

Louvain School of Management

Derivative Pricing under Funding Costs and Bilateral Counterparty Risk

A Burgard-Kjaer and XVA Comparison

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Abstract

This thesis studies the impact of funding costs and bilateral counterparty risk on derivative valuation within the Burgard–Kjaer framework. While classical Black–Scholes pricing assumes frictionless and default-free markets, modern OTC derivative markets are significantly affected by credit and funding considerations.

The analysis compares unilateral derivatives, represented by a European call option, with bilateral derivatives such as an equity forward and an interest rate forward. The Burgard–Kjaer pricing equations are implemented numerically using Crank–Nicolson finite differences and Monte Carlo simulations, and the results are compared with a simplified modern XVA framework combining CVA, DVA and FVA adjustments.

The results show that the Burgard–Kjaer and additive XVA approaches produce almost identical adjustments for unilateral derivatives. However, for bilateral derivatives, the Burgard–Kjaer adjustments become significantly larger, reaching differences close to a factor of three compared with the simplified XVA framework. These results suggest that additive XVA decompositions may remain acceptable for strongly unilateral products, but can materially underestimate valuation adjustments once bilateral exposure becomes significant.

Introduction

Derivative products play a central role in modern financial markets and are widely used for hedging, speculation and risk management purposes. Over the past decades, the development of quantitative finance has led to major advances in derivative pricing theory, among which the Black–Scholes–Merton framework (*Black and Scholes 1973; Merton 1973*) remains one of the most important. By assuming frictionless and default-free markets, continuous trading and perfect replication, the Black–Scholes model established a rigorous mathematical framework for derivative valuation.

Despite its importance, the assumptions underlying the Black–Scholes framework appear increasingly restrictive in modern over-the-counter markets. In practice, derivative transactions expose counterparties to credit risk, funding costs and liquidity constraints that are not captured by traditional risk-free pricing models. The 2007–2008 financial crisis particularly highlighted the importance of these effects and accelerated the development of valuation adjustments commonly grouped under the term XVA, including Credit Valuation Adjustment (CVA), Debit Valuation Adjustment (DVA) and Funding Valuation Adjustment (FVA) (*Gregory 2015*).

Within this context, the framework developed by Burgard and Kjaer (*Burgard and Kjaer 2011*) represents a major extension of classical derivative pricing theory. Their approach incorporates bilateral counterparty risk and funding costs directly into the replication framework underlying derivative pricing. By introducing default intensities, recovery rates and funding spreads into the pricing partial differential equation, the Burgard–Kjaer framework provides a consistent methodology to account for both credit and funding effects. An important feature of the framework is the treatment of close-out conventions at default, which can lead to either linear or nonlinear pricing equations depending on the assumptions considered.

This thesis studies derivative valuation under bilateral counterparty risk and funding costs within the Burgard–Kjaer framework. Particular attention is given to the distinction between unilateral and bilateral derivatives. In this context, unilateral derivatives mainly generate one-sided positive mark-to-market exposures, whereas bilateral deriva-

tives can generate both positive and negative exposures throughout their lifetime. This distinction is important because the impact of funding costs and bilateral credit effects strongly depends on the exposure structure of the derivative considered.

The analysis mainly builds upon the framework introduced by Burgard and Kjaer (*Burgard and Kjaer 2011*), while also relying on the broader literature on derivative pricing, counterparty credit risk and Monte Carlo methods (*Glasserman 2004*). The Burgard–Kjaer pricing equations are implemented numerically using finite-difference methods, particularly the Crank–Nicolson scheme (*Crank and Nicolson 1947*), as well as Monte Carlo simulations. The objective is to study how funding costs, bilateral credit risk and close-out assumptions affect derivative valuation, and how the Burgard–Kjaer framework relates to more standard modern XVA approaches.

This thesis is organised as follows. Chapter 1 presents the market context, focusing on the post-2008 evolution of OTC derivative markets, funding costs and regulation. Chapter 2 implements the Burgard–Kjaer framework numerically for a unilateral European call option and compares Crank–Nicolson and Monte Carlo methods. Chapter 3 extends the analysis to bilateral derivatives through an equity forward and an interest rate forward. Chapter 4 compares the Burgard–Kjaer valuation results with a simplified modern XVA framework. The thesis then concludes with the main results, a discussion of the limitations and possible extensions, and a final reflection on the research process.

Chapter 1

Market Context

1.1 Crisis of 2008 and the Emergence of OTC Risk Monitoring

The 2008 crisis exposed a structural weakness in over-the-counter (OTC) derivatives markets: risk was bilateral, opaque, and poorly monitored. Large counterparty exposures had accumulated through private contracts, while collateral, reporting, and operational controls remained fragmented (*Federal Reserve Bank of New York 2010*).

The crisis also showed that market size is not a measure of risk. In OTC derivatives, notional amounts capture volume, not exposure. The relevant metric is replacement cost after netting and collateral, since counterparty risk depends on the residual liability left after contractual offsets (*Segoviano and Singh 2008*).

This distinction is critical because OTC risk is not purely bilateral. Counterparty losses can propagate across institutions through common exposures and distress dependence. The IMF shows that once contagion is considered, system-wide losses can rise far beyond the direct loss generated by a single default (*Segoviano and Singh 2008*).

The crisis further revealed that opacity is itself a source of systemic risk. Firms could estimate their own exposure, but not the exposures of their counterparties to the rest of the market. This uncertainty reduced liquidity, impaired price discovery, and weakened the ability of institutions to hedge under stress (*Heitfield 2016*).

The ECB reaches the same conclusion from a market functioning perspective: when counterparty confidence breaks, institutions shorten funding horizons, reduce credit lines, delay settlement, and demand additional liquidity buffers. Counterparty uncertainty therefore becomes a direct transmission channel for liquidity stress (*European Central Bank 2009*).

Post-crisis OTC reform followed directly from these failures. Risk monitoring shifted from price observation to exposure surveillance, through trade repositories, central clearing, collateral management, and stress testing (*Heitfield 2016; Federal Reserve Bank of*

New York 2010). This shift is the starting point of this thesis: once counterparty and funding frictions become material, derivative pricing must move beyond frictionless Black–Scholes replication.

1.2 Current OTC Derivative Environment

1.2.1 Interest Rate Regime and Funding Cost

The post-2022 macro regime has been defined by the end of zero rates and the return of positive funding costs across major jurisdictions. After a decade of near-zero policy rates, central banks tightened aggressively to contain inflation, restoring interest rates as a first-order pricing variable in financial markets (*European Central Bank 2026b*; *International Monetary Fund 2025*).

In the euro area, the ECB deposit facility rate moved from -0.50% in 2021 to 4.00% in 2023 before easing back to 2.00% in 2025, confirming a structurally higher rate environment than the pre-inflation decade (*European Central Bank 2026b*). In the United States, the Federal Reserve followed the same path, lifting policy rates sharply and maintaining a restrictive stance for longer as inflation proved more persistent than initially expected (*International Monetary Fund 2025*).

This regime shift matters directly for OTC derivatives because interest rates determine both discounting and funding. Higher policy rates raise the cost of unsecured funding, widen the spread between internal funding curves and collateral discount curves, and increase the valuation impact of funding adjustments in uncollateralised or imperfectly collateralised trades (*Piterbarg 2010*). In practice, this makes funding no longer a negligible implementation detail, but a material pricing input.

The effect is strongest in interest rate derivatives, which remain the core of the OTC market. By mid-2025, OTC interest rate derivatives represented 79% of global OTC notional outstanding, and their growth was the main driver of the increase in total OTC activity (*Bank for International Settlements 2025*). The BIS explicitly links this expansion to elevated uncertainty around monetary policy, rates, and macro conditions, which increased demand for hedging and rate optionality (*Bank for International Settlements 2025*).

This has two implications for derivative pricing. First, higher rates increase the economic value of funding asymmetries embedded in OTC replication. Second, rate uncertainty increases the use of swaps, swaptions, and other OTC hedges, making funding, collateral, and discounting assumptions more central to valuation. In such an environment, pricing frameworks that ignore funding frictions become materially less realistic.

1.2.2 OTC Regulatory Framework

The post-2008 regulatory framework significantly changed the structure of OTC derivatives markets. The main objective was to reduce systemic counterparty risk through mandatory clearing, higher transparency, and stricter collateral requirements.

At the global level, the 2009 G20 commitments introduced three core principles: standardized OTC derivatives should be centrally cleared, transactions should be reported to trade repositories, and non-centrally cleared derivatives should face stronger margin requirements (*G20 Leaders 2009*). These principles became the foundation of modern OTC regulation.

In the United States, these reforms were implemented through Title VII of the Dodd–Frank Act, which imposed central clearing obligations, swap reporting requirements, and enhanced supervision of swap dealers (*U.S. Congress 2010*). In Europe, the European Market Infrastructure Regulation (EMIR) introduced similar requirements for clearing, reporting, and bilateral margining (*European Securities and Markets Authority 2024*).

For non-centrally cleared derivatives, the Basel Committee and IOSCO introduced mandatory variation and initial margin requirements. As a result, OTC trading now requires significant collateral mobilisation and liquidity management, especially during stressed market conditions (*Basel Committee on Banking Supervision and IOSCO 2024*).

These regulations directly reinforce the relevance of this thesis. Once collateral, margin calls, and funding costs become mandatory components of OTC trading, derivative pricing can no longer rely on frictionless Black–Scholes assumptions. Funding liquidity, collateral conventions, and counterparty exposure become integral parts of valuation.

Chapter 2

Numerical Methods Comparison for an Unilateral Option

2.1 Economic Setting and Trade Assumptions

Airline companies are highly exposed to fuel price fluctuations, as kerosene costs represent a significant share of their operating expenses. During periods of geopolitical uncertainty, particularly in the Middle East where a large part of global oil production is concentrated, crude oil prices may increase significantly due to supply disruption fears and market tensions. In such an environment, Air France-KLM may seek protection against a potential rise in future fuel costs.

We therefore consider a European call option written on crude oil. Air France-KLM is denoted by B and BNP Paribas Fortis by C . The option is interpreted as a transaction in which Air France-KLM buys a European call option from BNP Paribas Fortis in order to hedge against rising oil prices. Throughout the thesis, derivative values are expressed from the perspective of Air France-KLM, following the Burgard-Kjaer convention, so that positive values correspond to assets for Air France-KLM and negative values to liabilities.

The underlying asset is crude oil, with initial spot price

$$S_0 = 101.17.$$

This value is fixed as of 15 May 2026 using market data from Yahoo Finance (*Yahoo Finance 2026*). The volatility parameter is estimated from one year of historical crude oil returns, giving

$$\sigma = 50.12\%.$$

The option has maturity $T = 1$ year and strike $K = 110$. The risk-free rate is proxied by the one-year EUR overnight indexed swap (OIS) rate based on €STR, consistently with the one-year maturity of the derivative. This choice is more appropriate than using the spot €STR fixing, since the latter is an overnight rate, whereas the derivative considered in this numerical example has a one-year maturity. OIS rates are commonly used as reference rates for discounting because they represent fixed rates exchanged against an overnight floating benchmark and are widely used in yield-curve construction and mark-to-market valuation.

Using the Cbonds index (*Cbonds 2026c*), with index identifier 217603, the one-year EUR OIS rate on 15 May 2026 is

$$r = 2.52016\%.$$

Therefore, in the numerical implementation, the constant risk-free rate is set equal to

$$r = 0.0252016.$$

For the risky yield r_B , corresponding to Air France-KLM, we use an Air France-KLM EUR senior unsecured bond with coupon 3.875%, maturing on 1 July 2026, with ISIN FR0014004AF5. This bond is issued by Air France-KLM, is denominated in EUR, and has an outstanding amount of EUR 500 million (*Cbonds 2026a*). The coupon rate is not the yield used in the model. The coupon is fixed contractually, while the relevant input for the Burgard–Kjaer framework is the market yield-to-maturity. Public bond market data reports a yield close to 3.36% for this bond, which is therefore used as the risky yield (*Boursorama 2026*). We therefore set

$$r_B = 3.36\%.$$

For the counterparty risky yield r_C , corresponding to BNP Paribas Fortis, we use a BNP Paribas Fortis EUR senior preferred bond with coupon 3.125%, maturing on 2 February 2027, with ISIN BE6348272103. This bond is issued by BNP Paribas Fortis and is denominated in EUR (*Cbonds 2026b*). Since the bond is short-dated and trades close to par, its yield-to-maturity is proxied by a value close to its coupon. We therefore set

$$r_C = 3.10\%.$$

The risky yields used in the numerical implementation are therefore

$$r_B = 3.36\%, \quad r_C = 3.10\%.$$

These yields are not interpreted as exact default intensities. Instead, following the zero-recovery risky bond assumption of the Burgard–Kjaer framework, the effective default

intensities are approximated by the spread between the risky bond yield and the risk-free rate:

$$\lambda_B = r_B - r, \quad \lambda_C = r_C - r.$$

This gives

$$\lambda_B = 3.36\% - 2.52016\% = 0.8398\%,$$

$$\lambda_C = 3.10\% - 2.52016\% = 0.5798\%.$$

In decimal form,

$$\lambda_B = 0.008398, \quad \lambda_C = 0.005798.$$

We assume a recovery rate of 40% for both Air France-KLM and BNP Paribas Fortis:

$$R_B = 40\%, \quad R_C = 40\%.$$

Finally, the funding spread is defined as the difference between the valuation party's unsecured funding rate and the risk-free rate:

$$s_f = r_F - r.$$

Following the Burgard–Kjaer specification in the case where the derivative cannot be posted as collateral, the funding rate is

$$r_F = r + (1 - R_B)\lambda_B.$$

Therefore,

$$s_f = (1 - R_B)\lambda_B.$$

Using $R_B = 40\%$ and $\lambda_B = 0.008398$, we obtain

$$s_f = (1 - 0.40) \times 0.008398 = 0.005039.$$

Hence,

$$s_f = 0.5039\%.$$

The final parameters used in the numerical implementation are therefore

$$r = 0.0252016, \quad r_B = 0.0336, \quad r_C = 0.0310,$$

$$\lambda_B = 0.008398, \quad \lambda_C = 0.005798, \quad s_f = 0.005039,$$

$$R_B = 0.40, \quad R_C = 0.40.$$

2.2 Numerical Implementation of the Burgard–Kjaer PDE

2.2.1 Crank–Nicolson Finite-Difference Scheme

Throughout the thesis, derivative values are computed from the perspective of the party entering the hedge position. In the present chapter, the valuation is therefore performed from the perspective of Air France–KLM as the buyer of the European call option.

The Burgard–Kjaer equation is solved numerically using a finite-difference approach. Finite-difference methods are standard tools for solving option-pricing PDEs in financial engineering, since they transform the continuous pricing problem into a discrete grid problem that can be solved numerically (*Duffy 2006, Tavella and Randall 2000*). The objective is to compare the standard Black–Scholes value of a European call option with the risky value obtained under the nonlinear Burgard–Kjaer framework. The implementation therefore contains two pricing engines: a benchmark Crank–Nicolson solver for the Black–Scholes PDE and a nonlinear Crank–Nicolson solver for the Burgard–Kjaer PDE with close-out $M = \hat{V}$.

The European call payoff is defined as

$$\Phi(S_T) = \max(S_T - K, 0).$$

The stock price domain is truncated to a finite interval $[0, S_{\max}]$, which is discretised into M equally spaced points. Time is discretised into N steps between 0 and T . The numerical grid is therefore given by

$$S_j = j\Delta S, \quad j = 0, \dots, M,$$

and

$$t_n = n\Delta t, \quad n = 0, \dots, N,$$

where

$$\Delta S = \frac{S_{\max}}{M}, \quad \Delta t = \frac{T}{N}.$$

The terminal condition is imposed at maturity:

$$V(T, S) = \max(S - K, 0).$$

For the Black–Scholes benchmark, the lower boundary is set to zero,

$$V(t, 0) = 0,$$

while the upper boundary uses the standard deep in-the-money call approximation:

$$V(t, S_{\max}) \approx S_{\max} e^{-q(T-t)} - K e^{-r(T-t)}.$$

The Black–Scholes PDE is solved using the Crank–Nicolson scheme, originally introduced by Crank and Nicolson for parabolic partial differential equations (*Crank and Nicolson 1947*). The method averages the implicit and explicit finite-difference schemes, which provides a balance between stability and accuracy and makes it widely used in financial PDE applications (*Duffy 2006, Tavella and Randall 2000*). The resulting tridiagonal linear system is solved backward in time from maturity to the valuation date. The analytical Black–Scholes formula is also computed and used as a benchmark to verify the accuracy of the finite-difference implementation.

The nonlinear Burgard–Kjaer PDE is then solved using the same grid and Crank–Nicolson structure. In the case $M = \hat{V}$, the PDE can be written in sign-dependent form. When the risky value is positive, meaning that the derivative represents an asset for Air France–KLM, the equation behaves as

$$\hat{V}_t + \mathcal{A}\hat{V} - [r + (1 - R_C)\lambda_C + s_F] \hat{V} = 0.$$

When the risky value is negative, meaning that the derivative represents a liability for Air France–KLM, the equation behaves as

$$\hat{V}_t + \mathcal{A}\hat{V} - [r - (1 - R_B)\lambda_B] \hat{V} = 0.$$

The nonlinear feature comes from the fact that the effective discounting term depends on the sign of the unknown value $\hat{V}$.

To handle this nonlinearity, a Picard iteration is used at each time step. Starting from an initial guess for the new time layer, the sign of the solution is frozen, the corresponding effective rate is computed at each grid point, and a linear Crank–Nicolson system is solved. This procedure is repeated until the maximum difference between two consecutive iterates falls below a fixed tolerance. In the implementation, the maximum number of Picard iterations is set to 50 and the convergence tolerance is set to 10^{-10} .

A specific upper boundary condition is used for the Burgard–Kjaer call. For large

stock prices, the call is deep in-the-money and the risky value is positive. The relevant effective adjustment is therefore

$$k_+ = (1 - R_C)\lambda_C + s_F.$$

The upper boundary is consequently written as

$$\hat{V}(t, S_{\max}) \approx S_{\max} e^{-(q+k_+)(T-t)} - K e^{-(r+k_+)(T-t)}.$$

This avoids imposing the Black–Scholes deep in-the-money slope on the Burgard–Kjaer solution and reduces boundary-induced distortions in the computed Greeks.

The numerical outputs are then compared in three ways. First, the Crank–Nicolson Black–Scholes value is compared with the analytical Black–Scholes value to validate the finite-difference scheme. Second, the nonlinear Burgard–Kjaer value is compared with the Black–Scholes benchmark to measure the valuation impact of funding costs and counterparty risk. Third, the difference is reported both in absolute terms and in basis points of the option premium, which makes the adjustment easier to interpret economically.

Greeks Computation

The Crank–Nicolson grid is also used to compute option sensitivities. Once the PDE has been solved, the option value is available across the stock-price and time grid, which allows delta, gamma and theta to be extracted directly from the numerical solution. Delta and gamma are computed from the option value curve at $t = 0$ using centred finite differences around the initial underlying price S_0 :

$$\Delta \approx \frac{V(S_0 + \Delta S) - V(S_0 - \Delta S)}{2\Delta S},$$

$$\Gamma \approx \frac{V(S_0 + \Delta S) - 2V(S_0) + V(S_0 - \Delta S)}{(\Delta S)^2}.$$

Theta is computed from the first two time layers of the grid:

$$\Theta \approx \frac{V(t + \Delta t, S_0) - V(t, S_0)}{\Delta t}.$$

In practice, cubic interpolation is used to evaluate the option value around S_0 when S_0 does not fall exactly on a grid point. In addition to these grid-based sensitivities, vega, rho and dividend rho are computed using a bump-and-revalue approach. The relevant parameter is shifted upward and downward, the full Crank–Nicolson solver is rerun, and the sensitivity is approximated using a centred finite difference. Vega is reported per one volatility point, while rho and dividend rho are reported per one basis point.

