove that $h = \eta_P$.
By the universal property of η_P, there exists a unique $\phi : F(P) \to X$ such that $\phi \circ \eta_P = h$. This morphism ϕ is a regular epimorphism. Hence, we have to prove that ϕ is a monomorphism.
Let us prove that $K[P] \cong K[h]$ in the following diagram.

$$\begin{tikzcd}
 & K[h] \arrow[r] \arrow[dd, tail, "ker(h)"] & K[Y] \arrow[dd, tail] \\
K[P] \arrow[dr, tail, "k_P"] & & \\
 & P \arrow[r, "g"] \arrow[dl, "\eta_P"] \arrow[dd, "h"] & Y \arrow[dd, "\eta_Y"] \\
UF(P) \arrow[dr, "\phi"] & & \\
 & X \arrow[r, "f"] & UF(Y)
\end{tikzcd}$$

Firstly, we have to construct a morphism $\alpha : K[P] \to K[h]$.

$$\begin{tikzcd}
K[P] \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "k_P"] \arrow[ddr, bend right] & & \\
 & K[h] \arrow[r, "ker(h)"] \arrow[d] & P \arrow[d, "h"] \\
 & 0 \arrow[r] & X
\end{tikzcd}$$

By definition of the morphism ϕ, we have $\phi \circ \eta_P = h$,
therefore $h \circ k_P = \phi \circ \eta_P \circ k_P = 0$. And since $ker(h)$ is the kernel of h, there exists a unique morphism α such that $ker(h) \circ \alpha = k_P$.
Secondly, we have to construct a morphism $\beta : K[h] \to K[P]$.

Let us consider the following commutative diagram:

$$\begin{tikzcd}
K[h] \arrow[r, "ker(h)"] \arrow[d, "\eta_{K[h]}"'] & P \arrow[d, "\eta_P"] \\
F(K[h]) \arrow[r, "F(ker(h))"'] & F(P)
\end{tikzcd}$$

By Lemma 7, we know that $K[h] \cong K[Y]$ and (2) gives us that $F(K(Y)) = 0$.

$$\begin{tikzcd}
K[h] \arrow[r, "ker(h)"] \arrow[d, "\eta_{K[h]}"'] & P \arrow[d, "\eta_P"] \\
F(K[h]) \cong 0 \arrow[r, "F(ker(h))"'] & F(P)
\end{tikzcd}$$

Therefore, $\eta_P \circ ker(h)$ is the zero-morphism.

$$\begin{tikzcd}
K[h] \arrow[dr, dashed, "\beta"] \arrow[drr, bend left, "ker(h)"] \arrow[ddr, bend right] & & \\
& K[P] \arrow[r, "k_P"] \arrow[d] & P \arrow[d, "\eta_P"] \\
& 0 \arrow[r] & UF(P)
\end{tikzcd}$$

Since $\eta_P \circ ker(h) = 0$ and k_P is the kernel of η_P, there exists a unique β such that $k_P \circ \beta = ker(h)$.
In order to prove that $K[P] \cong K[h]$, we still have to prove that $\alpha \circ \beta = 1_{K[h]}$ and that $\beta \circ \alpha = 1_{K[P]}$.
Indeed, $ker(h) \circ \alpha \circ \beta = k_P \circ \beta = ker(h) = ker(h) \circ 1_{K[h]}$ hence, since $ker(h)$ is a monomorphism, we have that $\alpha \circ \beta = 1_{K[P]}$.
Similarly, $k_P \circ \beta \circ \alpha = ker(h) \circ \alpha = k_P = k_P \circ 1_{K[P]}$ implies $\beta \circ \alpha = 1_{K[P]}$.
Since $\eta_P \circ k_P = 0$, we have $\phi \circ \eta_P \circ k_P = 0$ and by the property of the kernel of ϕ there exists a unique α such that $ker(\phi) \circ \alpha = \eta_P \circ k_P = 0$.

$$\begin{tikzcd}
K[h] \arrow[r, "k_P"'] \arrow[d, dotted, "\alpha"] & P \arrow[d, "\eta_P"] \arrow[dr, "h"] & \\
K[\phi] \arrow[r, "ker(\phi)"'] & UF(P) \arrow[r, "\phi"] & X
\end{tikzcd}$$

In order to prove that ϕ is a monomorphism, we will prove that $K[\phi] \cong 0$. We already know that $ker(\phi) \circ \alpha = 0$ and that $ker(\phi)$ is a monomorphism. By uniqueness of the epi-mono factorization proved in Proposition 11, it suffices to show that α is a regular epimorphism to prove that $K[\phi]$ is the zero object.

