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Spherical Implicit Neural Representations

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Abstract

Representing and processing data in spherical domains presents unique challenges, primarily due to the curvature of the domain, which complicates the application of classical Euclidean techniques. Implicit neural representations (INRs) have emerged as a promising alternative to high-fidelity data representation; however, to effectively handle spherical domains, these methods must be adapted to the inherent geometry of the sphere to maintain both accuracy and stability. In this thesis, we introduce Herglotz-NET (HNET), a novel INR architecture that employs harmonic positional encoding based on complex Herglotz mappings. This approach yields a robust representation of spherical domains, offering interpretable and scalable spectral properties. Furthermore, we provide a unified expressivity analysis showing that any spherical-based INRs satisfying a mild condition exhibits a predictable spectral expansion that scales with network depth. Additionally, we extend HNET from the sphere to full three-dimensional spaces and evaluate it with the challenging task of reconstructing the gravitational potential of celestial bodies. Our results position HNET as a versatile, computationally efficient framework for spherical data, and demonstrate promising performance on gravitational-potential reconstruction.

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Notations

Symbol	Definition
$\mathbb{R}, \mathbb{C}$	Sets of real and complex numbers.
$\mathbb{N}, \mathbb{N}_{\geq 0}, \mathbb{Z}$	Natural numbers, non-negative integers, and integers.
$\mathbb{R}^d$	d -dimensional Euclidean space, $d \in \mathbb{N}_{\geq 1}$.
$\mathbb{K}$	Output space of an implicit neural representation (INR), either $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.
Θ	Parameters (weights) of an implicit neural representation (INR).
$f_{\Theta}(\mathbf{x})$	Implicit neural representation parameterized by Θ , evaluated at spatial point $\mathbf{x}$.
$\psi(\mathbf{x})$	Positional encoding function of an INR, with input spatial point $\mathbf{x}$.
$\sigma^{(q)}(z)$	Activation function at the q -th layer of an INR, $z \in \mathbb{K}$, $q \in \mathbb{N}_{\geq 0}$.
MLP	Multilayer perceptron mapping $\text{MLP} : \mathbb{K}^m \rightarrow \mathbb{K}$, with positional encoding dimension m .
$\mathbb{S}^{d-1}$	Unit sphere embedded in $\mathbb{R}^d$.
$\mathbb{C}[x_1, \dots, x_d]$	Space of complex-coefficient polynomials in d variables.
Δf	Laplacian of function f in $\mathbb{R}^d$.
$\Delta_{\mathbb{S}^{d-1}}$	Laplace–Beltrami operator on $\mathbb{S}^{d-1}$.
r, θ, φ	Spherical coordinates (radius, colatitude, longitude) on $\mathbb{S}^2$ with $r \geq 0$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$.
$SO(d)$	Special orthogonal group in dimension d : $SO(d) = \{\mathbf{Q} \in \mathbb{R}^{d \times d} : \mathbf{Q}^\top \mathbf{Q} = \mathbf{I}, \det \mathbf{Q} = 1\}.$
$d\sigma(\mathbf{x})$	Surface measure on $\mathbb{S}^{d-1}$; specifically for $d = 3$, $d\sigma(\mathbf{x}) = r \sin \theta \, dr \, d\theta \, d\varphi$.
$L^2(\mathbb{S}^{d-1})$	Space of square-integrable functions on $\mathbb{S}^{d-1}$: $L^2(\mathbb{S}^{d-1}) = \left\{ f : \mathbb{S}^{d-1} \rightarrow \mathbb{C} : \int_{\mathbb{S}^{d-1}} f(\mathbf{x}) ^2 d\sigma(\mathbf{x}) < \infty \right\}.$
$\bar{g}$	Complex conjugate of function the g .
$\sigma(\mathbb{S}^{d-1})$	Total surface area of the unit sphere $\mathbb{S}^{d-1}$.