For the Burgard–Kjaer model, additional credit and funding sensitivities are computed

in the same way. The counterparty default intensity λ_C , the own default intensity λ_B , the funding spread s_F , and the recovery rates R_C and R_B are bumped separately. The model is then repriced after each perturbation. The resulting sensitivities measure how the nonlinear risky value reacts to changes in counterparty credit risk, own default risk, funding costs and recovery assumptions.

Finally, the full delta, gamma and theta curves are plotted across the stock price grid for both the Black–Scholes and Burgard–Kjaer models. This allows the comparison to go beyond the single value at S_0 and shows how funding and credit risk affect the shape of the risk profile across different moneyness regions.

2.2.2 Monte Carlo

As a second numerical approach, the option value is computed using Monte Carlo simulation. Monte Carlo methods are standard tools in financial engineering for estimating risk-neutral expectations by simulation (*Glasserman 2004*). The purpose here is to provide an independent benchmark for the Crank–Nicolson results and to verify that the valuation adjustment obtained under the Burgard–Kjaer framework is not driven by the finite-difference discretisation.

Under the risk-neutral measure, the underlying follows a geometric Brownian motion:

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t.$$

Since the payoff of a European call depends only on the terminal stock price, the implementation simulates the terminal distribution directly:

$$S_T = S_0 \exp \left[\left(r - q - \frac{1}{2}\sigma^2 \right) T + \sigma\sqrt{T}Z \right],$$

where

$$Z \sim \mathcal{N}(0, 1).$$

This remains a Monte Carlo estimator, since the option value is computed as the sample average of discounted simulated payoffs. It is also more efficient than simulating the full path, because intermediate time steps are not needed for a European payoff. The final implementation uses 1,000,000 simulated terminal values, with a fixed random seed to ensure reproducibility.

For the Black–Scholes benchmark, the European call payoff is computed at maturity as

$$\Phi(S_T) = \max(S_T - K, 0),$$

and the Monte Carlo value is obtained by discounting the average payoff at the risk-free

rate:

$$V_{MC}^{BS} = e^{-rT} \frac{1}{N_{\text{sim}}} \sum_{i=1}^{N_{\text{sim}}} \max(S_T^{(i)} - K, 0).$$

For the Burgard–Kjaer value, the implementation relies on the fact that the bought European call remains almost entirely in the positive exposure region from the perspective of Air France–KLM. The derivative therefore mainly behaves as an asset for the valuation party, so the relevant effective discounting rate is

$$r_{BK} = r + (1 - R_C)\lambda_C + s_F.$$

The Monte Carlo Burgard–Kjaer value is then computed as

$$\hat{V}_{MC}^{BK} = e^{-r_{BK}T} \frac{1}{N_{\text{sim}}} \sum_{i=1}^{N_{\text{sim}}} \max(S_T^{(i)} - K, 0).$$

This Monte Carlo implementation does not require Picard iterations because the sign of the exposure is known in advance for the European call considered here. The Burgard–Kjaer discounting term therefore remains fixed at the positive-exposure rate r_{BK} . This makes the Monte Carlo implementation a useful consistency check against the nonlinear Crank–Nicolson solver.

The standard error of the Monte Carlo estimator is also computed to quantify the sampling uncertainty of the simulated value (*Glasserman 2004*):

$$SE = \frac{\text{Std}\left(e^{-rT}\Phi(S_T)\right)}{\sqrt{N_{\text{sim}}}}.$$

A 95% confidence interval is then constructed as

$$V_{MC} \pm 1.96 SE.$$

The Black–Scholes Monte Carlo value is compared with the analytical Black–Scholes value to validate the simulation engine. The Burgard–Kjaer Monte Carlo value is then compared with both the Black–Scholes Monte Carlo value and the Crank–Nicolson Burgard–Kjaer value. The differences are reported in absolute terms, in percentage terms, and in basis points of the option premium.

Greeks Computation

The Monte Carlo implementation is also extended to compute sensitivities using a bump-and-revalue approach. This approach consists in perturbing one input parameter, repricing the instrument, and approximating the derivative of the value by a finite difference. For each Greek, the relevant input parameter is shifted upward and downward while

keeping all other parameters unchanged. The value is then recomputed under both perturbations, and the sensitivity is approximated using a central finite difference. To reduce Monte Carlo noise, the same random seed is used across the base, upward-bumped and downward-bumped simulations. This corresponds to using common random numbers, a standard variance-reduction technique in simulation-based pricing (*Glasserman 2004*).

The standard Greeks are computed for both the Black–Scholes and Burgard–Kjaer Monte Carlo values. Delta and gamma are obtained by bumping the initial underlying price S_0 :

$$\Delta \approx \frac{V(S_0 + \Delta S) - V(S_0 - \Delta S)}{2\Delta S},$$

$$\Gamma \approx \frac{V(S_0 + \Delta S) - 2V(S_0) + V(S_0 - \Delta S)}{(\Delta S)^2}.$$

Theta is computed by reducing the maturity by one calendar day:

$$\Theta \approx \frac{V(T - \Delta T) - V(T)}{\Delta T}, \quad \Delta T = \frac{1}{365}.$$

Vega, rho and dividend rho are computed by bumping respectively the volatility, the risk-free rate and the dividend yield:

$$\text{Vega} \approx \frac{V(\sigma + \Delta\sigma) - V(\sigma - \Delta\sigma)}{2\Delta\sigma},$$

$$\rho \approx \frac{V(r + \Delta r) - V(r - \Delta r)}{2\Delta r},$$

$$\rho_q \approx \frac{V(q + \Delta q) - V(q - \Delta q)}{2\Delta q}.$$

In the implementation, the bumps are set to $\Delta S = 1$, $\Delta\sigma = 0.01$, $\Delta r = 0.0001$, and $\Delta q = 0.0001$. Vega is reported per one volatility point, while rho and dividend rho are reported per one basis point.

For the Burgard–Kjaer model, additional credit and funding sensitivities are computed. The counterparty default intensity λ_C , the own default intensity λ_B , the funding spread s_F , and the recovery rates R_C and R_B are bumped separately. The model is then repriced after each perturbation. The resulting sensitivities measure how the risky option value reacts not only to standard market inputs, but also to changes in counterparty credit risk, own default risk, funding costs and recovery assumptions introduced by the Burgard–Kjaer framework.

2.3 Numerical Methods Results Comparison

2.3.1 Value Comparison

Table 2.1: Crank–Nicolson and Monte Carlo value comparison for the European call option

Method	Model	Value	Adjustment (%)
Analytical	Risk-Free	17.712964	0.000000
Crank–Nicolson	Risk-Free	17.712963	0.000000
	Burgard–Kjaer	17.562723	-0.848190
Monte Carlo	Risk-Free	17.703283	0.000000
	Burgard–Kjaer	17.553126	-0.848190
Difference between models (%)			-0.054646

The value comparison confirms that both the Crank–Nicolson and Monte Carlo implementations are consistent with the analytical Black–Scholes benchmark.

Comparing the Burgard–Kjaer values to their respective risk-free values, both numerical methods produce an adjustment of approximately -84.81 basis points. This negative adjustment reflects the impact of counterparty risk and funding costs on the value of the call option from the perspective of Air France–KLM. Since the Monte Carlo and Crank–Nicolson risk-free values are not exactly identical, the two Burgard–Kjaer values differ slightly, by approximately 0.054% .

The Crank–Nicolson risk-free value remains closer to the analytical Black–Scholes benchmark. This difference can be explained by the fact that Monte Carlo simulations are inherently noisier and depend on the number of simulated paths. In this implementation, $1\,000\,000$ paths were simulated, producing the following 95% confidence interval:

$$[17.4784114341, 17.6278398019].$$

These results suggest that the Crank–Nicolson method provides more precise values while also being less computationally intensive for the valuation problem considered in this thesis.

2.3.2 Greeks

Table 2.2: Comparison between Crank–Nicolson and Monte Carlo Greeks

Model	Greek	CN	MC	MC – CN	Difference (%)
Black–Scholes	Delta	0.553267	0.553068	-0.000199	-0.035965
	Gamma	0.007797	0.007652	-0.000145	-1.859690
	Theta	-0.030113	-0.030127	-0.000014	0.046492
	Vega	0.400004	0.400063	0.000059	0.014750
	Rho	0.003826	0.003823	-0.000003	-0.078411
	Dividend rho	-0.005597	-0.005594	0.000003	-0.053600
Burgard–Kjaer	Delta	0.548575	0.548377	-0.000198	-0.036093
	Gamma	0.007731	0.007587	-0.000144	-1.862631
	Theta	-0.029448	-0.029463	-0.000015	0.050937
	Vega	0.396612	0.396669	0.000057	0.014371
	Rho	0.003794	0.003790	-0.000004	-0.105430
	Dividend rho	-0.005550	-0.005547	0.000003	-0.054054

Table 2.2 shows that the Greeks obtained from the Crank–Nicolson and Monte Carlo implementations remain very close and globally consistent for both the Black–Scholes and Burgard–Kjaer frameworks. Delta, Gamma, Theta and Rho are slightly higher for the Crank–Nicolson method, while Vega and Dividend Rho are marginally lower. This behaviour is consistent with the valuation results obtained previously, where the Crank–Nicolson values were themselves slightly higher than the Monte Carlo estimates.

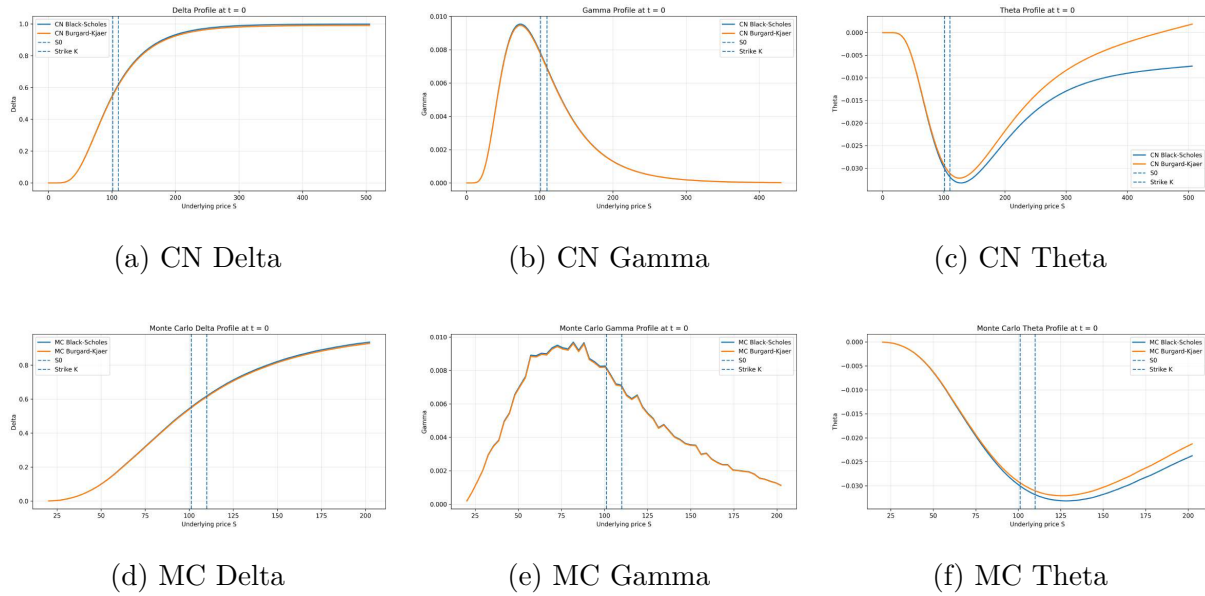


Figure 2.1: Comparison of Greeks profiles obtained from the Crank–Nicolson and Monte Carlo implementations.

The Greeks profiles presented in Figure 2.1 highlight the main numerical differences between the two methodologies. Delta, Gamma and Theta are shown because they are directly linked to the finite-difference structure of the PDE and therefore provide the most relevant comparison between the deterministic and simulation-based approaches. The remaining Greeks are omitted from the graphical comparison since they lead to similar conclusions without providing significant additional insight.

The Crank–Nicolson Greeks profiles remain smooth and stable across the stock-price domain because the PDE solver is deterministic once the numerical grid is fixed. In contrast, the Monte Carlo Greeks exhibit visible numerical noise, particularly for Gamma. This can be explained by the fact that Monte Carlo sensitivities are estimated statistically from simulated paths, while Gamma, being a second-order derivative, amplifies simulation noise significantly.

The computation of Monte Carlo Greeks also proved substantially more computationally intensive than the Crank–Nicolson implementation. A full bump-and-revalue approach for every grid node would have exceeded the available memory constraints. A lighter implementation based on local finite-difference approximations and common random numbers was therefore used to reduce the computational burden.

Overall, the Greeks comparison suggests that the Crank–Nicolson method provides more stable and computationally efficient sensitivities for the valuation problems considered in this thesis.

2.3.3 Methodological Conclusion

The numerical comparison shows that both the Crank–Nicolson and Monte Carlo implementations produce coherent values and Greeks for the Black–Scholes and Burgard–Kjaer frameworks. The valuation differences between the two methods remain very limited, while the Greeks obtained from both approaches are globally consistent.

However, important differences appear regarding numerical stability and computational efficiency. The Crank–Nicolson implementation produces smoother and more stable Greeks profiles, particularly for second-order sensitivities such as Gamma, while the Monte Carlo approach exhibits visible statistical noise due to the simulation process. In addition, the computation of Monte Carlo Greeks proved substantially more computationally intensive.

These results suggest that both methods remain valid for valuation purposes, but that the Crank–Nicolson method is better suited for the sensitivity analysis and repeated numerical computations performed throughout the remainder of this thesis. The following chapters therefore mainly rely on the Crank–Nicolson implementation for the analysis of bilateral derivatives and the comparison with modern XVA approaches.

2.4 Sensitivity Analysis

The previous sections focused on the numerical implementation and valuation of the Burgard–Kjaer framework under a fixed market calibration. However, beyond the computation of a single price, an important objective is to understand how the risky derivative value reacts to changes in the main market and credit parameters driving the model. Sensitivity analysis therefore provides additional economic intuition regarding the respective roles of interest rates, funding conditions and counterparty credit risk within the Burgard–Kjaer framework.

Particular attention is given to the risk-free rate r , the bank risky bond yield r_B and the counterparty risky bond yield r_C , since these quantities directly determine the default intensities and funding spreads embedded in the pricing equation. Analysing their impact also helps distinguish the standard Black–Scholes pricing effects from the additional adjustments introduced by bilateral counterparty risk and funding costs.

The sensitivity analysis is performed by shocking one market input at a time and recomputing the Burgard–Kjaer price with the Crank–Nicolson solver. The objective is to isolate the effect of the risk-free rate r , the bank risky bond yield r_B , and the counterparty risky bond yield r_C . After each shock, the dependent parameters are recomputed as

$$\lambda_B = r_B - r, \quad \lambda_C = r_C - r, \quad s_F = (1 - R_B)\lambda_B.$$

2.4.1 Shock on the Risk-Free Rate

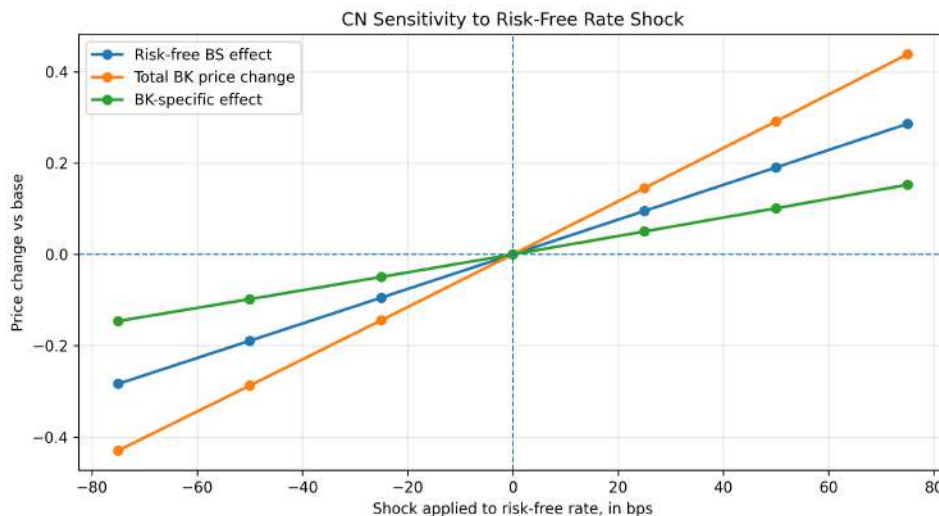


Figure 2.2: Sensitivity of the Burgard–Kjaer value with respect to the risk-free rate.

The first sensitivity shocks the risk-free rate r , while keeping the risky bond yields r_B and r_C fixed. Table C.2 shows that the Burgard–Kjaer value increases from 16.6951 under a -75 bps shock to 17.5627 under a $+75$ bps shock. Around the base case, the local sensitivity is approximately

$$\frac{\partial \hat{V}}{\partial r} \approx 0.00584$$

per basis point.

Part of this increase corresponds to the standard Black–Scholes rho effect. The risk-free Crank–Nicolson value itself increases from 17.1438 to 17.7130 across the shock interval.

This interpretation nevertheless requires caution. In practice, a rise in r would likely also affect the risky yields r_B and r_C . If all yields increased together, the implied default intensities and funding spreads could remain partially unchanged, reducing the BK-specific effect isolated in this sensitivity analysis.

The decomposition between the risk-free valuation effect and the Burgard–Kjaer-specific effect is illustrated in Figure 2.2. The figure shows that both components contribute materially to the total sensitivity of the Burgard–Kjaer value.

2.4.2 Shock on Bank Risky Bond Yield

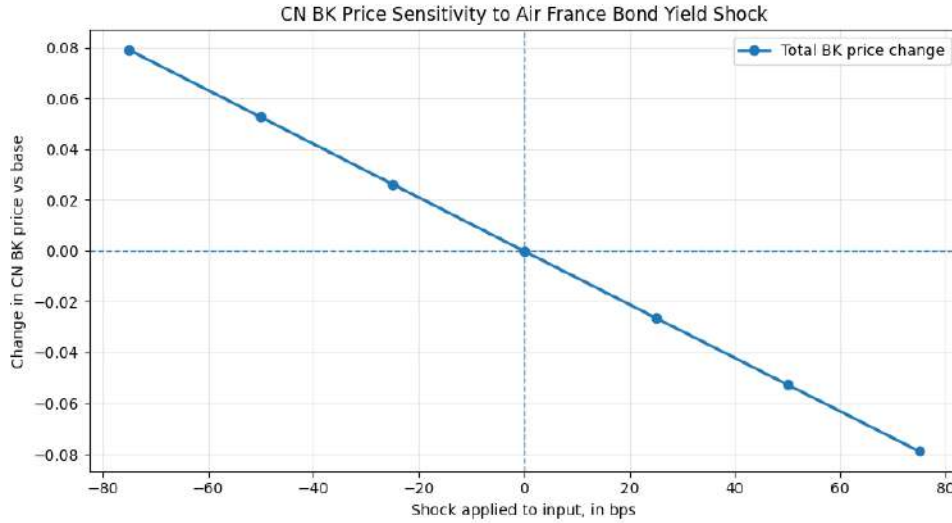


Figure 2.3: Sensitivity of the Burgard–Kjaer value with respect to own risky yield.

The second sensitivity shocks the risky yield r_B of Air France–KLM while keeping r and r_C fixed. The Burgard–Kjaer value decreases from 17.6419 under a -75 bps shock to 17.4839 under a $+75$ bps shock, with local sensitivity

$$\frac{\partial \hat{V}}{\partial r_B} \approx -0.001054$$

per basis point.