Let us consider the following commutative diagram:

$$
\begin{tikzcd}
K[h] \arrow[rrr, "\alpha"] \arrow[dr, "k_P"] \arrow[dd] \arrow[drrr, phantom, "(1)"] \arrow[ddr, phantom, "(2)"] & & & K[\phi] \arrow[dr, "ker(\phi)"] \arrow[dd] \arrow[ddr, phantom, "(4)"] & \\
& P \arrow[rrr, "\eta_P"] \arrow[dd] & & & UF(P) \arrow[dd, "\phi"] \\
0 \arrow[rrr, Rightarrow] \arrow[dr] \arrow[drrr, phantom, "(3)"] & & & 0 \arrow[dr] & \\
& X \arrow[rrr, Rightarrow] & & & X
\end{tikzcd}
$$

Observe that (2)+(3) is a pullback. Therefore, (1)+(4) is a pullback. Hence, since (4), is a pullback, (1) is also a pullback.
Since (1) is a pullback and η_P is a regular epimorphism, we know that α is also a regular epimorphism. And by the previous argument, $K[\phi] \cong 0$ and ϕ is a monomorphism.
We can therefore conclude that ϕ is an isomorphism (since it is both a regular epimorphism and a monomorphism).
Hence, h is η_P (up to isomorphism).

(3) $\implies$ (2)
Let us consider, for $X \in \mathcal{C}$, the following rectangle:

$$
\begin{tikzcd}
K(X) \arrow[r, equal] \arrow[d, "\eta_{K(X)}"'] & K(X) \arrow[r, "k_X"] \arrow[d] & X \arrow[d, "\eta_X"] \\
F(K(X)) \arrow[r] & 0 \arrow[r] & F(X)
\end{tikzcd}
$$

We have to prove that the morphism $F(K(X)) \to 0$ is an isomorphism.
But since the reflection is fibered, the whole rectangle is a pullback.
Hence, the left-hand square is a pullback which prove that $F(K(X)) \to 0$ is an isomorphism by Proposition 13 and that $F : \mathcal{C} \to \mathcal{F}$ is normal. □

Definition 28. *The **fibre** of $F : \mathcal{F} \to \mathcal{E}$ a functor at I an object in $\mathcal{E}$ is the subcategory $\mathcal{F}_I$ of $\mathcal{F}$ defined in the following way:*

- $X \in \mathcal{F}$ *is in* $\mathcal{F}_I \iff F(X) = I$.
- $X, Y \in \mathcal{F}_I,\ f : X \to Y \in \mathcal{F}_I \iff F(f) = 1_I$.

Definition 29. *Let $F : \mathcal{F} \to \mathcal{E}$ be a functor and $\alpha : J \to I$ be a morphism in $\mathcal{E}$.*
$f : Y \to X$ of $\mathcal{F}$ is ***cartesian*** *over α if :*

1. *$F(f) = \alpha$,*

2. *If $g : Z \to X$ is a morphism of $\mathcal{F}$ such that $F(g)$ factors as $\alpha \circ \beta$, there exists a unique morphism $h : Z \to Y$ in $\mathcal{F}$ such that $F(h) = \beta$ and $g = f \circ h$.*

Definition 30. *A functor $F : \mathcal{F} \to \mathcal{E}$ is* ***fibered*** *if and only if $\forall \alpha : J \to I$ in $\mathcal{E}$ and $\forall X \in \mathcal{F}_I$, there exists a cartesian morphism $f : Y \to X$ over α. In this case, we call $\mathcal{F}$ a "category fibered over $\mathcal{E}$".*

Proposition 31. *Let $U,\ F : \mathcal{F} \leftrightarrows \mathcal{C}$ be a full reflective subcategory of a normal category $\mathcal{C}$. We have that F is fibered if and only if the pullback of a unit η_A of the adjunction along a morphism $f \in \mathcal{F}$ is again a unit.*

Proof. Let us take a morphism $f : M \to L$, since F is fibered, there exists a cartesian morphism $g : B \to A$ over f. By definition, $F(g) = f$ and the following diagram is commutative.