Symbol	Definition
$\langle \cdot, \cdot \rangle_{L^2(\mathbb{S}^{d-1})}$	Inner product on $L^2(\mathbb{S}^{d-1})$: $\langle f, g \rangle_{L^2(\mathbb{S}^{d-1})} = \frac{1}{\sigma(\mathbb{S}^{d-1})} \int_{\mathbb{S}^{d-1}} f(\mathbf{x}) \bar{g}(\mathbf{x}) d\sigma(\mathbf{x}).$
$\ f\ _{L^2(\mathbb{S}^{d-1})}$	Norm of f in $L^2(\mathbb{S}^{d-1})$.
$L^2_{\mathbb{R}}(\mathbb{S}^{d-1})$	Space of real-valued square-integrable functions on $\mathbb{S}^{d-1}$.
$\bigoplus_{\ell=0}^{\infty} \mathcal{H}_{\ell}(\mathbb{S}^{d-1})$	Direct sum of spherical harmonics spaces of degree ℓ on $\mathbb{S}^{d-1}$, $\ell = 0, 1, \dots$.
$\mathcal{H}_{\ell}(\mathbb{R}^d)$	Space of harmonic homogeneous polynomials of degree ℓ in $\mathbb{R}^d$.
$\mathcal{H}_{\ell}(\mathbb{S}^{d-1})$	Restriction to $\mathbb{S}^{d-1}$ of harmonic homogeneous polynomials of degree ℓ .
$\xi \in \mathbb{S}^2$	Generic point on $\mathbb{S}^2$ with spherical coordinates (θ, φ) .
$\mathbf{a} \in \mathbb{C}^3$	Herglotz vector; $a_{\Re}, a_{\Im}$ its real and imaginary components.
$\Re u, \Im u$	Real and imaginary parts of complex vector $u \in \mathbb{C}^d$.
$\mathbf{a}_{\Re}, \mathbf{a}_{\Im} \in \mathbb{R}^3$	Real and imaginary components of Herglotz vector $\mathbf{a}$.
ℓ	Spherical harmonic degree ($\ell \geq 0$).
$\delta_{m,m'}$	Kronecker delta: equals 1 if $m = m'$, otherwise 0.
$P_{\ell}, P_{\ell}^m, P_{\ell m}$	Legendre, associated Legendre, and normalized associated Legendre polynomials of degree ℓ and moment m .
$Y_{\ell m}, Y_{\ell m}^{\text{real}}$	Complex and real spherical harmonics of degree ℓ and moment m .
$\ \cdot\ $	Euclidean norm.
$\varpi_0 \in \mathbb{R}$	Frequency parameter of HNET.
$\sim$	"is distributed as" notation.
$\mathcal{N}(\bar{\mu}, \sigma^2)$	Normal distribution with mean $\bar{\mu}$ and variance σ^2 .
$\mathcal{U}(a, b)$	Uniform distribution on the interval $[a, b]$.
$\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix}$	Wigner 3-j symbol
G	Gravitational constant.
μ	$\mu = Gm$ is the gravitational parameter, where m is the mass of a celestial body and G is the gravitational constant.
$p_{\mathbf{z} \mathbf{x}}$	Probability density function of the random variable $\mathbf{z}$ given $\mathbf{x}$.
$ \mathbf{x} $	Magnitude of the vector $\mathbf{x} \in \mathbb{R}^3$.
$R_{\ell m}(r, \theta, \varphi)$	Un-normalized regular solid harmonic defined by $R_{\ell m}(r, \theta, \varphi) = r^{\ell} Y_{\ell m}(\theta, \varphi)$

where $Y_{\ell m}$ is the spherical harmonic of degree ℓ and order m .
 $I_{\ell m}(r, \theta, \varphi)$ Un-normalized irregular solid harmonic defined by

$$I_{\ell m}(r, \theta, \varphi) = r^{-(\ell+1)} Y_{\ell m}(\theta, \varphi)$$

where $Y_{\ell m}$ is the spherical harmonic of degree ℓ and order m .

Symbol	Definition
$\mathcal{O}(\cdot)$	Big-O notation, indicating the asymptotic behavior of a function.
r^*, a^*, Φ^*	Reference value of radius, acceleration, and potential.
$C_{\ell m}, S_{\ell m}$	Stokes coefficients of the spherical harmonic expansion of some potential with $\ell \geq 0$ and $0 \leq m \leq \ell$; these coefficients are real-valued.