The negative sign is intuitive from the perspective of Air France–KLM. A higher r_B increases both the implied own default intensity

$$\lambda_B = r_B - r,$$

and the funding spread

$$s_F = (1 - R_B)\lambda_B.$$

Within the Burgard–Kjaer framework, this higher funding spread increases the effective discounting applied to the future positive payoff of the call option. As a result, the present value of the derivative decreases.

This effect isolates a pure widening of Air France–KLM’s funding or credit spread, since the risk-free rate r is held fixed. In practice, if r_B increased because the whole risk-free curve moved upward, part of the negative funding effect could be offset by the positive sensitivity of the option value to the risk-free rate itself.

2.4.3 Shock on Counterparty Risky Bond Yield

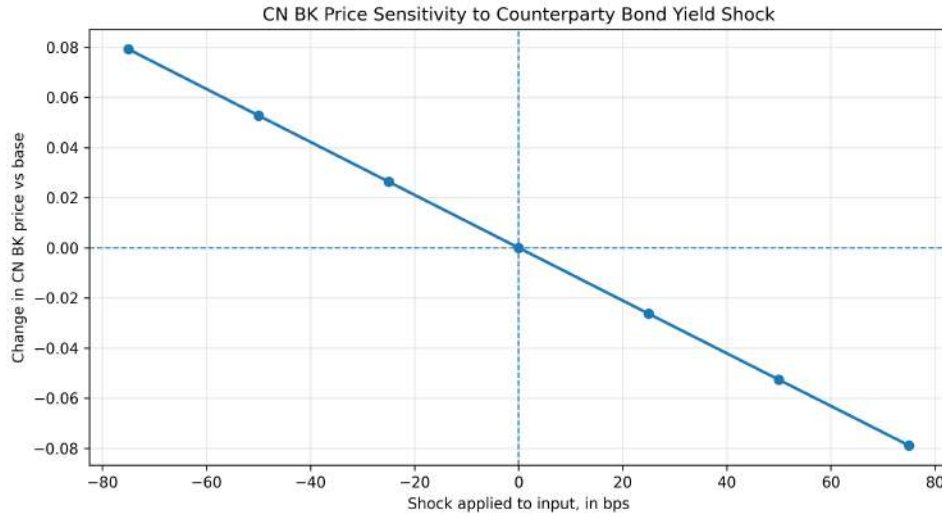


Figure 2.4: Sensitivity of the Burgard–Kjaer value with respect to the counterparty risky yield.

The third sensitivity shocks the counterparty risky yield r_C , corresponding to BNP Paribas Fortis, while keeping r and r_B fixed. Table C.4 shows that the Burgard–Kjaer value decreases from 17.6419 under a -75 bps shock to 17.4839 under a $+75$ bps shock. The local sensitivity is again

$$\frac{\partial \hat{V}}{\partial r_C} \approx -0.001054$$

per basis point.

An increase in r_C raises the implied counterparty default intensity

$$\lambda_C = r_C - r.$$

For Air France–KLM, the bought call option mainly represents a positive exposure to BNP Paribas Fortis. Therefore, a higher counterparty default intensity reduces the expected recoverable value of this positive exposure, which lowers the risky Burgard–Kjaer value.

In this calibration, the sensitivities to r_B and r_C are identical. This is not a numerical error. Since $R_B = R_C = 40\%$, both risky yields enter the positive-exposure adjustment with the same loss-given-default weight. Economically, however, r_B affects the value through Air France–KLM’s own funding spread s_F , while r_C affects it through BNP Paribas Fortis’ counterparty default intensity λ_C .

As for r_B , the interpretation depends on what drives the change in r_C . If r_C increases because BNP Paribas Fortis’ credit spread widens while the risk-free curve remains unchanged, the value decreases as shown above. If the increase mainly comes from a parallel upward shift of the risk-free curve, part of the effect would be offset by the positive sensitivity to r .

Chapter 3

Extending the Burgard–Kjaer Framework to Bilateral Exposure Derivatives

The previous chapter analysed the Burgard–Kjaer framework using a European call option. Although the Burgard–Kjaer framework itself incorporates bilateral counterparty risk, the bought call option considered previously remained largely unilateral from an exposure perspective, since its value stays mostly positive for Air France–KLM over the life of the contract.

In their paper, Burgard and Kjaer (*Burgard and Kjaer 2011*) suggested that the interaction between funding costs and bilateral counterparty risk could become more significant for derivatives whose value may switch between positive and negative regions. This chapter explores that intuition by applying the framework to an equity forward and an interest rate forward, two contracts that naturally generate bilateral exposure profiles.

3.1 Equity Forward Contract

An airline company may sometimes prefer to lock in a future purchase price rather than simply protect itself against extreme price increases. In such a situation, a forward contract becomes more appropriate than a call option, since it guarantees a fixed future price for fuel purchases but also removes the possibility of benefiting from lower market prices. This makes forward contracts particularly relevant for studying genuinely bilateral exposure profiles.

We consider a long forward contract written on the same underlying as in the previous chapter and valued from the perspective of Air France–KLM. The terminal payoff is defined as

$$\Phi(S_T) = S_T - K_F,$$

where K_F denotes the forward delivery price. In order to isolate the effect of the

Burgard–Kjaer adjustment, the delivery price is chosen so that the risk-free value of the forward is zero at inception. The forward strike is therefore set equal to

$$K_F = S_0 e^{(r-q)T}.$$

Using the calibration of the previous chapter, with

$$S_0 = 101.17, \quad r = 2.5202\%, \quad q = 0, \quad T = 1,$$

the forward delivery price is

$$K_F = 103.7520.$$

The risk-free value of the forward at time t is

$$V(t, S) = S e^{-q(T-t)} - K_F e^{-r(T-t)}.$$

At inception, this gives

$$V(0, S_0) = 0.$$

This specification is useful because any deviation from zero in the Burgard–Kjaer value can then be directly interpreted as the impact of counterparty risk, own funding effects and bilateral exposure dynamics.

3.1.1 Numerical Implementation

The forward contract is valued using the same Crank–Nicolson finite-difference framework as in the previous chapter.

The terminal condition is changed to the forward payoff:

$$\hat{V}(T, S) = S - K_F.$$

The risk-free benchmark is computed with the same Crank–Nicolson grid and compared with the analytical forward value. The Burgard–Kjaer forward is then obtained by applying the nonlinear sign-dependent equation. As before, a Picard iteration is used at each time step. The sign of the current iterate determines whether each grid point is treated as a positive or negative exposure region. The corresponding Burgard–Kjaer adjustment is then applied locally before solving the tridiagonal system.

This implementation is particularly suited to the forward extension because the Crank–Nicolson grid provides the full value function across the stock-price domain. It therefore makes it possible to observe directly where the derivative represents an asset or a liability for Air France–KLM, and how the Burgard–Kjaer adjustment changes.

3.1.2 Pricing Results

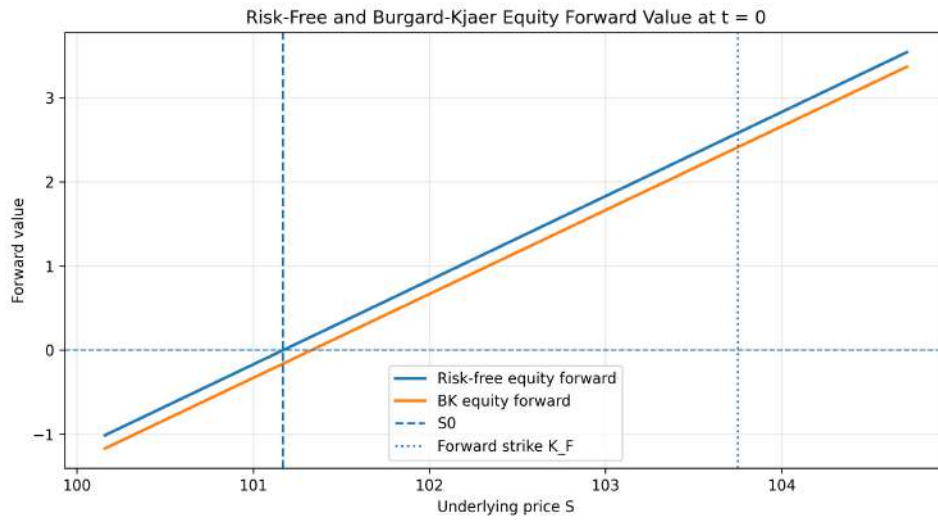


Figure 3.1: Risk-free and Burgard–Kjaer values of the equity forward contract.

Table 3.1: Equity forward Burgard–Kjaer adjustment

Metric	Value
Risk-free Analytical value	0.000000
Risk-free CN value	0.000000
BK value	-0.181375
BK adjustment	-0.181375
BK adjustment as % of S_0	-0.179278
BK adjustment in bps of S_0	-17.927777

The risk-free forward is correctly initialized at zero. The analytical risk-free value is

$$V(0, S_0) = 0.0000,$$

and the Crank–Nicolson risk-free value is also approximately zero:

$$V^{CN}(0, S_0) = 0.0000.$$

This confirms that the forward strike has been correctly chosen and that the risk-free numerical implementation is consistent with the analytical benchmark.

The Burgard–Kjaer value of the same forward is

$$\hat{V}^{BK}(0, S_0) = -0.1814.$$

3.1.3 Exposure Regions and Bilaterality

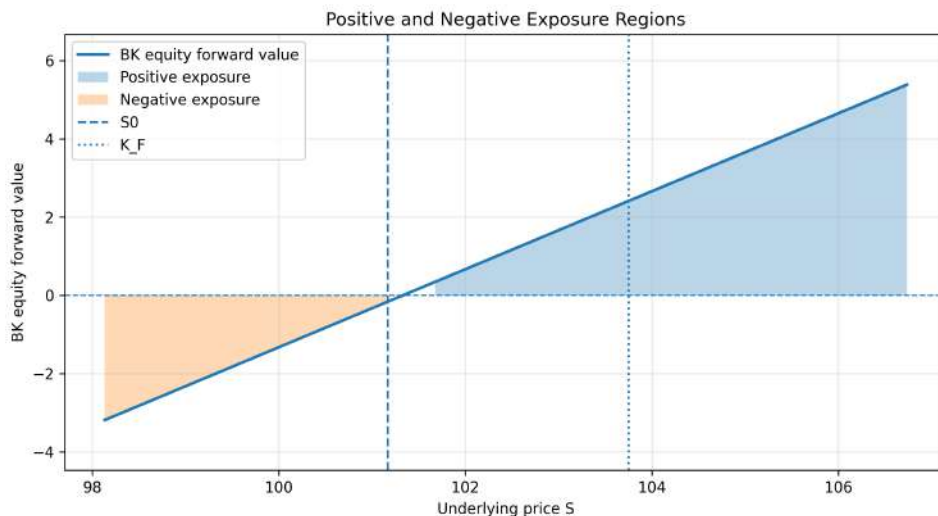


Figure 3.2: Positive and negative exposure regions of the equity forward contract.

This figure illustrates the positive and negative exposure regions of the Burgard–Kjaer equity forward. The contract value crosses zero at a stock price slightly above the initial spot price.

Below this level, the forward value is negative and the contract behaves as a liability for Air France–KLM, activating the negative component $\hat{V}^-$ of the Burgard–Kjaer equation. Above it, the value becomes positive and the contract behaves as an asset for Air France–KLM, activating the positive component $\hat{V}^+$.

Unlike the European call option studied previously, the equity forward therefore generates genuinely bilateral exposure regions where both CVA/FVA and DVA contribute to the valuation (*Gregory 2015*). The presence of DVA partially offsets the negative impact of CVA and funding costs, which limits the overall Burgard–Kjaer adjustment.

3.2 Interest Rate Forward Contract

The contract is interpreted as a simplified forward rate agreement between Air France–KLM and BNP Paribas. For example, Air France–KLM may know that part of its aircraft fleet will need to be renewed within one year, requiring future debt financing to purchase additional planes. Since the financing cost of this future borrowing depends on interest rates at that date, a rise in rates could significantly increase the cost of the investment and complicate budget planning. In order to reduce this uncertainty and secure more stable financing conditions, Air France–KLM enters a forward rate agreement allowing it to lock in a future borrowing rate today. The contract is assumed to have a notional amount of 1 million.

The contract is valued from the perspective of Air France–KLM. The terminal payoff is written as

$$\Phi(r_T) = N\Delta(r_T - K),$$

where r_T denotes the short rate observed at maturity and K is the fixed contractual rate of the forward.

3.2.1 Numerical Implementation

The first step of the implementation consists in modelling the short-rate dynamics in order to obtain the parameters required for the finite-difference resolution of the Burgard–Kjaer PDE. Unlike the equity forward case, where the underlying asset follows a Black–Scholes diffusion, the state variable of the PDE is now the short rate itself. A stochastic short-rate model is therefore required to define the drift and volatility terms entering the Crank–Nicolson discretisation.

Vasicek Calibration

The short-rate dynamics are modelled using the Vasicek process

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r dW_t.$$

The parameters are calibrated using historical observations of the euro short-term rate (€STR) obtained from the European Central Bank database *European Central Bank 2026a*. The dataset contains 1699 daily observations covering the period from October 2019 to May 2026.

The calibration produces the following parameters:

$$r_0 = 1.9320\%, \quad K = 2.1905\%,$$

$$\kappa = 0.1368, \quad \theta = 3.9678\%, \quad \sigma_r = 0.6528\%.$$

These calibrated parameters are used to construct the Vasicek zero-coupon bond prices, the implied forward short rate, and the drift and volatility coefficients entering the Burgard–Kjaer PDE.

The interest rate forward is then valued using the same Crank–Nicolson methodology as in the previous sections, with the stock-price dynamics replaced by the Vasicek short-rate process. The terminal condition becomes

$$\hat{V}(T, r) = N\Delta(r - K).$$

As in the equity forward extension, the nonlinear Burgard–Kjaer equation is solved using a Picard iteration at each time step. The sign of the current iterate determines whether each grid point belongs to a positive or negative exposure region, which locally activates the corresponding Burgard–Kjaer adjustment.

3.2.2 Pricing Results

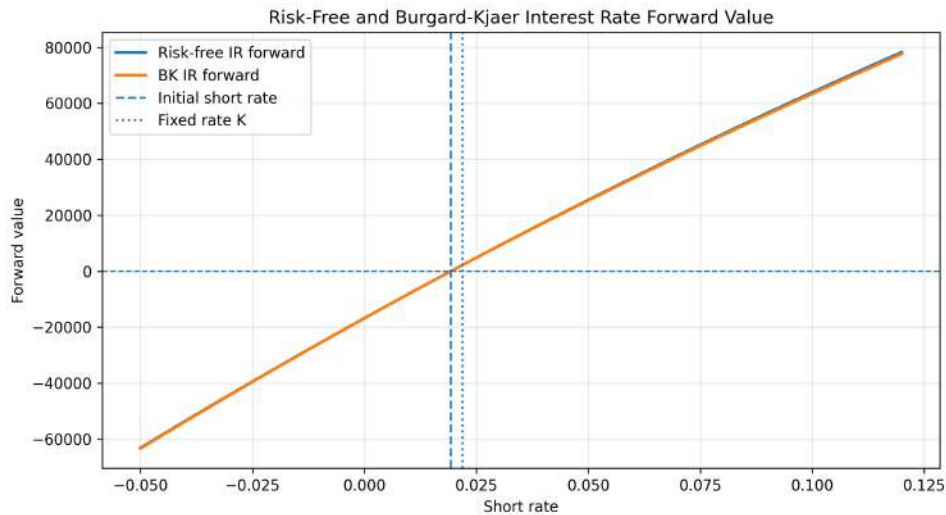


Figure 3.3: Risk-free and Burgard–Kjaer values of the interest rate forward contract.

Table 3.2: Interest rate forward Burgard–Kjaer adjustment diagnostics

Metric	Value
Risk-free Analytical value	0.000000
Risk-free CN value	-0.014371
BK value	-20.956101
BK adjustment	-20.941730
BK adjustment as % of notional	-0.002094
BK adjustment in bps of notional	-0.209417

As expected, the analytical risk-free value of the forward is approximately zero at inception. The Crank–Nicolson implementation produces only a very small numerical deviation due to the discretisation of the Vasicek PDE.

The Burgard–Kjaer value is significantly lower than the corresponding risk-free value, leading to an adjustment of approximately -20.94 . Although this difference appears large relative to the near-zero initial value of the contract, it remains economically small compared with the 1,000,000 notional, corresponding to approximately -0.21 basis points of notional.

3.2.3 Exposure Regions and Bilaterality

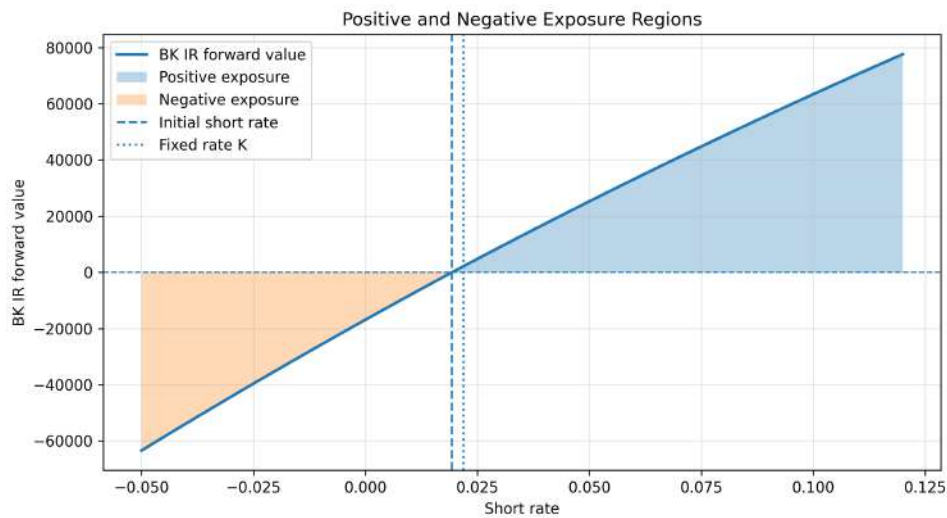


Figure 3.4: Positive and negative exposure regions of the interest rate forward contract.

This figure illustrates the positive and negative exposure regions of the Burgard–Kjaer interest rate forward. As for the equity forward, the contract dynamically switches between asset and liability regions depending on the level of the underlying state variable, here the short rate rather than the stock price.

The forward value crosses zero at a short rate slightly below the contractual fixed rate K . Above this level, the contract behaves as an asset for Air France–KLM and the positive exposure component $\hat{V}^+$ dominates the valuation. Below it, the contract becomes a liability and the negative exposure component $\hat{V}^-$ becomes active.

The presence of both positive and negative exposure regions confirms the genuinely bilateral structure of the interest rate forward, where DVA partially offsets the negative impact of CVA and funding costs.

3.3 Comparison Between the Equity and Interest Rate Forwards

Both forward contracts exhibit genuinely bilateral exposure profiles since their value may become either positive or negative depending on market conditions. As a consequence, both products activate the positive and negative branches of the Burgard–Kjaer equation.

For the equity forward, the Burgard–Kjaer adjustment reaches approximately

$$-17.93 \text{ basis points relative to } S_0,$$

whereas the interest rate forward adjustment corresponds to approximately

−0.21 basis points relative to the notional amount.

The adjustment of the interest rate forward therefore appears economically much smaller in relative terms. This can mainly be explained by the scale used for the comparison. The interest rate forward adjustment is expressed relative to a 1.000.000 notional, while the equity forward adjustment is measured relative to the underlying price. In addition, the mean-reverting Vasicek dynamics generate a smoother exposure profile than the equity forward, limiting the magnitude of the Burgard–Kjaer adjustment.