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "\eta_B"'] & A \arrow[d, "\eta_A"] \\
M \arrow[r, "f"'] & L
\end{tikzcd}$$

We have to prove that the previous diagram is a pullback.
Let us take two morphisms $m : C \to A$ and $n : C \to M$ in $\mathcal{C}$ such that $\eta_A \circ m = f \circ n$.
By the universal property of η_C, n factors uniquely through η_C via a certain morphism s.

$$\begin{tikzcd}
C \arrow[rrd, bend left, "m"] \arrow[rdd, bend left, "n"] \arrow[ddd, bend right, "\eta_C"'] & & \\
& B \arrow[r, "g"] \arrow[d, "\eta_B"'] & A \arrow[d, "\eta_A"] \\
& M \arrow[r, "f"'] & L \\
UF(C) \arrow[ru, dashed, "s"] \arrow[rru, bend right, "UF(m)"'] & &
\end{tikzcd}$$

But η is a natural transformation, therefore $UF(m) \circ \eta_C = \eta_A \circ m$. Moreover, $\eta_A \circ m = f \circ n = f \circ s \circ \eta_C$.
Therefore, $UF(m) \circ \eta_C = f \circ s \circ \eta_C$, then $f \circ s = UF(m)$.

Since g is cartesian over f, there exists a unique h such that $UF(h) = s$ and $g \circ h = m$.

$$\begin{tikzcd}
C \arrow[drr, bend left, "m"] \arrow[dr, dashed, "h"] \arrow[ddr, bend left, "n"] \arrow[ddd, bend right, "\eta_C"'] & & \\
& B \arrow[r, "g"] \arrow[d, "\eta_B"'] & A \arrow[d, "\eta_A"] \\
& M \arrow[r, "f"'] & L \\
UF(C) \arrow[ur, "s"'] \arrow[urr, bend right, "UF(m)"'] & &
\end{tikzcd}$$

In order to prove that (B, g, η_S) is the pullback of f and η_A, we have to prove that $F(h) = s$ is equivalent to $\eta_B \circ h = n$.
Firstly, if $F(h) = s$, since η is a natural transformation, $\eta_B \circ h = UF(h) \circ \eta_C = s \circ \eta_C = n$.
Secondly, if $\eta_B \circ h = n$, since η is a natural transformation, $UF(h) \circ \eta_C = \eta_B \circ h = n = s \circ \eta_C$. Therefore, $UF(h) = s$.
Conversely, let us take $f : M \to L$ in $\mathcal{L}$ and $X \in \mathcal{C}_L$.
We consider the following pullback:

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "\eta_B"'] & A \arrow[d, "\eta_A"] \\
M \arrow[r, "f"'] & L
\end{tikzcd}$$

We have to prove that g is the cartesian morphism over f.

1. $F(g) = f$ since $\eta_A \circ g = f \circ \eta_B$.

2. If $m : C \to A$ is a morphism of $\mathcal{C}$ such that $UF(m) = f \circ s$.

$$\begin{tikzcd}
C \arrow[drr, bend left, "m"] \arrow[dr, dashed, "\exists\,!\,h"] \arrow[ddr, bend right, "s \circ \eta_C"'] & & \\
& B \arrow[r, "g"] \arrow[d, "\eta_B"] & A \arrow[d, "\eta_A"] \\
& M \arrow[r, "f"'] & L
\end{tikzcd}$$

   Since $f \circ s \circ \eta_C = UF(m) \circ \eta_C = \eta_A \circ m$, there exists a unique h such that $g \circ h = m$ and $\eta_B \circ h = s \circ \eta_C$, i.e. $UF(h) = s$.

$\square$

We have just observed that the fact that the unit morphisms in a reflection $F : \mathcal{C} \to \mathcal{F}$ are stable under pullbacks along a morphism lying in the subcategory $\mathcal{F}$, is equivalent to the fact that $F : \mathcal{C} \to \mathcal{F}$ is semi-left-exact.
This precisely means that for a functor $F : \mathcal{C} \to \mathcal{F}$, being a fibered reflector is equivalent to being a semi-left-exact reflector.

Conclusion

We have generalized the concept of torsion theory to the context of normal categories. It would not be useful to extend the concept if the most important properties of torsion theories were not verified on normal categories. Fortunately, all the properties of torsion theories from [5] in the context of homological categories have been shown to hold in the context of normal categories.
The key elements making this possible were three properties of normal categories that allowed us to avoid the property of protomodularity that is part of the definition of homological categories [5]. The first one was Proposition 11 which allowed us to have a unique normal epi-mono factorization of every normal epimorphism. The second one was Proposition 10 which stated that every normal epimorphism is the cokernel of its kernel. The last one, Proposition 12, stated that if the kernel of a morphism is zero then this morphism is a monomorphism.
The final part of this master thesis showed the equivalence between torsion-free subcategory, fibered reflection and normal reflection.
This master thesis brings some more details of the different proofs found in the literature. It also provides a unique text where the basics of torsion theories in normal categories are thoroughly presented, together with all the categorical notions and properties needed to understand the main results.

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