Abbreviations

Abbr.	Meaning
INR	Implicit Neural Representation (coordinate-based network used to model a continuous signal)
PE	Positional Encoding (embedding of spatial coordinates fed to the INR)
MLP	Multilayer Perceptron (fully-connected feed-forward sub-network of the INR)
HNET	<i>Herglotz-NET</i> – harmonic INR introduced in this thesis
SIREN	<i>Sinusoidal Representation Network</i> (Euclidean baseline with sinusoidal activations)
SPH-SIREN	Spherical SIREN (SIREN equipped with spherical-harmonic PE)
FFN	Fourier-Feature Network (uses random Fourier features as PE)
SH	Spherical Harmonics
DFS	Double-Fourier Sphere
OD	Orbit Determination (iterative estimation of spacecraft trajectory & gravity field)
PDE	Partial Differential Equation
NN	Neural Network
PINN	Physics-Informed Neural Network (NN trained with PDE residuals in the loss)
LS	Least-Squares inversion
RMSE	Root-Mean-Square Error
MAE	Mean Absolute Error
MAPE	Mean Absolute Percentage Error
SNR	Signal-to-Noise Ratio (decibels)
RMSE Power Spectrum	Root Mean Square spectrum of the prediction error
MEX	Mars Express spacecraft (eccentric orbit)
MEX-LS	Least-Square inversion applied on the Mars Express spacecraft (MSE)-like orbit
MRO	Mars Reconnaissance Orbiter (near-circular orbit)
MRO-LS	Least-Square inversion applied on the Mars Reconnaissance Orbiter (MRO)-like orbit

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Chapter 1

Introduction

In modern applications, we process an ever-growing range of data modalities—from 2-D images, video, and audio to higher-dimensional forms like 3-D meshes, point clouds, volumetric scans, and various multimodal sensor data.

Traditionally, despite their continuous nature, these types of data are represented discretely—for instance, pixel grids for images or point clouds for 3-D shapes [1]. Although intuitive and very popular, discrete representations have inherent limitations due to finite hardware storage capacity, which restricts the achievable resolution and detail. Additionally, the computational costs associated with processing these discrete forms increase significantly as resolution improves.

Recently, there has been a shift toward novel deep learning-based [2] representations designed to overcome the constraints of traditional discrete formats. Implicit Neural Representations (INRs) [3], in particular, have emerged as a powerful solution. The core principle of an INR is that it does not explicitly store signal values; rather, it learns a mapping from input coordinates to those target values and retains only the network’s weights. Because the size of this parameter set is fixed, storage becomes decoupled from data resolution, which yields a continuous, memory-efficient, and fully differentiable representation that readily adapts to diverse data modalities and applications.

INRs have been successfully applied across various tasks, such as image compression, solving PDEs, signal super-resolution, detailed 3D scene reconstruction via signed distance fields, generative shape modeling, and more [4–11].

However, most existing INRs are designed specifically for Euclidean domains. Adapting INRs effectively to non-Euclidean domains requires careful consideration of the geometry involved to maintain accuracy and numerical stability [12, 13]. This thesis therefore focuses on establishing a robust INR framework for data defined on the sphere which is a fundamental non-Euclidean domain.

Contributions The main contributions of this work are threefold:

- We introduce a novel implicit neural representation for spherical data named Herglotz-NET (HNET). HNET utilizes a harmonic positional encoding based on complex Herglotz mappings, yielding a well-defined, easily implementable,

and computationally efficient INRs for spherical data. Furthermore, Herglotz mappings offer a flexible framework for integrating spherical geometry into INRs, paving the way for future extensions.

- We perform a unified analysis of spherical INRs expressivity, demonstrating that spherical INRs under mild conditions exhibit a predictable spectral expansion scaling with network depth. This extends spectral analysis previously established for Euclidean INRs in [14].
- We extend the Herglotz-NET framework to the entire $\mathbb{R}^3$ domain, specifically for gravitational potential reconstruction problems, and show its promising capabilities through experimental evaluations.

Outline The thesis is organized as follows: Chap. 1 introduces the fundamentals of INRs and spherical harmonics. Chap. 2 details the design, implementation, spectral properties and evaluates the capabilities of HNET against established baselines. Finally, Chap. 3 discusses the extension of HNET to $\mathbb{R}^3$, focusing on gravitational potential reconstructions and presenting experimental outcomes.

1.1 Implicit Neural Representations (INRs)

In this section, we first introduce the general concept and taxonomy of INRs. We then specifically focus on Euclidean-based INRs, examining their properties and highlighting their theoretical and practical implications. This discussion provides essential foundational knowledge for our spherical INR.