3.4 Impact of Bilaterality

Comparing the unilateral European call option with the two bilateral forward contracts highlights an important effect of bilaterality within the Burgard–Kjaer framework. The European call generated a Burgard–Kjaer adjustment of approximately 84 basis points, compared with approximately 17.93 basis points for the equity forward and only 0.21 basis points for the interest rate forward.

This reduction is mainly explained by the appearance of negative exposure regions in bilateral derivatives. Unlike the bought call option, whose value remains mostly positive for Air France–KLM, the forward contracts may also become liabilities during part of their lifetime. As a consequence, the valuation must incorporate both positive and negative exposure components within the Burgard–Kjaer framework.

The introduction of this bilateral structure activates the DVA component, which affects the derivative value in the opposite direction of the CVA and funding adjustments. The presence of DVA therefore partially offsets the negative impact generated by counterparty risk and funding costs, reducing the overall Burgard–Kjaer adjustment compared with the unilateral option case (*Gregory 2015*). The role of this offsetting effect and its implications for derivative valuation will be analysed in more detail in the next chapter through the comparison with the modern XVA framework.

Chapter 4

Comparison of Burgard–Kjaer and XVA Frameworks

The previous chapters computed risky derivative values using the Burgard–Kjaer framework. In this framework, counterparty default risk, own default risk and funding costs are embedded directly into the pricing equation. In current market practice, similar economic effects are often analysed through the XVA framework (*Gregory 2015*), where the risk-free value is adjusted through separate valuation adjustments such as CVA, DVA and FVA.

The objective of this chapter is to compare the Burgard–Kjaer values obtained with the Crank–Nicolson implementation with values computed using a simplified risk-free plus XVA approach. The purpose is not to reproduce a full industrial XVA engine, which would require collateral agreements, netting sets, margining rules, funding policies, capital charges and potentially stochastic credit spreads. Instead, the objective is to compare the economic content of the Burgard–Kjaer adjustment with a standard exposure-based XVA approximation.

4.1 XVA Framework

The XVA value is computed by starting from the risk-free value and then applying CVA, DVA and FVA adjustments. The simplified valuation relation is written as

$$V^{RF+XVA} = V^{RF} - CVA + DVA - FVA.$$

The CVA captures the expected loss due to counterparty default on positive exposure. In its simplest form, CVA can be written as

$$CVA = LGD_C \times PD_C \times EE^{disc}.$$

For a derivative whose exposure changes over time, this expression is applied over the

full exposure profile. The time-discretised approximation becomes

$$CVA \approx (1 - R_C) \sum_{i=1}^N DF(t_i) EPE(t_i) \Delta PD_C(t_i),$$

where $EPE(t_i)$ is the expected positive exposure at time t_i , $DF(t_i)$ is the discount factor, and $\Delta PD_C(t_i)$ is the marginal default probability of the counterparty over the corresponding time interval.

Assuming a constant counterparty default intensity λ_C , the marginal default probability is approximated by

$$\Delta PD_C(t_i) \approx \lambda_C \Delta t.$$

The CVA therefore becomes

$$CVA \approx (1 - R_C) \lambda_C \int_0^T e^{-rt} EPE(t) dt.$$

It is approximated numerically using the trapezoidal rule.

Similarly, the DVA captures the expected gain associated with the own default of the valuation party on negative exposure. It is computed as

$$DVA \approx (1 - R_B) \lambda_B \int_0^T e^{-rt} ENE(t) dt,$$

where $ENE(t)$ is the expected negative exposure.

Finally, the FVA captures the cost of funding positive exposure. In this simplified setting, it is computed as

$$FVA \approx s_F \int_0^T e^{-rt} EPE(t) dt.$$

The default intensities and funding spread are kept consistent with the Burgard–Kjaer implementation:

$$\lambda_B = r_B - r, \quad \lambda_C = r_C - r, \quad s_F = (1 - R_B) \lambda_B.$$

The exposure profiles are generated using Monte Carlo simulations under the same Black–Scholes dynamics used throughout the thesis:

$$dS_t = (r - q) S_t dt + \sigma S_t dW_t.$$

The volatility, risk-free rate, risky bond yields, default intensities and funding spread are assumed constant over the one-year horizon. This keeps the comparison focused on the valuation framework itself rather than on the modelling of stochastic volatility, interest rates or credit spreads.

4.2 Results

4.2.1 Unilateral exposure derivative: EU call option

Metric	Value
Risk-free price	17.712963
BK price	17.562723
Risk-free + XVA price	17.562040
BK adjustment (bps)	-84.819033
XVA adjustment (bps)	-85.204774
Difference XVA – BK (bps)	-0.385741
BK vs XVA difference (%)	-0.452722

Table 4.1: XVA vs Burgard–Kjaer comparison for the European call option

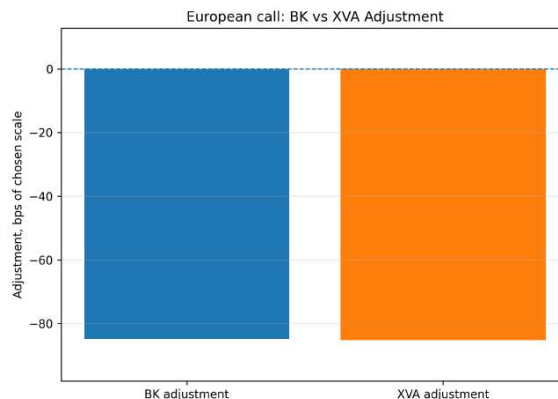


Figure 4.1: Comparison between Burgard–Kjaer and simplified XVA adjustments for the European call option.

The results show that the simplified XVA framework and the Burgard–Kjaer framework produce almost identical adjustments for the European call option. The difference between the two approaches is only -0.39 basis points, confirming that for a strongly unilateral exposure derivative the additive XVA decomposition provides a very close approximation of the full BK valuation.

This result is explained by the exposure profile of the bought call option held by Air France–KLM, which remains almost entirely positive over the life of the trade. Since the contract rarely becomes a liability for Air France–KLM, no meaningful DVA effect appears and the adjustment is mainly driven by counterparty risk and funding costs.

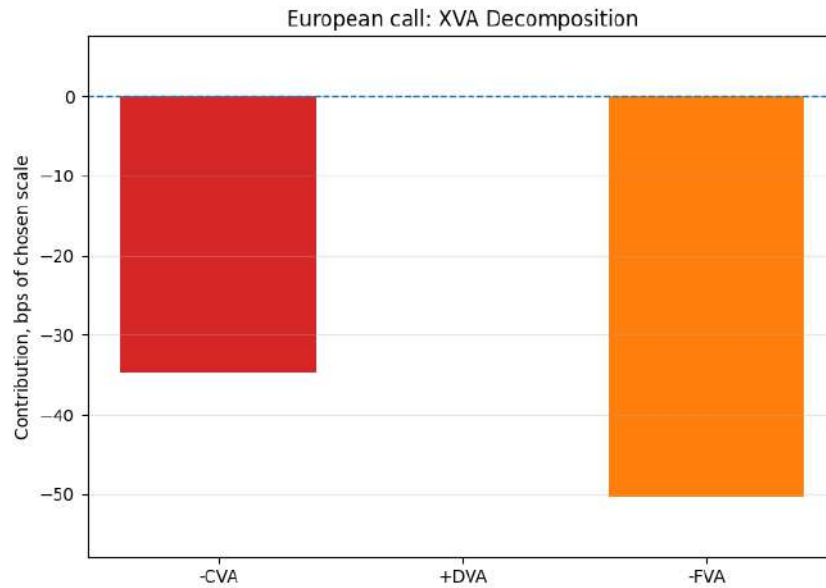


Figure 4.2: Simplified XVA decomposition for the European call option.

The simplified XVA decomposition confirms this interpretation. Figure 4.2 shows that the total adjustment is mainly driven by the negative CVA contribution of approximately -35 basis points and the FVA contribution of approximately -50 basis points, while the DVA component remains negligible.

The bought European call held by Air France–KLM therefore remains largely unilateral from an exposure perspective. Since the contract rarely enters negative exposure regions, the interaction between CVA, DVA and funding effects remains limited, which explains why the Burgard–Kjaer adjustment stays extremely close to the simplified additive XVA adjustment.

4.2.2 Bilateral exposure derivative: Equity Forward contract

Metric	Value
Risk-free value	0.000000
BK value	-0.181375
Risk-free + XVA value	-0.046675
BK adjustment (bps)	-17.927777
XVA adjustment (bps)	-4.613500
Difference XVA – BK (bps)	13.314277
BK vs XVA difference (%)	288.593835

Table 4.2: XVA vs Burgard–Kjaer comparison for the equity forward contract

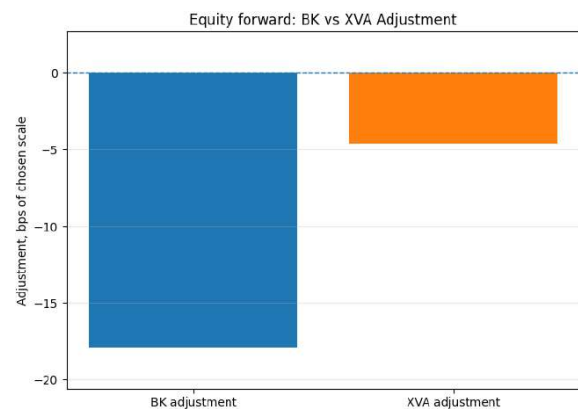


Figure 4.3: Comparison between Burgard–Kjaer and simplified XVA adjustments for the equity forward contract.

The equity forward produces materially different results from the European call because the contract generates bilateral exposure. Unlike the bought call option, the forward can become both an asset and a liability for Air France–KLM depending on the evolution of the underlying stock price. As a result, the DVA component becomes significant and partially offsets the negative effects of CVA and FVA.

Figure 4.3 shows that the simplified XVA adjustment reaches only approximately -4.61 basis points, whereas the Burgard–Kjaer framework produces a significantly larger adjustment of approximately -17.93 basis points. The difference between the two approaches therefore becomes economically significant, with the Burgard–Kjaer adjustment exceeding the simplified XVA adjustment by approximately 289%.

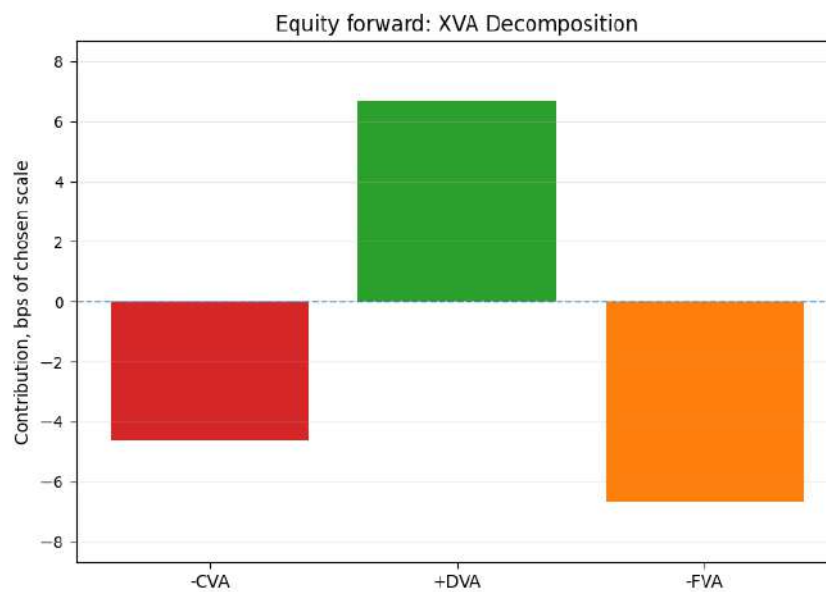


Figure 4.4: Simplified XVA decomposition for the equity forward contract.

The simplified XVA decomposition confirms this bilateral behaviour. Figure 4.4 shows that the negative CVA contribution of approximately -4.6 basis points and the negative FVA contribution of approximately -6.7 basis points are partially offset by a positive DVA contribution of approximately $+6.7$ basis points. As a result, the total additive XVA adjustment remains relatively limited.

Unlike the unilateral European call, the bilateral structure of the equity forward creates significant interactions between positive and negative exposure regions. These interactions are captured directly within the Burgard–Kjaer PDE but only approximated in the additive XVA decomposition, which explains the much larger divergence between the two approaches.

4.2.3 Bilateral exposure derivative: Interest Rate Forward contract

Metric	Value
Risk-free value	-0.014371
BK value	-20.956101
Risk-free + XVA value	-5.427860
BK adjustment (bps)	-0.209417
XVA adjustment (bps)	-0.054135
Difference XVA – BK (bps)	0.155282
BK vs XVA difference (%)	286.843522

Table 4.3: XVA vs Burgard–Kjaer comparison for the interest rate forward contract

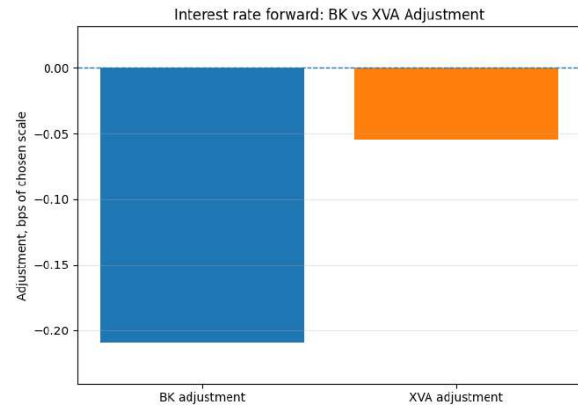


Figure 4.5: Comparison between Burgard–Kjaer and simplified XVA adjustments for the interest rate forward contract.

The interest rate forward leads to conclusions similar to those obtained for the equity forward. Since the contract generates bilateral exposure, the positive DVA component partially offsets the negative CVA and FVA contributions.

Figure 4.5 again shows that the simplified XVA framework materially underestimates the Burgard–Kjaer adjustment. The additive XVA adjustment reaches only approximately -0.054 basis points of notional, whereas the Burgard–Kjaer framework produces an adjustment of approximately -0.209 basis points. The BK adjustment therefore exceeds the simplified XVA adjustment by approximately 287%.

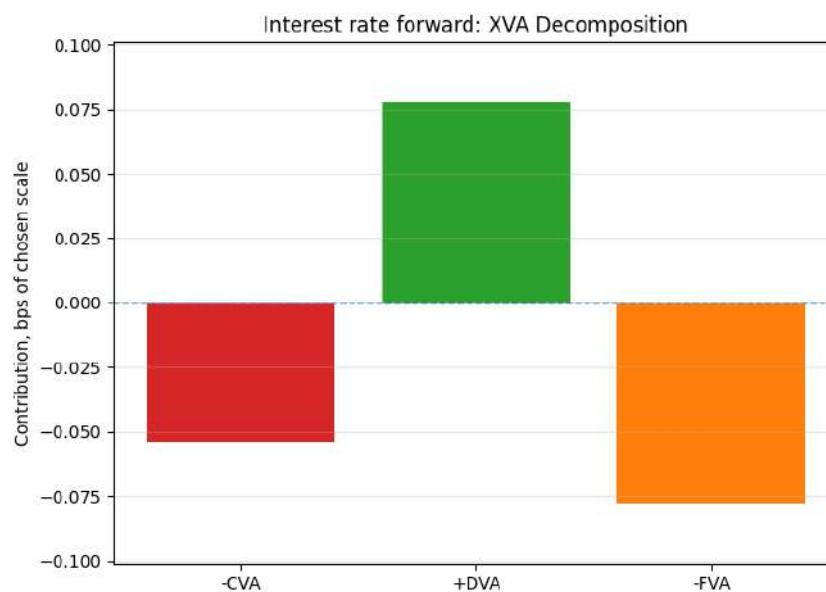


Figure 4.6: Simplified XVA decomposition for the interest rate forward contract.

The decomposition therefore confirms the results obtained for the equity forward. Figure 4.6 shows that the negative CVA contribution of approximately -0.054 basis points and the negative FVA contribution of approximately -0.078 basis points are partially offset by a positive DVA contribution of approximately $+0.078$ basis points.

Once both positive and negative exposure regions coexist, the additive XVA approximation no longer fully captures the valuation effects embedded directly in the nonlinear Burgard–Kjaer PDE.

4.3 Frameworks Comparison Conclusion

The comparison between the Burgard–Kjaer framework and the simplified additive XVA decomposition highlights the importance of exposure bilaterality in derivative valuation.

For the unilateral European call option, both approaches produce almost identical results. Since the exposure remains mostly positive for Air France–KLM throughout the life of the trade, the adjustment is mainly driven by counterparty credit risk and funding costs, while the DVA component remains negligible. As a consequence, the simplified additive XVA framework provides a very close approximation of the Burgard–Kjaer valuation.

The conclusions become materially different for bilateral derivatives. Both the equity forward and the interest rate forward generate positive and negative exposure regions, introducing a significant DVA component that partially offsets the negative CVA and FVA contributions. However, despite this compensation effect, the simplified XVA decomposition systematically underestimates the Burgard–Kjaer adjustment.

For the equity forward, the Burgard–Kjaer adjustment exceeds the simplified XVA adjustment by approximately 289%. A similar result is obtained for the interest rate forward, where the difference reaches approximately 287%. These results suggest that once bilateral exposure appears, the interaction between funding costs, counterparty risk and own default risk becomes more complex than a simple additive decomposition.

The graphical decompositions confirm this interpretation. While the additive XVA framework computes CVA, DVA and FVA separately from simulated exposure profiles, the Burgard–Kjaer framework embeds these effects directly within the pricing PDE through the replication dynamics. The resulting nonlinear interactions remain limited for strongly unilateral products but become materially significant for bilateral derivatives.

The results therefore support the intuition suggested by Burgard and Kjaer that funding costs and bilateral counterparty risk cannot always be fully captured through a simplified additive decomposition. In practice, the additive XVA approximation may remain acceptable for vanilla unilateral products, but becomes less reliable for OTC derivatives with substantial two-sided exposure profiles such as forward contracts.

Chapter 5

Conclusion

This thesis investigated the impact of bilateral counterparty risk and funding costs on derivative valuation within the Burgard–Kjaer framework, and compared these results with more classical additive XVA approaches.

The first objective of the thesis was to study the numerical implementation of the Burgard–Kjaer framework. Both Crank–Nicolson finite differences and Monte Carlo simulations produced globally consistent results for prices and Greeks. However, the Crank–Nicolson implementation proved substantially more stable for second-order sensitivities such as Gamma, while also requiring significantly lower computational effort. The numerical differences between both methods remained very limited, with Burgard–Kjaer prices differing by less than 0.06% between the two implementations for the European call option.

The second objective was to analyse the impact of bilaterality on derivative valuation. The results showed that the unilateral European call option generated a Burgard–Kjaer adjustment of approximately -84.8 basis points relative to the option premium. In contrast, the bilateral derivatives produced substantially smaller relative adjustments, approximately -17.9 basis points for the equity forward and -0.21 basis points for the interest rate forward. This reduction is explained by the appearance of negative exposure regions, which activate the DVA component and partially offset the negative impact of CVA and funding costs.

The final objective was to compare the Burgard–Kjaer framework with a simplified additive XVA decomposition. For the unilateral European call option, both approaches produced almost identical results, with a difference of only -0.39 basis points between the Burgard–Kjaer and XVA adjustments. However, the conclusions became materially different for bilateral derivatives. For the equity forward, the Burgard–Kjaer adjustment exceeded the additive XVA adjustment by approximately 289%, while for the interest rate forward the difference reached approximately 287%.

These results suggest that additive XVA decompositions may remain acceptable approximations for strongly unilateral derivatives, where exposure remains almost entirely

one-sided. However, once bilateral exposure becomes significant, the interaction between positive and negative exposure regions creates nonlinear valuation effects that are directly embedded in the Burgard–Kjaer PDE but only approximated in the additive XVA framework. Under the retained calibration, this led to differences approaching a factor of three between the Burgard–Kjaer and additive XVA adjustments for both bilateral forward contracts.