1.1.1 Principle & Architecture

We are interested in learning *continuous* functions

$$f : \mathcal{E} \subset \mathbb{R}^n \rightarrow \mathbb{K}, \quad \text{where } \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C},$$

that satisfy, a *general implicit relation*

$$F(\mathbf{x}, f, \nabla_{\mathbf{x}} f, \nabla_{\mathbf{x}}^2 f, \dots) = 0, \quad \forall \mathbf{x} \in U \subseteq \mathcal{E}, \quad (1.1)$$

the subset U collects the locations at which the relation is enforced. The function F can encode anything from a physical-based differential equation to a purely data-driven regularization. An implicit neural representation (INR), refers to a neural network $f_{\Theta} : \mathcal{E} \rightarrow \mathbb{K}$ that, through its weights $\Theta \in \mathbb{R}^d$, parametrized the implicitly defined function class in (1.1).

A typical example is the task of learning a continuous signal f from its sampling $\mathcal{D} = \{(\mathbf{x}_i, y_i)\}_{i=1}^N$ on a set of N discrete points $\{\mathbf{x}_i\}_{i=1}^N \subset \mathcal{E}$. This is done by *training* the set of parameters Θ such that the network interpolates the target function f at the sampled points. In that framework, the implicit relation (1.1) can be casted as:

$$F(\mathbf{x}, f) = f(\mathbf{x}) - y_i, \quad \text{for } \mathbf{x} = \mathbf{x}_i, \quad i = 1, \dots, N.$$

where $U = \cup_{i=1}^N \{\mathbf{x}_i\}$.

Properties We highlight three fundamental properties of INRs that distinguish them from traditional signal representations:

- (i) **Continuous.** INRs yield a *continuous* representation, allowing the learned function to be evaluated at any point $\mathbf{x} \in \mathcal{E}$. This continuity removes the resolution limits (see, Fig. 1.1), which enables INRs to capture arbitrarily fine details without the memory-intensive storage associated with discrete representations.
- (ii) **Grid Independence.** Unlike traditional methods that depend on fixed grids or structured sampling schemes, INRs can naturally handle arbitrary discrete samples. This flexibility lets the model adapt to the data structure at hand, making INRs particularly valuable for irregularly sampled data or applications without uniform data distribution.
- (iii) **Differentiability.** With appropriate activation functions, the neural network f_{Θ} is fully differentiable. Gradients with respect to input coordinates $\mathbf{x}$ can

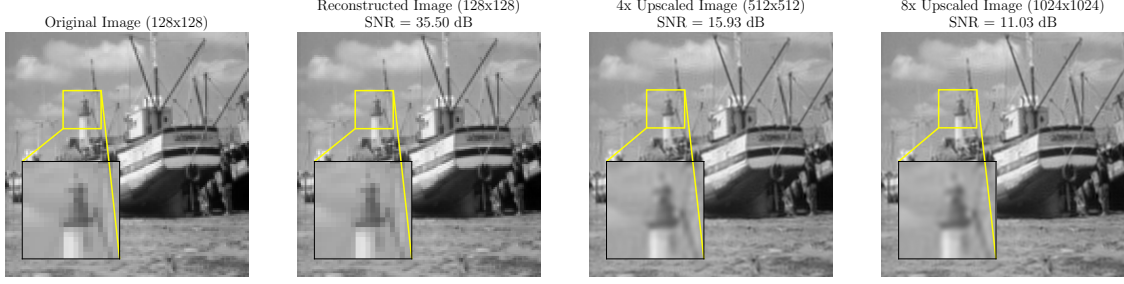


Figure 1.1: Image interpolation and super-resolution using a SIREN [6] INR. From left to right: original image at a resolution of 128×128 pixels, reconstructed image at the same resolution, and reconstructions evaluated at increasingly higher resolutions: $4\times$ upscaled (512×512) and $8\times$ upscaled (1024×1024).

thus be computed precisely using automatic differentiation [15]. This allows us to add PDE constraints, regularization terms, or any implicit relation involving derivatives of f , avoiding the need for finite-difference approximations.

These three properties represent only the tip of the iceberg in explaining the remarkable success of INRs. In Sec. 1.1.2, we investigate an additional specific aspect underlying the success of INRs.

Taxonomy In this work, we follow the unified architecture framework introduced by Yüce et al. [14], decomposing an INR into two main components (see Fig. 1.2): a positional encoding (PE) and a multilayer perceptron (MLP).

The positional encoding $\psi : \mathcal{E} \rightarrow \mathbb{K}^m$ transforms input coordinates into an m -dimensional embedding. This embedding is then processed by an MLP composed of Q layers, with weights $\mathbf{W}^{(q)} \in \mathbb{K}^{d_q \times d_{q-1}}$, biases $\mathbf{b}^{(q)} \in \mathbb{K}^{d_q}$, and activation functions $\sigma^{(q)} : \mathbb{K} \rightarrow \mathbb{K}$, applied element-wise. Denoting by $\mathbf{z}^{(q)}$ the activated output of layer q , the INR architecture is described mathematically as:

$$\begin{cases} \mathbf{z}^{(0)} = \psi(\mathbf{x}), \\ \mathbf{z}^{(q)} = \sigma^{(q)} \left(\mathbf{W}^{(q)} \mathbf{z}^{(q-1)} + \mathbf{b}^{(q)} \right), & q = 1, \dots, Q-1, \\ f_{\Theta}(\mathbf{x}) = \mathbf{W}^{(Q)} \mathbf{z}^{(Q-1)} + \mathbf{b}^{(Q)}. \end{cases} \quad (1.2)$$

which we will often write compactly as:

$$f_{\Theta}(\mathbf{x}) = \text{MLP}(\psi(\mathbf{x})) \quad (1.3)$$

where $\text{MLP} : \mathbb{K}^m \rightarrow \mathbb{K}$ denotes the function defined by the sequence of operations in (1.2) from $q = 1$ to $q = Q$.

1.1.2 Euclidean INRs

We define a *Euclidean Implicit Neural Representation* as an INR whose positional encoding (PE) functions $\psi_i(\mathbf{x})$ are continuous and differentiable functions on a

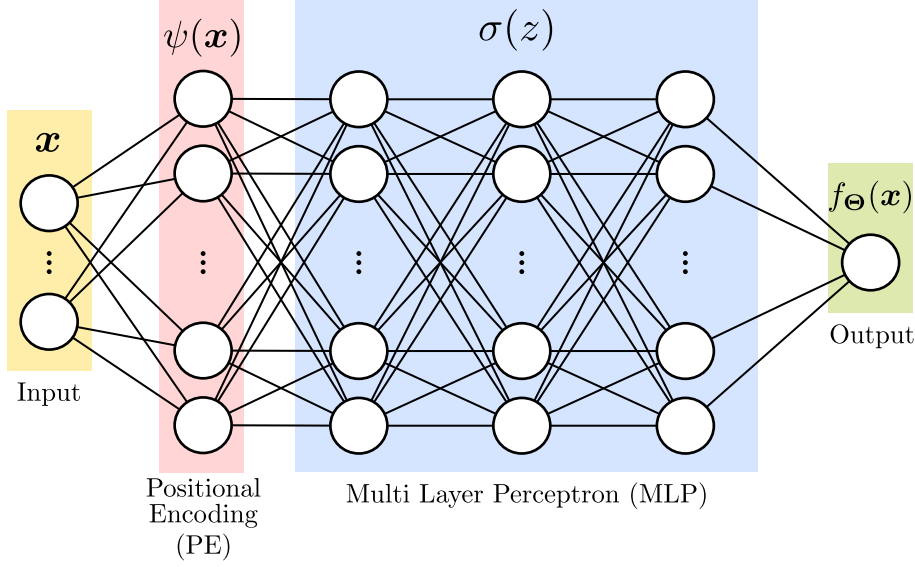


Figure 1.2: Different components of the INR architecture. The positional encoding (PE) transforms input coordinates into an m -dimensional embedding, which is then processed by a multilayer perceptron (MLP) composed of Q layers. Each layer applies a linear transformation followed by an activation function $\sigma(\cdot)$, resulting in the final output $f_{\Theta}(\mathbf{x})$.

Euclidean domain $\mathcal{E} \subset \mathbb{R}^n$. This differentiation is essential because the Euclidean INRs presented here do not directly generalize to spherical domains, which is the primary focus of this thesis. Nevertheless, a thorough understanding of these Euclidean architectures provides valuable insights toward our generalization goal.