Chapter 6

Discussion

Despite these results, several limitations remain. The analysis was conducted under simplified assumptions regarding default intensities, funding spreads and collateralization, while the products studied remained relatively vanilla in nature. Future research could therefore extend the framework to stochastic default intensities, collateralized environments, wrong-way risk and larger netted derivative portfolios in order to better reflect modern OTC market practices.

Additional limitations also arise from the numerical implementation itself. The Crank–Nicolson framework, while stable and computationally efficient, remains sensitive to discretization choices in both time and space dimensions, particularly in the nonlinear Burgard–Kjaer setting. Similarly, Monte Carlo simulations may require a very large number of paths to obtain sufficiently stable estimates for exposure profiles and valuation adjustments, especially for long-maturity derivatives.

Another important limitation concerns the modelling assumptions inherited from the Black–Scholes framework. The volatility of the underlying assets was assumed constant and several market frictions, such as liquidity effects, were not explicitly incorporated into the valuation process. In practice, volatility surfaces, jump dynamics and liquidity constraints can materially affect both derivative values and exposure profiles.

Furthermore, the analysis focused primarily on valuation adjustments rather than on dynamic hedging performance. While the Burgard–Kjaer framework is derived through replication arguments, the practical implementation of such hedging strategies may become significantly more complex in real market conditions due to transaction costs, discrete hedging frequencies and imperfect market liquidity.

The treatment of recovery rates also remained simplified throughout the thesis. Recovery assumptions were considered deterministic and constant, whereas in practice recovery values may vary across counterparties, products and market conditions. This may introduce additional uncertainty into valuation adjustments.

Future extensions could therefore include stochastic volatility models, jump-diffusion dynamics and liquidity-adjusted pricing frameworks in order to better capture realistic

market behaviour. Another relevant extension would be the incorporation of dynamic collateral agreements with rehypothecation effects and margin period of risk considerations.

From a numerical perspective, future research could also investigate alternative methodologies such as adaptive finite-difference schemes or regression-based Monte Carlo approaches for large multidimensional portfolios. Such methods may become particularly relevant when extending the framework to highly complex OTC portfolios involving multiple correlated market factors.

Finally, a more extensive empirical comparison between Burgard–Kjaer pricing and the valuation methodologies currently used by financial institutions could provide further insight into the practical significance of the differences observed throughout this thesis.

Chapter 7

Research Process, Challenges and Professional Development

The initial objective of this thesis was primarily focused on implementing the Burgard-Kjaer framework numerically in order to study the impact of bilateral counterparty risk and funding costs on derivative valuation. However, as the research progressed, the scope of the work progressively evolved toward a broader comparison between Burgard-Kjaer pricing and more modern XVA methodologies. The distinction between unilateral and bilateral derivatives also became increasingly central to the analysis, as it appeared that the structure of the exposure profile itself strongly influenced the magnitude and interpretation of valuation adjustments. This evolution of the research question helped position the thesis more clearly within modern derivative pricing and risk management practices.

One of the main challenges encountered during the project concerned the numerical implementation of the Burgard-Kjaer framework. Depending on the close-out convention considered, the pricing partial differential equation can become non-linear, which required careful discretisation choices and stability analysis. Several iterations of the Crank-Nicolson implementation were necessary in order to obtain stable and economically coherent results, particularly when analysing products capable of generating both positive and negative mark-to-market exposures. Additional work was also required to ensure consistency between finite difference results and Monte Carlo simulations.

Another important difficulty concerned the interpretation of the valuation adjustments themselves. While modern XVA approaches often decompose valuation adjustments into additive components such as CVA, DVA and FVA, the Burgard-Kjaer framework integrates funding and bilateral credit effects directly into the replication argument and the pricing equation. As a result, direct comparisons between both frameworks were not always straightforward and required additional analysis regarding close-out assumptions, funding conventions and exposure structures. This progressively shifted the thesis from a

purely numerical implementation exercise toward a more critical analysis of the conceptual differences between pricing frameworks.

The project also highlighted the trade-off between analytical tractability and market realism. In order to maintain a manageable scope, several simplifying assumptions were retained throughout the analysis, including deterministic hazard rates, simplified funding spreads and relatively vanilla derivative products. Although these assumptions allowed for clearer interpretation of the results and more stable numerical implementations, they also illustrated the limitations of simplified academic pricing frameworks when compared to real OTC derivative environments. This naturally led to discussions regarding possible extensions such as stochastic hazard rates, collateral modelling, wrong-way risk and larger netted portfolios.

From both an academic and professional perspective, this thesis allowed me to explore in depth the evolution of OTC derivatives, from their historical development to the modern regulatory framework that now surrounds derivative markets. This helped create an important bridge between the theoretical derivative pricing concepts studied during my master's degree, particularly in the derivative pricing course of Professor Frédéric Vrin, and the practical reality of OTC derivatives in professional financial markets. Understanding this broader regulatory and market context was essential to better understand how derivatives are effectively priced, managed and traded in practice.

The thesis also allowed me to significantly strengthen my quantitative and programming skills through the implementation of numerical pricing methods, PDE solvers and XVA-related valuation frameworks. In addition, the project required developing project management and analytical skills by combining financial theory, numerical methods and economic interpretation within a coherent research framework.

Finally, the thesis gave me the opportunity to exchange with market professionals and practitioners, whose practical experience and feedback helped refine the analysis and better connect the theoretical framework to real market practices and risk management considerations.

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Appendix A

Literature Review

A.1 Black–Scholes–Merton Framework

A.1.1 Framework and Assumptions

The Black–Scholes–Merton (BSM) framework constitutes the classical benchmark for modern derivative pricing and serves as the theoretical foundation of the Burgard–Kjaer extension developed later in this thesis. The original model was introduced by Black and Scholes (1973) and further generalized by Merton (1973), who extended the framework to assets paying continuous dividends and provided a broader continuous-time theory of rational option pricing (*Black and Scholes 1973; Merton 1973*).

The framework assumes that the price process of the underlying asset follows a geometric Brownian motion under continuous trading:

$$dS_t = (\mu - q)S_t dt + \sigma S_t dW_t \tag{A.1}$$

where S_t denotes the asset price at time t , μ the expected return of the asset, q the continuous dividend yield, σ the constant volatility, and W_t a standard Brownian motion.

The dynamics imply that relative price variations are normally distributed and that the asset price remains strictly positive. Within the BSM framework, uncertainty originates exclusively from the diffusion risk associated with the underlying asset price process.

The model relies on the construction of a dynamic self-financing replication strategy. By continuously adjusting the quantity of underlying assets held in the hedge portfolio, it becomes possible to eliminate the local exposure to market risk. Under the no-arbitrage principle, any portfolio that is instantaneously riskless must therefore earn the risk-free rate. This replication argument is the central mechanism allowing the derivation of the pricing equation.

The framework further relies on several simplifying assumptions. Markets are assumed to be frictionless, implying the absence of transaction costs, taxes, liquidity constraints,

or bid-ask spreads. Trading is continuous, borrowing and lending take place at the same constant risk-free rate r , and both volatility σ and interest rate remain constant through time. The model also assumes perfect divisibility of assets, unrestricted short selling, and the absence of counterparty or funding risk (*Black and Scholes 1973; Merton 1973; Vries 2025*).

These assumptions considerably simplify the pricing problem and allow the derivation of closed-form valuation formulas for European-style derivatives. However, many of these assumptions become unrealistic in modern OTC derivatives markets, particularly after the 2007–2008 financial crisis, where funding costs, collateralization, liquidity constraints, and counterparty default risk became central pricing considerations.

The complete derivation of the Black–Scholes pricing equation is provided in Appendix A.

A.1.2 Black–Scholes Partial Differential Equation

For notational simplicity, the explicit time dependence will be omitted in the following derivations. In particular, when no ambiguity arises, the notations

$$S \equiv S_t, \quad W \equiv W_t, \quad V \equiv V_t = v(t, S_t)$$

will be used. Consider a derivative whose value is denoted by $V(S, t)$, where S represents the price of the underlying asset and t the current time. Applying Itô's lemma to the derivative value yields:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} (dS)^2 \quad (\text{A.2})$$

Substituting the geometric Brownian motion dynamics of the underlying asset gives:

$$dV = \left(\frac{\partial V}{\partial t} + (\mu - q)S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW \quad (\text{A.3})$$

A self-financing hedging portfolio is then constructed using:

- one derivative position worth V ,
- a short position of Δ units of the underlying asset.

The portfolio value is therefore:

$$\Pi = V - \Delta S \quad (\text{A.4})$$

Its differential becomes:

$$d\Pi = dV - \Delta dS \quad (\text{A.5})$$

Choosing the hedge ratio as:

$$\Delta = \frac{\partial V}{\partial S} \quad (\text{A.6})$$

eliminates the stochastic component proportional to dW_t . The portfolio becomes locally riskless over an infinitesimal time interval. Under the no-arbitrage condition, the return of this riskless portfolio must equal the risk-free rate:

$$d\Pi = r\Pi dt \quad (\text{A.7})$$

After simplification, this leads to the Black–Scholes partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S \frac{\partial V}{\partial S} - rV = 0 \quad (\text{A.8})$$

The equation must be solved subject to a terminal payoff condition determined by the derivative contract. For a European call option with strike K and maturity T , the terminal condition is:

$$V(S, T) = \max(S_T - K, 0) \quad (\text{A.9})$$

Solving Equation (A.8) together with the payoff condition yields the well-known Black–Scholes closed-form formula for European options (*Black and Scholes 1973; Merton 1973; Vrins 2025*).

The importance of the Black–Scholes PDE extends beyond the analytical formula itself. The equation establishes the fundamental principle that derivative pricing can be formulated as a replication problem under no-arbitrage conditions. This replication-based reasoning remains the starting point of most modern derivative pricing frameworks, including the Burgard–Kjaer model studied in this thesis.

A.1.3 Limits in OTC Derivatives Markets

Although the Black–Scholes framework remains one of the foundations of modern quantitative finance, its assumptions become restrictive in real OTC derivatives markets.

One major limitation is the assumption that market participants can continuously hedge without friction. In practice, trading involves transaction costs, liquidity constraints, bid-ask spreads, and discrete rebalancing frequencies. During periods of market stress, continuous replication may become impossible or prohibitively expensive.

More importantly, the original BSM framework assumes that all funding takes place at a unique risk-free rate and that default risk does not exist. These assumptions became increasingly unrealistic after the global financial crisis of 2007–2008. OTC derivative transactions expose financial institutions not only to market risk, but also to counterparty default risk, own default risk, collateral costs, and funding constraints.

As a consequence, the classical replication argument underlying the Black–Scholes PDE is no longer sufficient to describe the true economic cost of maintaining a derivative position. The funding strategy used to hedge the position itself becomes part of the pricing problem.

In practice, derivative desks now incorporate several valuation adjustments collectively referred to as XVA adjustments, including:

- Credit Valuation Adjustment (CVA),
- Debit Valuation Adjustment (DVA),
- Funding Valuation Adjustment (FVA),
- Capital Valuation Adjustment (KVA),
- and collateral-related adjustments (CollVA).

These effects introduce nonlinearities into the pricing framework because funding costs and default exposures may depend on the sign and value of the derivative position itself. As a result, the pricing equation can no longer always be reduced to the linear Black–Scholes PDE.

The Burgard–Kjaer framework extends the Black–Scholes replication approach precisely to account for these effects. By incorporating funding spreads, counterparty credit risk, and close-out conventions directly into the replication strategy, the BK framework provides a more realistic representation of OTC derivative pricing under modern market conditions (*Burgard and Kjaer 2011*).

A.2 Burgard–Kjaer Framework

A.2.1 Assumptions

The Black–Scholes framework presented in the previous section relies on strong simplifying assumptions: markets are frictionless, there is no credit risk, and funding takes place at the risk-free rate.

In practice, these assumptions are not realistic. Financial institutions are exposed to counterparty default risk, face their own default risk, and incur funding costs (through the funding spread) when hedging derivative positions. The Burgard–Kjaer (BK) framework extends the Black–Scholes model by incorporating these additional sources of risk and cost in a consistent no-arbitrage setting.

The objective of this section is to derive the Burgard–Kjaer pricing equation and to understand how it modifies the classical Black–Scholes partial differential equation.

A.2.2 Model Parameters

The Black–Scholes framework prices a derivative by constructing a self-financing strategy that eliminates the only source of uncertainty, namely the diffusion risk of the underlying asset. Once this market risk is hedged away, the remaining portfolio is locally riskless and must therefore earn the risk-free rate.

The Burgard–Kjaer framework keeps the same replication logic, but extends the market to account for features that are absent from Black–Scholes and are essential in OTC derivatives markets:

- the counterparty may default,
- the bank itself may default,
- borrowing and lending are not necessarily performed at the same rate,
- the derivative is terminated at default and replaced by a close-out value.

As a result, the value of the derivative is no longer driven only by the stock diffusion. It may also jump at the default time of either party, and the hedging strategy must explicitly incorporate both default hedges and funding cash flows.

A.2.3 Deriving the Burgard–Kjaer Partial Differential Equation

Following Burgard and Kjaer, we simplify the notation by omitting explicit time dependence whenever no ambiguity arises. In particular, we write

$$S \equiv S_t, \quad W \equiv W_t, \quad \hat{V} \equiv \hat{V}(t, S_t), \quad V \equiv V(t, S_t).$$

This convention is consistent with the notation used in the original BK paper and improves readability of the derivation (*Burgard and Kjaer 2011*). However, partial derivatives are still kept explicitly. For example, $\hat{V}_t$, $\hat{V}_S$, and $\hat{V}_{SS}$ denote the derivatives of the deterministic pricing function with respect to time and the underlying stock price.

Step 1: Market setup

Consider a derivative with risky value $\hat{V}$, written on an underlying asset S . Before any default, the stock follows

$$dS = \mu S dt + \sigma S dW,$$

where μ is the stock drift, σ is the stock volatility, and W is a Brownian motion.

In the BK framework, the stock-related terms of the pricing equation are later grouped into the operator

$$\mathcal{A}\hat{V} = \frac{1}{2}\sigma^2 S^2 \hat{V}_{SS} + (q_S - \gamma_S)S\hat{V}_S.$$

The first term,

$$\frac{1}{2}\sigma^2 S^2 \hat{V}_{SS},$$

captures the effect of stock price uncertainty and the curvature of the derivative value with respect to the stock price.

The second term,

$$(q_S - \gamma_S)S\hat{V}_S,$$

captures the net carry cost of the stock hedge. The rate q_S is the financing rate of the stock position, typically linked to the repo rate of the underlying, while γ_S is the dividend yield received on the stock. Therefore, $q_S - \gamma_S$ represents the cost of financing the stock hedge after deducting dividend income.

The rate q_S should not be confused with the bank's unsecured funding rate r_F . The former applies specifically to the stock hedge, whereas the latter applies to residual cash funding needs in the replication strategy.

Two default times are introduced:

$$\tau_B \quad (\text{bank default}), \quad \tau_C \quad (\text{counterparty default}).$$

Let J_B and J_C denote the corresponding default indicator jump processes. The derivative is terminated at the first default event and replaced by a close-out value.

The market also contains two risky zero-recovery bonds:

- P_B , issued by the bank,
- P_C , issued by the counterparty.

Their dynamics are

$$dP_B = r_B P_B dt - P_B dJ_B, \quad dP_C = r_C P_C dt - P_C dJ_C.$$

Here, r_B is the yield of the bank's risky bond, r_C is the yield of the counterparty's risky bond, and r is the risk-free rate. The jump terms dJ_B and dJ_C represent default events. Since the bonds are assumed to have zero recovery, the corresponding bond price jumps to zero at default.

The associated credit spreads are defined as

$$\lambda_B = r_B - r, \quad \lambda_C = r_C - r.$$

Thus, λ_B measures the bank's credit spread over the risk-free rate, while λ_C measures the counterparty's credit spread over the risk-free rate. In this simplified zero-recovery bond setup, these spreads can also be interpreted as default intensities. With positive recovery, the credit spread would no longer be equal to the default intensity.

Step 2: Close-out at default

A key element of the BK framework is the close-out rule. It defines what happens to the derivative value when either the bank or the counterparty defaults before maturity.

Let M denote the mark-to-market value used as reference for the close-out payment. This value is measured from the bank's point of view and can be positive or negative:

$$M^+ = \max(M, 0), \quad M^- = \min(M, 0).$$

A positive value M^+ means that the derivative is an asset for the bank: the counterparty owes money to the bank. A negative value M^- means that the derivative is a liability for the bank: the bank owes money to the counterparty.

If the bank defaults first, the post-default value of the derivative is

$$\hat{V}(t, S, 1, 0) = M^+ + R_B M^-.$$

This means that if the derivative is an asset for the bank, the bank still receives the full positive amount M^+ . If the derivative is a liability for the bank, the counterparty only recovers a fraction R_B of what the bank owes. Since $M^- < 0$, the term $R_B M^-$ represents the reduced payment made by the defaulting bank.

If the counterparty defaults first, the post-default value is

$$\hat{V}(t, S, 0, 1) = R_C M^+ + M^-.$$

In this case, if the derivative is an asset for the bank, the bank only recovers a fraction R_C of the positive exposure M^+ . If the derivative is a liability for the bank, the bank still pays the full negative amount M^- .

Burgard and Kjaer study two possible definitions of the close-out reference value M :

- **Risky close-out:** $M = \hat{V}$. The close-out is based on the risky value of the derivative itself.
- **Counterparty-riskless close-out:** $M = V$. The close-out is based on the corresponding default-free value.

This distinction is central. If $M = \hat{V}$, the close-out terms depend on the unknown risky value being solved for, which makes the pricing equation nonlinear. If $M = V$, the close-out terms depend on an external default-free value, which makes the equation linear in $\hat{V}$.

Step 3: Dynamics of the derivative

Before default, the risky derivative value $\hat{V}$ is assumed to be smooth in (t, S) . Applying Itô's Lemma with jumps gives

$$d\hat{V} = \left(\hat{V}_t + \frac{1}{2} \sigma^2 S^2 \hat{V}_{SS} + \mu S \hat{V}_S \right) dt + \sigma S \hat{V}_S dW + \Delta \hat{V}_B dJ_B + \Delta \hat{V}_C dJ_C,$$

where

$$\Delta \hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0),$$

and

$$\Delta \hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0).$$

The terms $\Delta \hat{V}_B$ and $\Delta \hat{V}_C$ represent the jumps in the derivative value if the bank or the counterparty defaults. Also note that, despite the simplified notation introduced above, $\hat{V}_t$ denotes the partial derivative of the risky value with respect to time. It should not be confused with the time-indexed process $\hat{V}(t, S_t)$, for which we write simply $\hat{V}$ in this section.

At this stage, the drift term still contains μ , the stock drift. This is only the raw Itô dynamics of the derivative. In the replication step, the stock diffusion risk is hedged away. Once the stock hedge and its cash flows are included, the drift μ disappears and the stock contribution is represented by the BK operator $\mathcal{A}\hat{V}$.

Step 4: Hedging instruments and replicating portfolio

The derivative is replicated through a self-financing portfolio composed of:

- δ units of stock S ,
- α_B units of the bank bond P_B ,
- α_C units of the counterparty bond P_C ,
- a cash account.

The portfolio value is written as

$$\Pi = \delta S + \alpha_B P_B + \alpha_C P_C + \beta.$$

The replication condition requires the derivative and the hedging portfolio to offset each other:

$$\hat{V} + \Pi = 0,$$

or equivalently,

$$-\hat{V} = \delta S + \alpha_B P_B + \alpha_C P_C + \beta.$$

This extends the standard Black–Scholes replication strategy by adding risky bonds to hedge default jumps.

Step 5: Funding cash flows

The treatment of funding is one of the main differences relative to Black–Scholes.

Positive cash balances are invested at the risk-free rate r , while negative cash balances are funded at the borrowing rate

$$r_F = r + s_F,$$

where s_F is the funding spread.

After accounting for the position in the bank bond, the residual funding balance is

$$-\hat{V} - \alpha_B P_B.$$

Its growth is

$$d\bar{\beta}_F = \left[r(-\hat{V} - \alpha_B P_B)^+ + r_F(-\hat{V} - \alpha_B P_B)^- \right] dt.$$

Equivalently,

$$d\bar{\beta}_F = r(-\hat{V} - \alpha_B P_B) dt + s_F(-\hat{V} - \alpha_B P_B)^- dt.$$

The second term is the additional cost generated when the replication strategy requires borrowing at the funding rate rather than investing at the risk-free rate.

Step 6: Elimination of diffusion and jump risks

The derivative and the hedging portfolio must offset each other.

Matching the diffusion terms gives the delta hedge

$$\delta = -\hat{V}_S.$$

The minus sign follows from the convention

$$\hat{V} + \Pi = 0.$$

The jump risks are eliminated by choosing the bond positions such that the jumps in the derivative are exactly offset by the jumps in the risky bonds:

$$\alpha_B = \frac{\Delta \hat{V}_B}{P_B}, \quad \alpha_C = \frac{\Delta \hat{V}_C}{P_C}.$$

Once these positions are chosen, both diffusion risk and jump-to-default risk are removed. The remaining terms are drift, credit, and funding contributions.

Resulting General Burgard–Kjaer Pricing Equation

Substituting the hedge ratios into the self-financing condition and collecting the drift terms yields the general Burgard–Kjaer pricing equation:

$$\hat{V}_t + \mathcal{A}\hat{V} - r\hat{V} = s_F(\hat{V} + \Delta\hat{V}_B)^+ - \lambda_B\Delta\hat{V}_B - \lambda_C\Delta\hat{V}_C.$$

Using the close-out relations, this can be rewritten as

$$\hat{V}_t + \mathcal{A}\hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-).$$

The left-hand side is the usual Black–Scholes operator applied to the risky value $\hat{V}$. The right-hand side contains the additional effects created by default risk and funding costs.

The term

$$(\lambda_B + \lambda_C)\hat{V}$$

captures the fact that the trade can be terminated by either the bank’s default or the counterparty’s default. It is the gross default-intensity effect.

The term

$$s_F M^+$$

is the funding cost. It applies when the close-out value is positive to the bank. A higher funding spread increases the cost of carrying positive exposure and therefore tends to reduce the derivative value.

The term

$$-\lambda_B(R_B M^- + M^+)$$

comes from the bank’s own default. It mainly affects negative exposure, because when the bank owes money, default reduces the amount effectively paid to the counterparty. This is the DVA-like effect.

The term

$$-\lambda_C(R_C M^+ + M^-)$$

comes from counterparty default. It mainly affects positive exposure, because when the counterparty owes money, the bank only recovers a fraction R_C . This is the CVA-like effect.

Therefore, positive exposure is mainly affected by counterparty risk and funding costs, while negative exposure is mainly affected by the bank’s own default risk. This is why a forward or a swap, whose value can be positive or negative, is more useful than a long call to illustrate bilaterality.

(Burgard and Kjaer 2011)

Nonlinear BK equation: $M = \hat{V}$

If the close-out is based on the risky value itself,

$$M = \hat{V},$$

then the general PDE becomes

$$\hat{V}_t + \mathcal{A}\hat{V} - r\hat{V} = (1 - R_B)\lambda_B\hat{V}^- + (1 - R_C)\lambda_C\hat{V}^+ + s_F\hat{V}^+.$$

This is the nonlinear BK equation. The nonlinearity comes from the positive and negative parts of the unknown value $\hat{V}$. The equation therefore depends on the sign of the solution being computed.

Linear BK equation: $M = V$

If the close-out is based on the counterparty-riskless value,

$$M = V,$$

then the close-out terms depend on the external default-free value V , not on the unknown risky value $\hat{V}$. The PDE becomes

$$\hat{V}_t + \mathcal{A}\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B\lambda_B + \lambda_C)V^- - (\lambda_B + R_C\lambda_C)V^+ + s_FV^+.$$

This is the linear BK equation. The unknown remains $\hat{V}$, but the close-out terms are expressed using V , which makes the equation linear in $\hat{V}$.

A.2.4 Methodology and Numerical Analysis

After deriving the pricing PDE, Burgard and Kjaer study the quantitative impact of bilateral credit risk and funding costs through numerical examples (*Burgard and Kjaer 2011*). Their analysis focuses on a bought European call option and compares the valuation effects generated by different close-out and funding assumptions.

The numerical analysis is structured around two modelling dimensions. The first is the close-out convention: the close-out value is either based on the risky value of the derivative, $M = \hat{V}$, or on the counterparty-riskless value, $M = V$. The second is the funding assumption: the hedge is either funded at the risk-free rate or at a funding rate including the institution's funding spread.

This leads to four cases in total:

- risky close-out, $M = \hat{V}$, with risk-free funding;
- risky close-out, $M = \hat{V}$, with funding spread;

- counterparty-riskless close-out, $M = V$, with risk-free funding;
- counterparty-riskless close-out, $M = V$, with funding spread.

The methodology therefore consists in pricing the same derivative contract under these four specifications while keeping the underlying market dynamics unchanged. The resulting price differences isolate the effects of the close-out convention and of funding frictions on bilateral CVA and derivative valuation.

A.2.5 Main Results

Using this methodology, Burgard and Kjaer show that bilateral CVA cannot be computed consistently without incorporating the funding costs generated by the hedging strategy itself (*Burgard and Kjaer 2011*). Since a bank's funding spread is directly linked to its own credit quality, funding costs and own default risk must therefore be treated jointly within the pricing equation.

Their results show that including realistic funding conditions can materially change derivative valuation compared to a framework assuming risk-free funding. In the numerical example of a bought European call option, the inclusion of funding costs increases the CVA by up to 100% relative to the risk-free funding case (*Burgard and Kjaer 2011*). This demonstrates that funding assumptions are not negligible implementation details, but potentially dominant valuation components.

The authors further argue that these effects are particularly relevant for liquid OTC products such as vanilla options and interest rate swaps. In these markets, pricing margins are often small relative to funding and credit spreads, meaning that ignoring these effects may absorb a substantial part of the economic profitability of a trade (*Burgard and Kjaer 2011*).

Appendix B

Python Code

This appendix presents the main Python implementations used throughout the thesis. The objective is not to reproduce every script in full detail, but rather to provide the core numerical implementations required to reproduce the pricing results, Greeks, and valuation adjustments presented in the main chapters.

The appendix is organised according to the different modelling blocks developed in the thesis:

- analytical Black–Scholes pricing,
- Crank–Nicolson finite-difference implementation,
- nonlinear Burgard–Kjaer PDE solver,
- Monte Carlo simulations,
- Greeks computations,
- bilateral forward extensions,
- Vasicek calibration,
- XVA exposure simulations.

B.1 Libraries and Global Parameters

The following libraries and global parameters are used throughout the numerical implementation. The calibration parameters correspond to the market assumptions presented in Chapter 2.

```
1 import numpy as np
2 import pandas as pd
3 import matplotlib.pyplot as plt
4
5 from scipy.stats import norm
6 from scipy.interpolate import CubicSpline
7
```

```
8 # Market parameters
9 S0 = 101.17
10 K = 110
11 T = 1.0
12 r = 0.0252016
13 sigma = 0.5012
14 q = 0.0
15
16 # Burgard-Kjaer parameters
17 lambda_C = 0.005798
18 lambda_B = 0.008398
19
20 R_B = 0.40
21 R_C = 0.40
22
23 s_F = 0.005039
24
25 # Numerical parameters
26 M = 400
27 N = 1000
28 S_max = 500
29
30 # Monte Carlo
31 n_sim = 1_000_000
32 seed = 42
```

B.2 Analytical Black–Scholes Pricing Functions

The analytical Black–Scholes formula is used as a benchmark to validate the numerical implementations. The closed-form solution is also used for consistency checks against the Crank–Nicolson and Monte Carlo results.

```
1 def bs_call_price(S, K, T, r, sigma, q=0.0):
2
3     d1 = (
4         np.log(S / K)
5         + (r - q + 0.5 * sigma**2) * T
6     ) / (sigma * np.sqrt(T))
7
8     d2 = d1 - sigma * np.sqrt(T)
9
```

```

10     price = (
11         S * np.exp(-q * T) * norm.cdf(d1)
12         - K * np.exp(-r * T) * norm.cdf(d2)
13     )
14
15     return price

```

B.3 Crank–Nicolson Solver for the Black–Scholes PDE

The Black–Scholes PDE is solved numerically using a Crank–Nicolson finite-difference scheme. The stock-price domain is discretised into a finite grid and the PDE is solved backward in time from maturity to inception.

```

1  dS = S_max / M
2  dt = T / N
3
4  S_grid = np.linspace(0, S_max, M + 1)
5
6  V = np.maximum(S_grid - K, 0)
7
8  for n in range(N):
9
10     tau = T - n * dt
11
12     V[0] = 0
13
14     V[-1] = (
15         S_max * np.exp(-q * tau)
16         - K * np.exp(-r * tau)
17     )
18
19     # Tridiagonal system coefficients
20     # Backward induction step

```

B.4 Nonlinear Burgard–Kjaer Crank–Nicolson Solver

The nonlinear Burgard–Kjaer PDE is solved using the same Crank–Nicolson structure, combined with a Picard iteration to handle the sign-dependent effective discounting rate.

The nonlinearity originates from the dependence of the PDE coefficients on the sign of the risky derivative value itself.

```

1  for iteration in range(max_iter):
2
3      V_old = V_guess.copy()
4
5      for j in range(1, M):
6
7          if V_guess[j] >= 0:
8
9              k_eff = (
10                  (1 - R_C) * lambda_C
11                  + s_F
12              )
13
14          else:
15
16              k_eff = -(
17                  (1 - R_B) * lambda_B
18              )
19
20          # Solve tridiagonal system
21
22          error = np.max(np.abs(V_guess - V_old))
23
24          if error < tolerance:
25              break

```

B.5 Greeks Extraction

Once the PDE has been solved, the Greeks are extracted directly from the numerical grid using finite-difference approximations.

```

1  delta = (
2      V[j + 1] - V[j - 1]
3  ) / (2 * dS)
4
5  gamma = (
6      V[j + 1]
7      - 2 * V[j]
8      + V[j - 1]
9  ) / (dS**2)
10

```

```

11 theta = (
12     V_next[j] - V[j]
13 ) / dt

```

Additional Greeks are computed using a bump-and-revalue approach.

```

1 vega = (
2     price_up
3     - price_down
4 ) / (2 * bump_sigma)

```

B.6 Monte Carlo Simulation Framework

As an independent numerical benchmark, the option price is also estimated using Monte Carlo simulation under the risk-neutral measure.

```

1 np.random.seed(seed)
2
3 Z = np.random.normal(size=n_sim)
4
5 ST = S0 * np.exp(
6     (
7         r - q - 0.5 * sigma**2
8     ) * T
9     + sigma * np.sqrt(T) * Z
10 )
11
12 payoff = np.maximum(ST - K, 0)
13
14 price_mc = np.exp(-r * T) * np.mean(payoff)

```

The standard error is computed as

```

1 std_error = (
2     np.std(
3         np.exp(-r * T) * payoff
4     )
5     / np.sqrt(n_sim)
6 )

```

B.7 Monte Carlo Greeks

The Monte Carlo Greeks are computed using a bump-and-revalue methodology with common random numbers in order to reduce simulation noise.

```

1 np.random.seed(42)
2
3 price_up = mc_price(
4     S0 + bump_S
5 )
6
7 np.random.seed(42)
8
9 price_down = mc_price(
10    S0 - bump_S
11 )
12
13 delta_mc = (
14     price_up - price_down
15 ) / (2 * bump_S)

```

B.8 Equity Forward Extension

The Burgard–Kjaer framework is then extended to a bilateral equity forward contract.

The forward payoff is defined as

$$\Phi(S_T) = S_T - K_F.$$

The numerical implementation uses the same nonlinear Crank–Nicolson methodology as for the European call option.

```

1 K_F = S0 * np.exp(
2     (r - q) * T
3 )
4
5 payoff = S_grid - K_F

```

The sign-dependent structure of the Burgard–Kjaer PDE allows the derivative to switch between positive and negative exposure regions depending on the stock price.

B.9 Vasicek Calibration and Interest Rate Simulation

The short-rate dynamics are modelled using the Vasicek process

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r dW_t.$$

The calibration is performed using historical observations of the euro short-term rate (STR).

```

1 for t in range(n_steps - 1):
2
3     Z = np.random.normal()
4
5     r_path[t + 1] = (
6         r_path[t]
7         + kappa * (
8             theta - r_path[t]
9         ) * dt
10        + sigma_r
11        * np.sqrt(dt)
12        * Z
13    )

```

B.10 Interest Rate Forward under Burgard–Kjaer

The Burgard–Kjaer framework is then adapted to a bilateral interest rate forward.

The terminal payoff is defined as

$$\Phi(r_T) = N\Delta(r_T - K).$$

```

1 payoff = notional * delta_t * (
2     r_grid - K
3 )

```

The nonlinear PDE is solved using the same sign-dependent Crank–Nicolson methodology.

B.11 XVA Exposure Simulation

The XVA framework is implemented through simulated exposure profiles generated under the same market dynamics as the Burgard–Kjaer implementation.

Expected Exposure (EE) and Expected Negative Exposure (ENE) are computed as

```

1 EE = np.mean(
2     np.maximum(
3         exposure_paths,
4         0
5     ),

```

```

6     axis=1
7 )
8
9 ENE = np.mean(
10     np.maximum(
11         -exposure_paths,
12         0
13     ),
14     axis=1
15 )

```

The valuation adjustments are then approximated using trapezoidal integration.

```

1 CVA = (
2     (1 - R_C)
3     * lambda_C
4     * np.trapz(
5         np.exp(-r * time_grid) * EE,
6         time_grid
7     )
8 )
9
10 DVA = (
11     (1 - R_B)
12     * lambda_B
13     * np.trapz(
14         np.exp(-r * time_grid) * ENE,
15         time_grid
16     )
17 )
18
19 FVA = (
20     s_F
21     * np.trapz(
22         np.exp(-r * time_grid) * EE,
23         time_grid
24     )
25 )

```

The final XVA-adjusted price is computed as

```

1 price_xva = (
2     price_rf
3     - CVA

```

```
4      + DVA
5      - FVA
6  )
```

Appendix C

Additional Numerical Results and Figures

This appendix gathers supplementary numerical results, sensitivity analyses and graphical illustrations supporting the main developments of the thesis. The figures presented here are intended to complement the discussion contained in the core chapters while preserving the readability of the main text.

C.1 Crank–Nicolson Greeks and Sensitivity Analyses

This section contains the additional Greeks profiles and parameter sensitivity analyses obtained from the Crank–Nicolson implementation of both the Black–Scholes and Burgard–Kjaer frameworks.

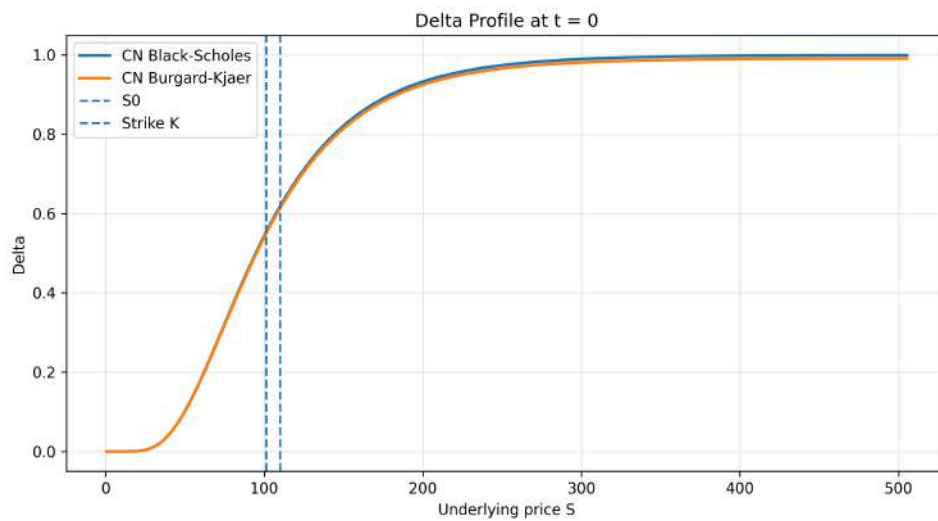


Figure C.1: Delta profile obtained from the Crank–Nicolson implementation.

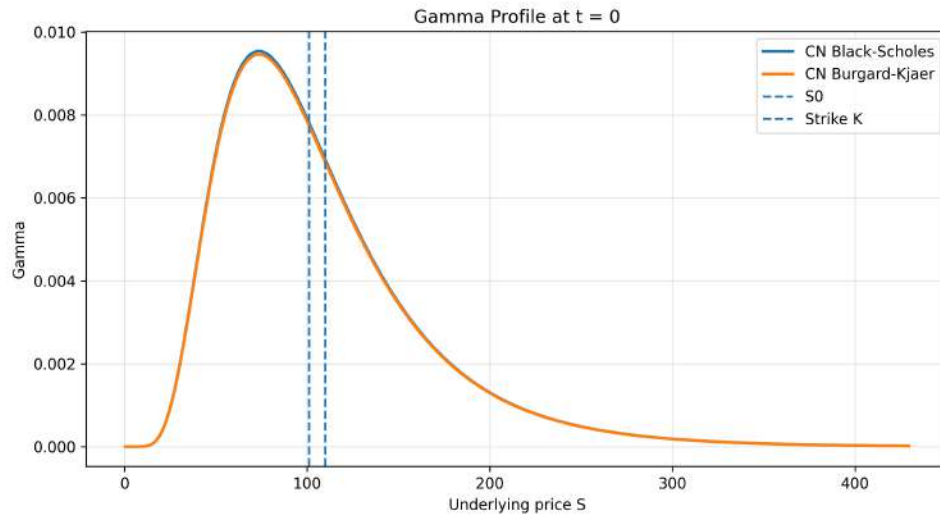


Figure C.2: Gamma profile obtained from the Crank–Nicolson implementation.

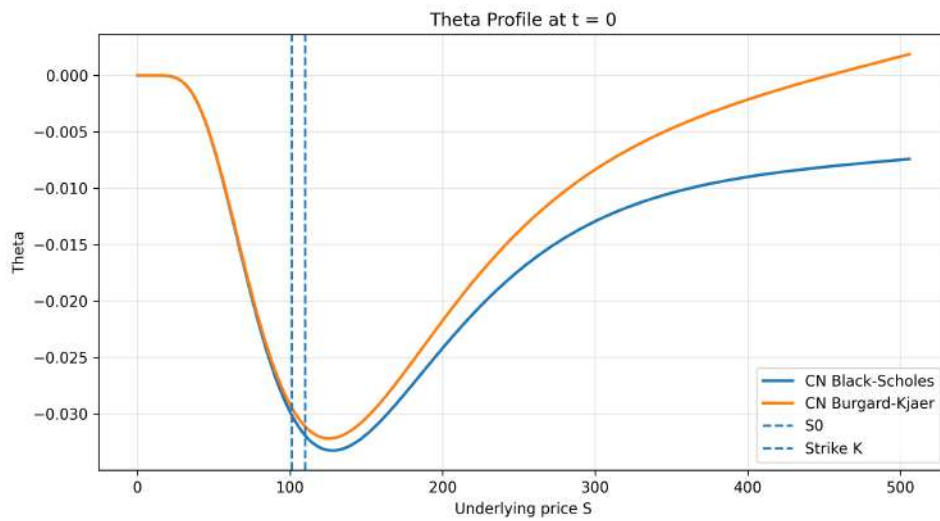


Figure C.3: Theta profile obtained from the Crank–Nicolson implementation.