Two popular Euclidean INR architectures are Sinusoidal Representation Networks (SIRENs) [6] and Fourier Feature Networks (FFNs) [16]. Both architectures share the sinusoidal positional encoding:

$$\psi_{\text{sin}}(\mathbf{x}) = \sin(\mathbf{\Omega}\mathbf{x} + \mathbf{b}) = \left(\sin(\boldsymbol{\omega}_i^T \mathbf{x} + b_i) \right)_{i=1}^m,$$

but differ in their implementation:

- In SIRENs, the frequency matrix is parameterized as $\mathbf{\Omega} = \omega_0 \cdot \mathbf{W}$, where ω_0 is a fixed hyperparameter, and $\mathbf{W}$ is learned during training. Additionally, SIRENs employ sinusoidal activation functions ($\sigma^{(q)}(\cdot) = \sin(\cdot)$) across all hidden layers.
- In contrast, FFNs utilize a randomly initialized and fixed frequency matrix $\mathbf{\Omega}$, where elements are drawn from a normal distribution $\mathcal{N}(0, \sigma^2)$, leading to the so-called Random Fourier Features [17]. FFNs typically use ReLU activation functions.

These architectures notably address the classical limitation of standard neural networks known as *spectral bias*. The spectral bias refers to the tendency of neural networks to learn low-frequency components of a target function more effectively than

high-frequency components, leading to suboptimal performance in tasks requiring high-frequency detail [18]. SIRENs and FFNs mitigate this bias and it has been shown than simply adjusting the frequency hyperparameters (ω_0 for SIRENs, σ^2 for FFNs) can significantly improve the learning of high-frequency features.

Yüce et al. [14] provided theoretical insight into this phenomenon by analyzing the spectral properties of these INRs. Assuming that activation functions can be approximated by truncated Taylor expansions:

$$\sigma^{(q)}(x) \approx \sum_{k=0}^K \alpha_k x^k,$$

they showed that passing a frequency through such nonlinearities expands the spectral content. For instance, considering $\psi(x) = e^{j\omega x}$, a single-layer activation transforms it into:

$$\sigma^{(1)}(e^{j\omega x}) = \sum_{k=0}^K \alpha_k e^{j\omega k x}.$$

This reveals that a single frequency ω has now expanded into multiple frequencies $\{0, \omega, 2\omega, \dots, K\omega\}$. At each subsequent nonlinear activation layer, these new frequencies are further combined and expanded, thus exponentially increasing the spectral content available within the network.

Formally, this property is described by the following theorem:

Theorem 1.1 (Yüce et al., 2021 [14]). *Let $f_{\Theta} : \mathbb{R}^n \rightarrow \mathbb{R}$ be an INR described by Eq. (1.2), with activations approximated by polynomials as above. Let the frequency matrix $\Omega \in \mathbb{R}^{m \times n}$ and phase vector $\phi \in \mathbb{R}^m$ define the positional encoding $\psi_{\sin}(\mathbf{x}) = \sin(\Omega \mathbf{x} + \phi)$. Then, this architecture can represent functions of the form*

$$f_{\Theta}(\mathbf{x}) = \sum_{\omega' \in \mathcal{H}(\Omega)} c_{\omega'} \sin(\langle \omega', \mathbf{x} \rangle + \phi_{\omega'}),$$

where

$$\mathcal{H}(\Omega) \subseteq \left\{ \sum_{t=0}^{m-1} s_t \Omega_t \mid s_t \in \mathbb{Z}, \sum_{t=0}^{m-1} |s_t| \leq K^{Q-1} \right\}.$$

Below we summarize the key insights of Th. 1.1 intuitively.

Firstly, the network output relies fundamentally on a frequency dictionary, denoted as the set $\mathcal{H}(\Omega)$. This dictionary contains the frequencies from which the network can select to represent the target function. We refer to this as the *frequency content*, or *spectral content*, of the network output. Similarly, we define the frequency content at any given layer of the network as the set of frequencies available up to that point. Initially, at the positional encoding, this frequency content is simply the frequency matrix Ω itself.