Table C.1: Local Burgard–Kjaer sensitivities per basis point

Input shocked	BS Sensitivity	BK Total Sensitivity	BK-Specific Sensitivity
r	0.003795	0.005784	0.001989
r_B	—	-0.001027	-0.001027
r_C	—	-0.001027	-0.001027

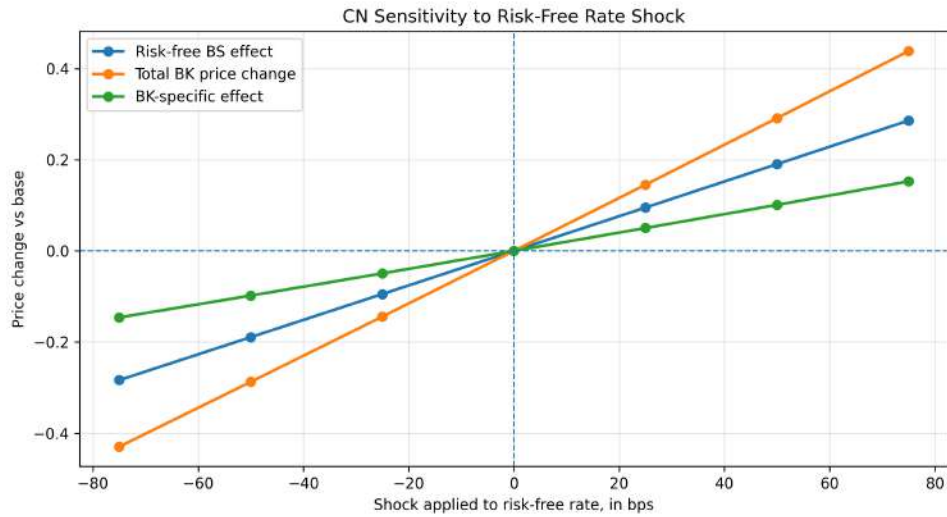


Figure C.4: Sensitivity of the Burgard–Kjaer value with respect to the risk-free rate.

Table C.2: Sensitivity of the Burgard–Kjaer value to risk-free rate shocks

Shock (bps)	CN BS Value	CN BK Value	BK Total Change	BS Change	BK-Specific Effect
-75	17.4272	17.1245	-0.4382	-0.2858	-0.1524
-50	17.5222	17.2696	-0.2931	-0.1908	-0.1023
-25	17.6174	17.4157	-0.1470	-0.0955	-0.0515
0	17.7130	17.5627	0.0000	0.0000	0.0000
25	17.8087	17.7107	0.1480	0.0958	0.0522
50	17.9048	17.8598	0.2970	0.1918	0.1052
75	18.0011	18.0098	0.4470	0.2881	0.1589

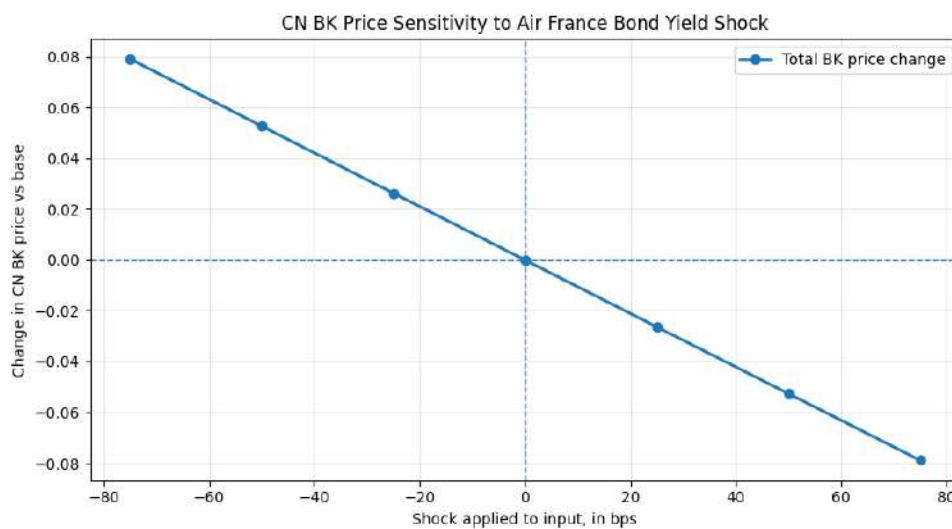
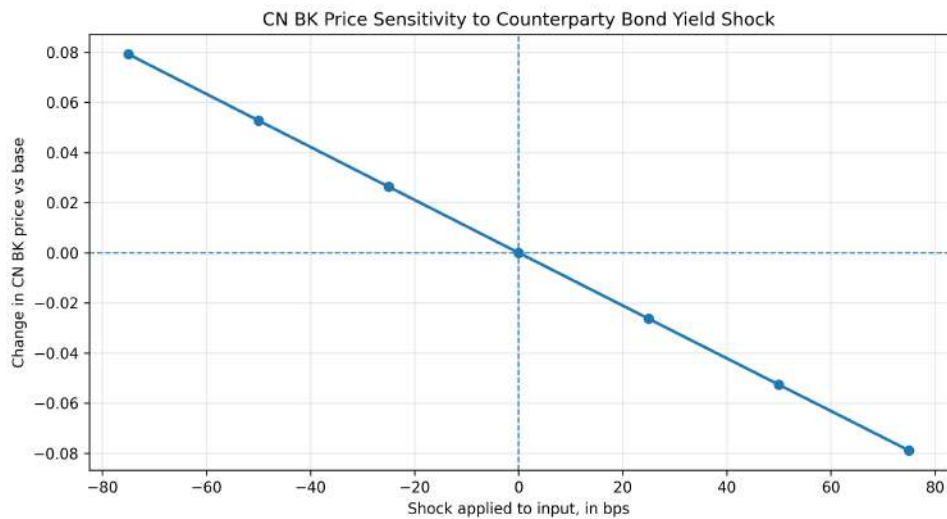


Figure C.5: Sensitivity of the Burgard–Kjaer value with respect to Air France risky yield.

Table C.3: Sensitivity of the Burgard–Kjaer value to own risky yield shocks

Shock (bps)	CN BK Value	BK Total Change
-75	17.6419	0.0792
-50	17.6155	0.0528
-25	17.5891	0.0264
0	17.5627	0.0000
25	17.5364	-0.0263
50	17.5101	-0.0526
75	17.4839	-0.0789

**Figure C.6:** Sensitivity of the Burgard–Kjaer value with respect to the counterparty risky yield.**Table C.4:** Sensitivity of the Burgard–Kjaer value to counterparty risky yield shocks

Shock (bps)	CN BK Value	BK Total Change
-75	17.6419	0.0792
-50	17.6155	0.0528
-25	17.5891	0.0264
0	17.5627	0.0000
25	17.5364	-0.0263
50	17.5101	-0.0526
75	17.4839	-0.0789

C.2 Monte Carlo Simulations and Greeks

This section presents the additional Monte Carlo simulations and finite-difference Greeks approximations used to validate the Crank–Nicolson implementation.

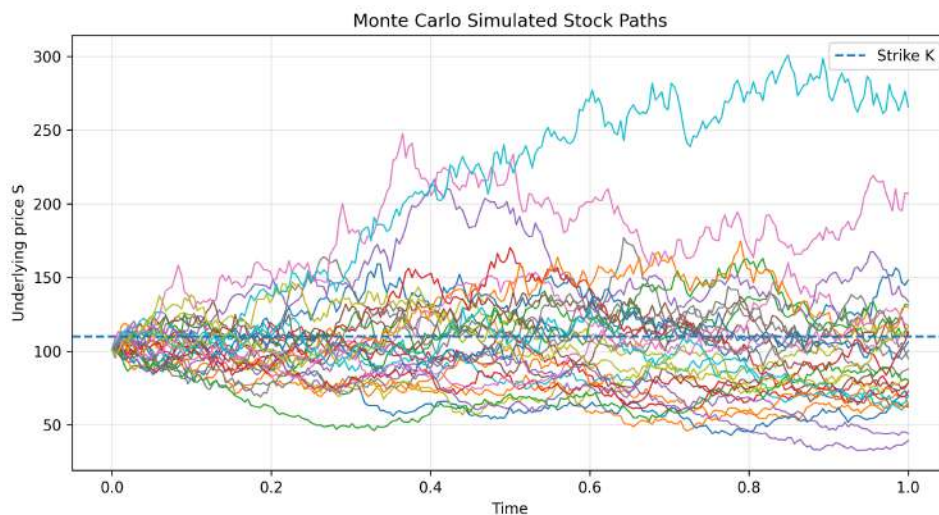


Figure C.7: Simulated Black–Scholes paths used in the Monte Carlo implementation.

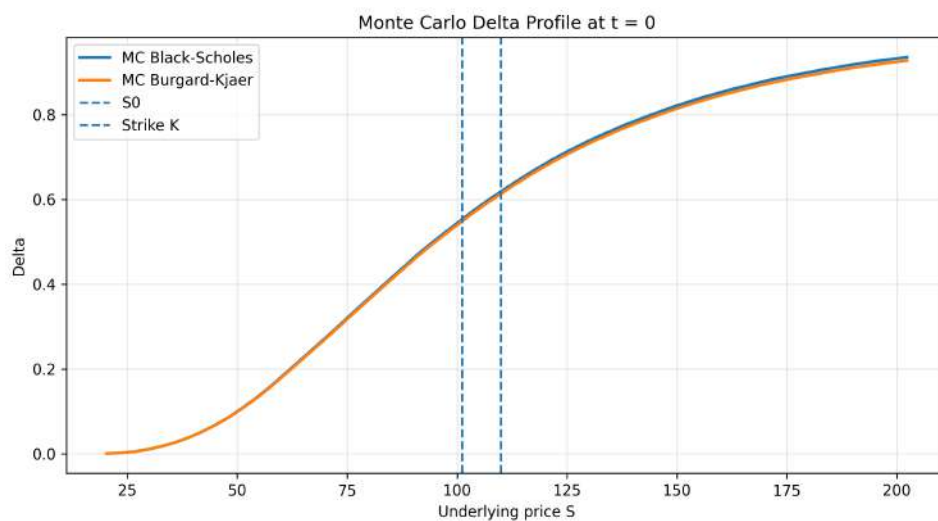


Figure C.8: Monte Carlo delta profile.

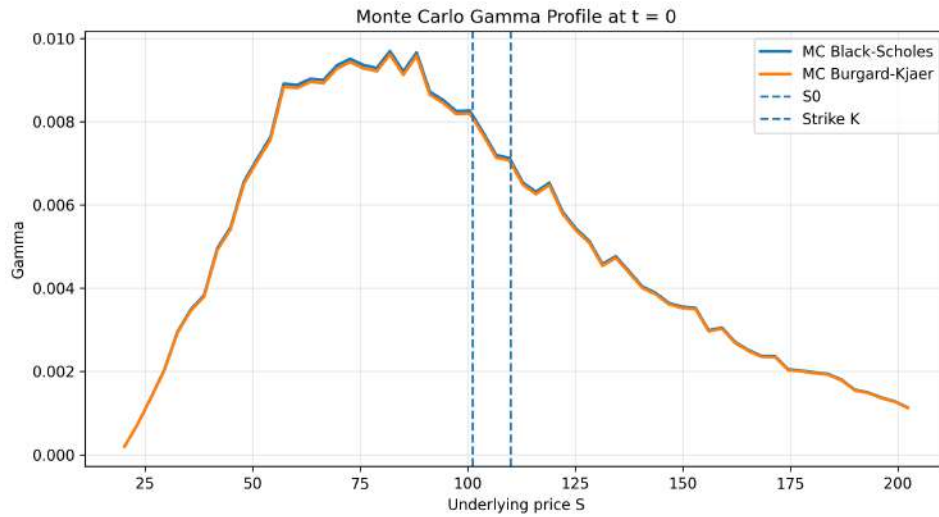


Figure C.9: Monte Carlo gamma profile.

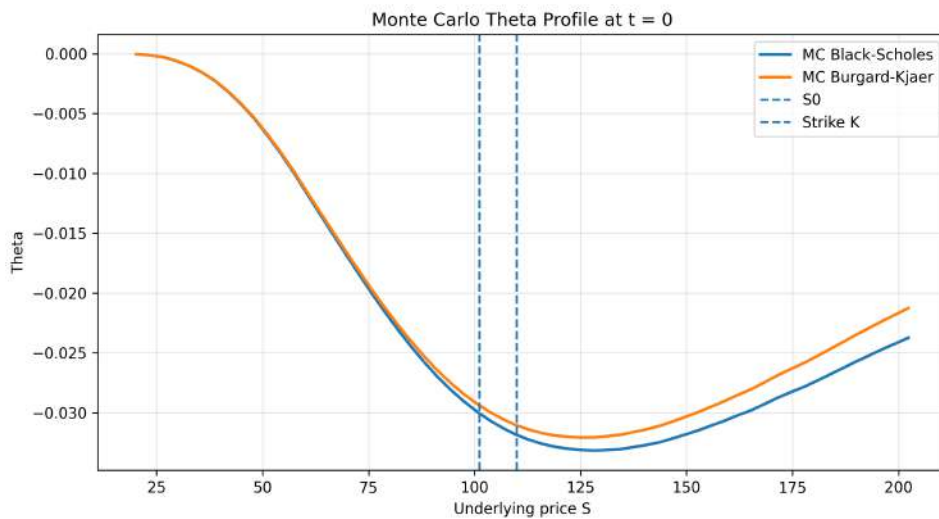


Figure C.10: Monte Carlo theta profile.

C.3 Equity Forward Extension

This section gathers the additional figures related to the extension of the Burgard–Kjaer framework to the bilateral equity forward contract.

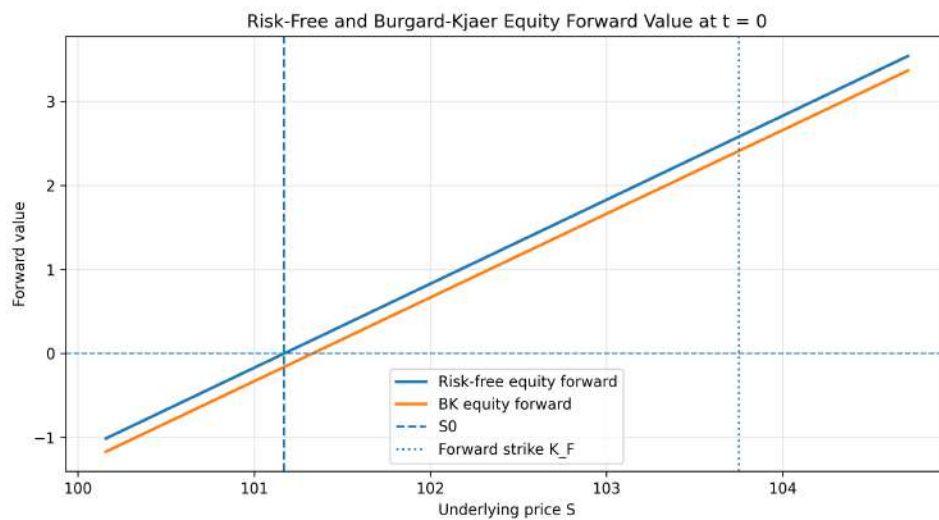


Figure C.11: Risk-free and Burgard-Kjaer values of the equity forward contract.

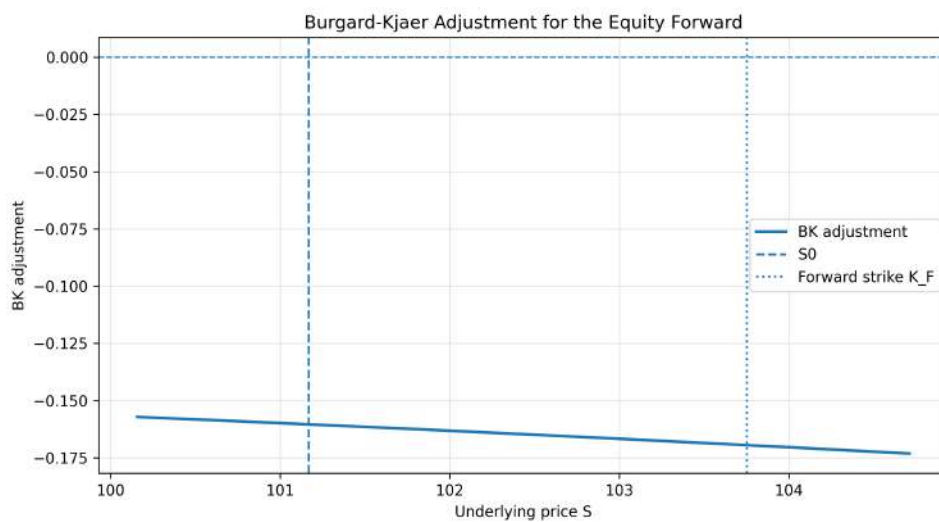


Figure C.12: Burgard-Kjaer adjustment for the equity forward contract.

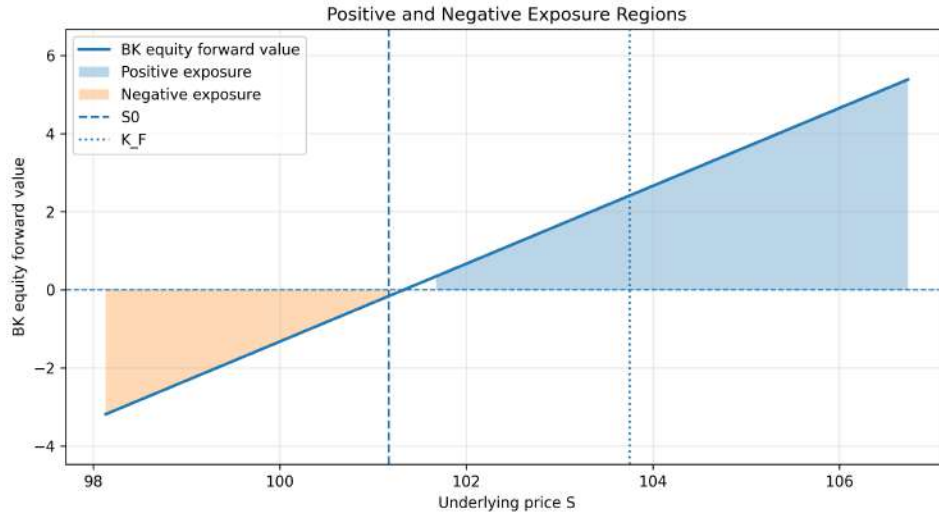


Figure C.13: Positive and negative exposure regions of the equity forward contract.

C.4 Interest Rate Forward Extension

This section presents the additional figures associated with the interest rate forward extension under Vasicek dynamics.

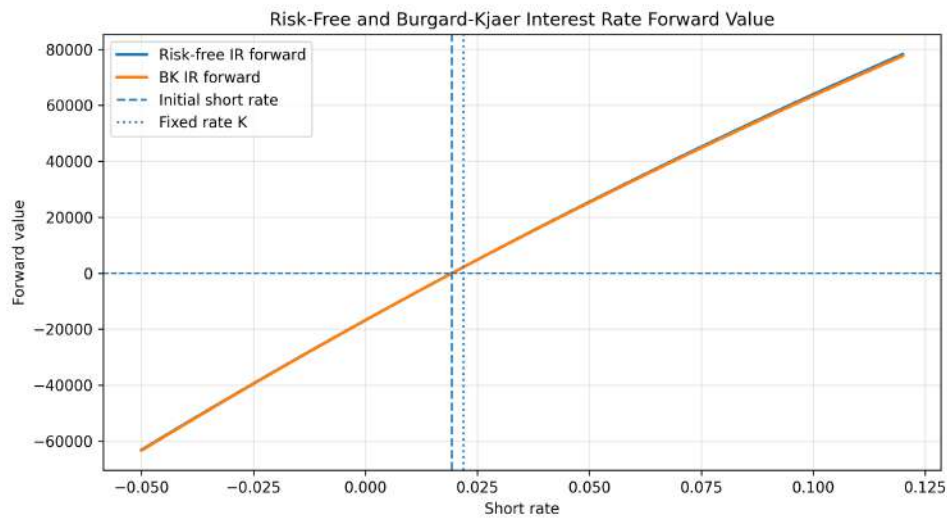


Figure C.14: Risk-free and Burgard–Kjaer values of the interest rate forward contract.

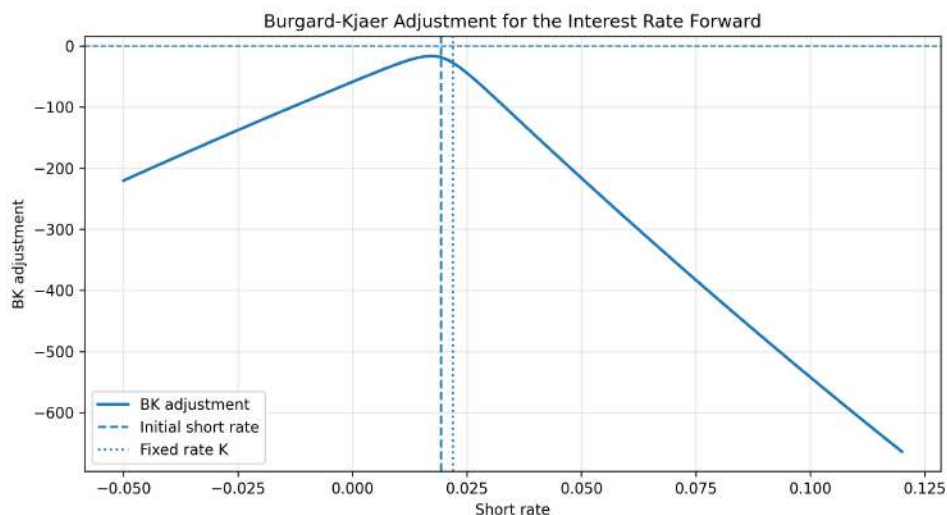


Figure C.15: Burgard–Kjaer adjustment for the interest rate forward contract.

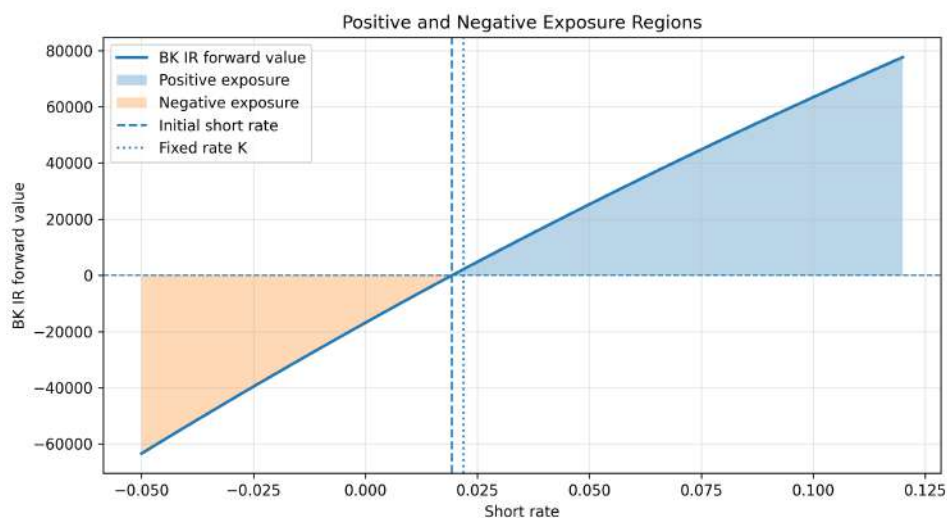


Figure C.16: Positive and negative exposure regions of the interest rate forward contract.

C.5 XVA Exposure Profiles

This section contains the expected exposure profiles used in the simplified XVA framework for the European call option, the equity forward and the interest rate forward.

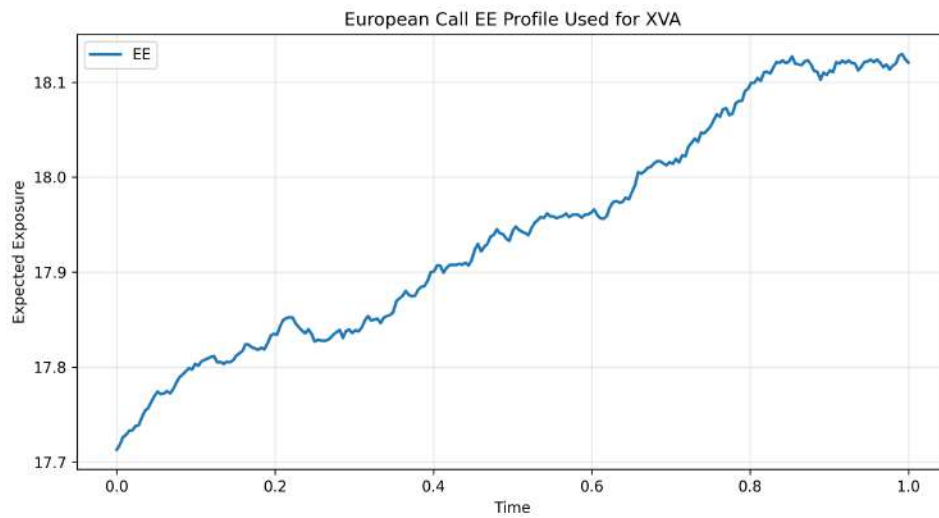


Figure C.17: Expected exposure profile of the European call option.

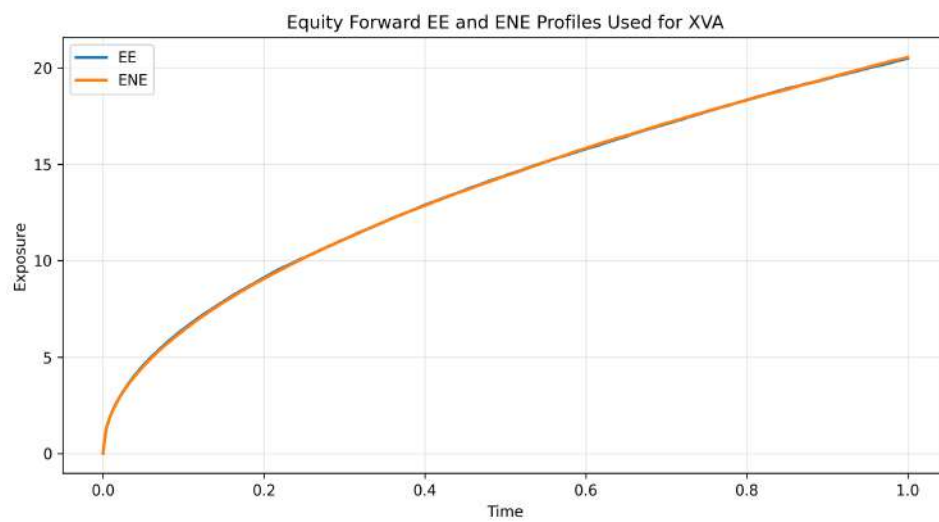


Figure C.18: Expected positive and negative exposure profiles of the equity forward contract.

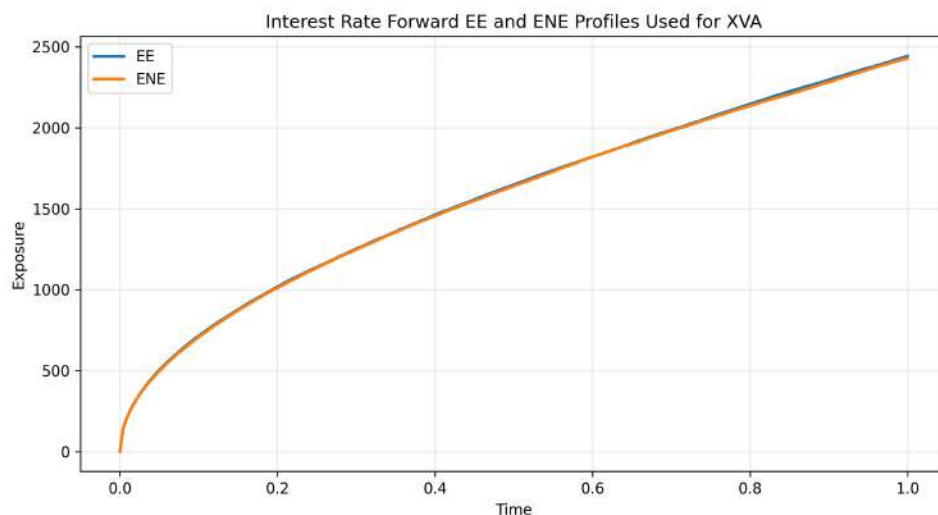


Figure C.19: Expected positive and negative exposure profiles of the interest rate forward contract.

C.6 Comparison Between Burgard–Kjaer and XVA Adjustments

This section gathers the additional comparison plots between the Burgard–Kjaer adjustments and the simplified XVA framework.

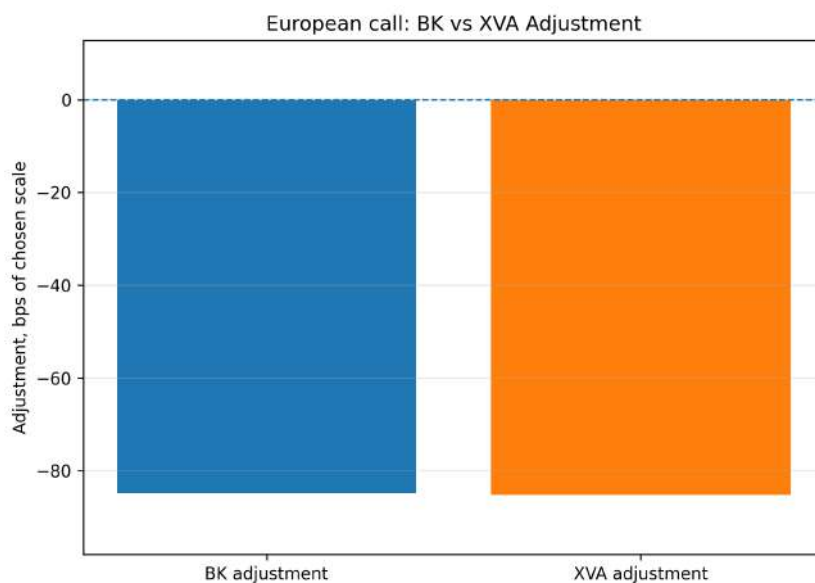


Figure C.20: Comparison between Burgard–Kjaer and simplified XVA adjustments for the European call option.

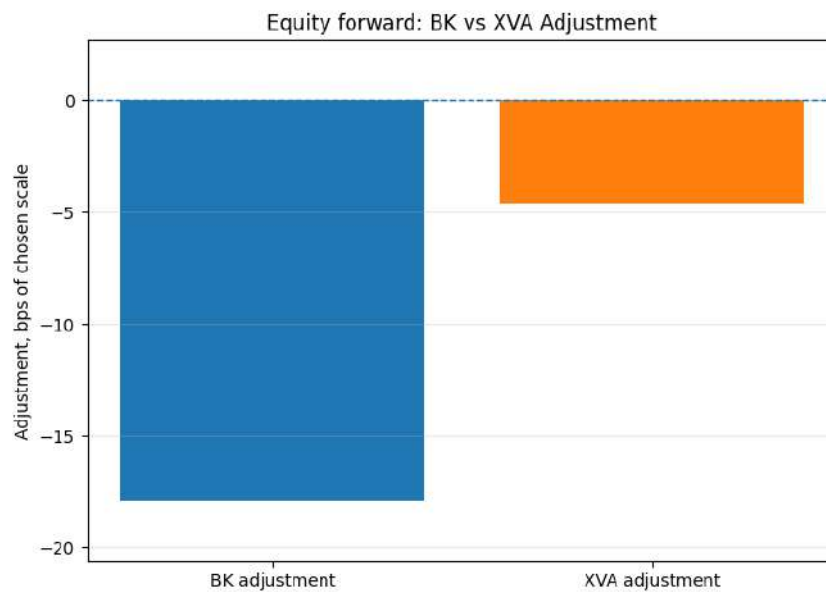


Figure C.21: Comparison between Burgard–Kjaer and simplified XVA adjustments for the equity forward contract.

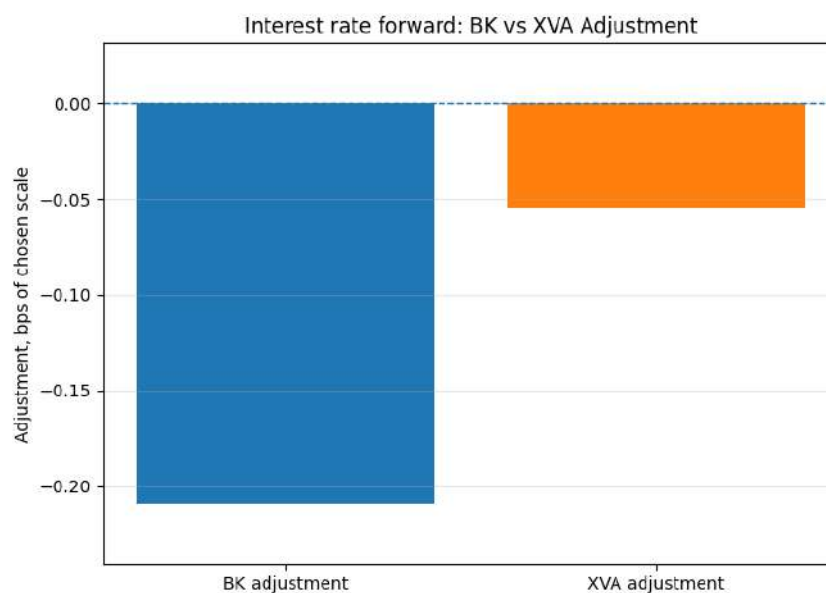


Figure C.22: Comparison between Burgard–Kjaer and simplified XVA adjustments for the interest rate forward contract.

C.7 XVA Decomposition

This section presents the decomposition of the simplified XVA adjustments into CVA, DVA and FVA contributions.

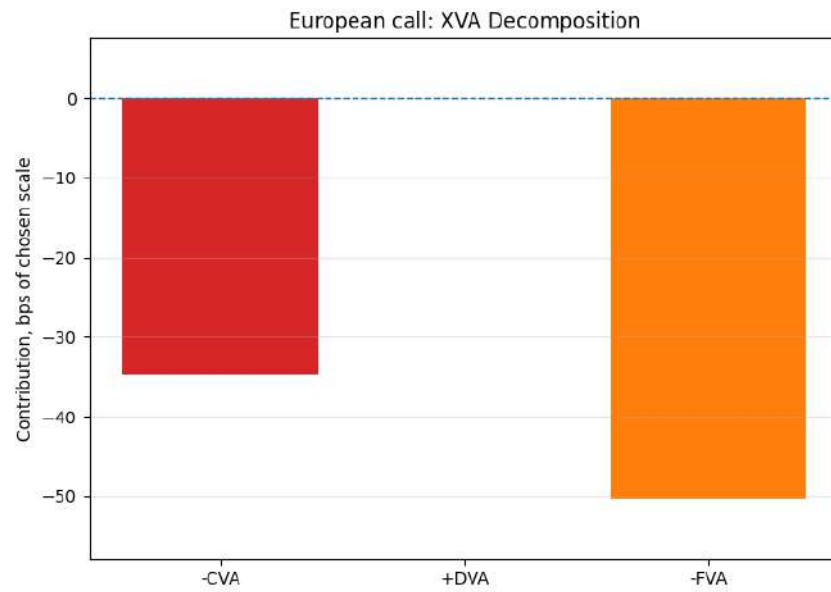


Figure C.23: Simplified XVA decomposition for the European call option.

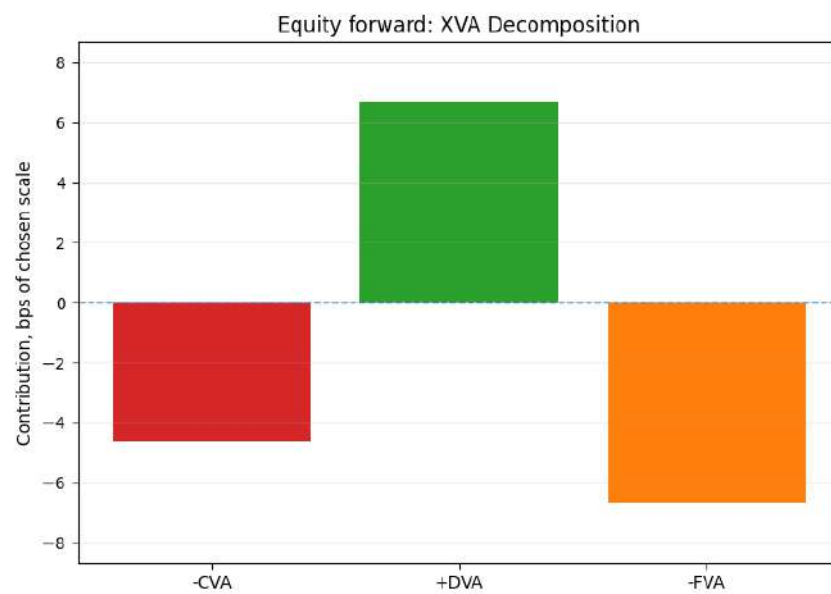


Figure C.24: Simplified XVA decomposition for the equity forward contract.

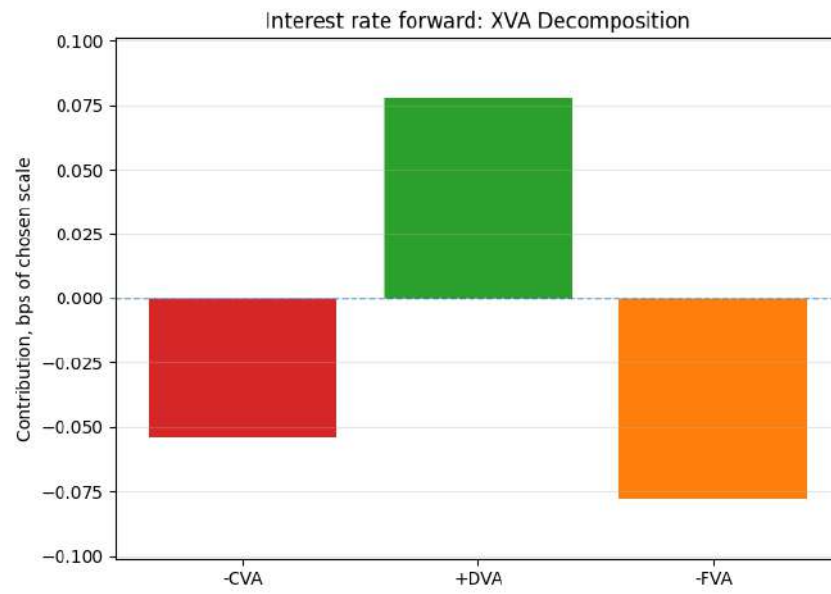


Figure C.25: Simplified XVA decomposition for the interest rate forward contract.