Secondly, the theorem highlights a crucial aspect: the frequency content available to the network expands significantly with each added layer. In other words, as the network grows deeper (with increasing depth Q), it progressively enriches its available frequency dictionary. Importantly, this growth in available frequencies is exponential

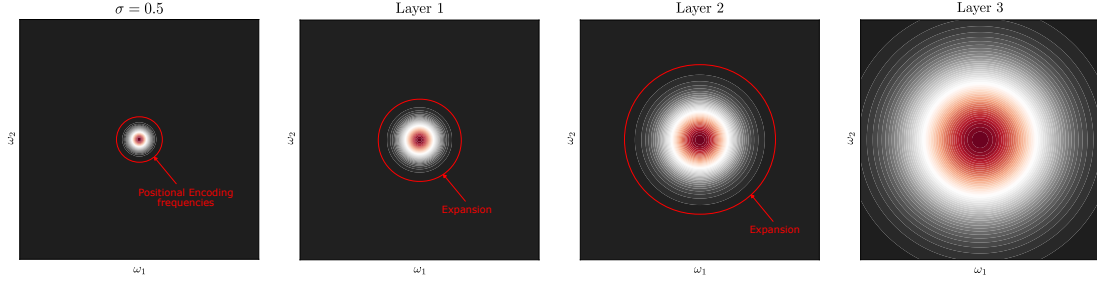


Figure 1.3: Illustrative visualization of spectral expansion in a hypothetical three-layer FFN with quadratic activation functions ($\sigma(x) = x^2$) and initial positional encoding frequencies drawn from a normal distribution $\mathcal{N}(0, 0.5^2)$. Each panel displays the superposed 2D Fourier spectra of neurons at a given layer. The visualization illustrates how the network’s frequency coverage theoretically expands from layer to layer.

with respect to the network depth, specifically scaling as K^{Q-1} , where K represents the number of terms in the polynomial expansion of the activation function.

For sinusoidal activation functions, this expansion is theoretically limitless because K is infinite, implying that, in principle, the model could capture frequencies without bound. However, in practice, higher-order terms from the Taylor expansion rapidly diminish due to their decreasing coefficients, ensuring that the predominant spectral components remain clustered around a manageable subset of frequencies.

Finally, the theorem underscores the central role of the initial frequency matrix Ω . All frequencies available at the network’s output are linear combinations of these initial frequencies. Consequently, the initial selection and magnitude of frequencies in Ω directly determine how rich and how detailed the frequency content of the INR can be. The larger the initial frequencies included in Ω , the higher the range of high-frequency details the network can represent.

To build intuition for this frequency expansion, Fig. 1.3 illustrates a simplified scenario showing how the frequency content expands across network layers. Each activated frequency shown in Fig. 1.3 is subsequently available for the network to build its approximation.

These fundamental properties are precisely what we aim to have with spherical INRs. However, the frequency content of Euclidean INRs is inherently defined within the classical Fourier context, which naturally applies to rectangular (Euclidean) domains. Consequently, we need to adapt this concept to the spherical setting. This adaptation gives rise to *spherical harmonics*, which form a Fourier-like basis on the sphere, allowing us to define and analyze frequency content in spherical domains rigorously. In the following section, we formally introduce the notion of spherical harmonics.

1.2 Spherical Harmonics

In this section, we establish the theoretical foundations of *spherical harmonics*. We first review harmonic function theory in general dimensions (Sec. 1.2.1), then

specialize to $\mathbb{S}^2$ to derive their explicit forms (Sec. 1.2.2), and conclude by summarizing the essential concepts (Sec. 1.2.3). The main references for this chapter are [19, 20].

We assume that the reader is familiar with the basic concepts of functional analysis, including Hilbert spaces and orthonormal bases. For a more detailed introduction to these topics, we refer the reader to [21].

1.2.1 Harmonic Function Theory

Before introducing the so-called *spherical harmonics* on the sphere $\mathbb{S}^2$, it is instructive to consider the broader context in which these functions arise, namely on any hypersphere $\mathbb{S}^{d-1}$. The goal in this section, is to briefly review the general theorems that make the spherical harmonics a reliable tool for representing functions on the spheres.

At the heart of the theory lies the concept of *harmonic functions*. We begin by recalling the following definition:

Definition 1.1. A function $u \in \mathcal{C}^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^d$, is said to be harmonic on Ω if it satisfies Laplace's equation

$$\Delta u = \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2} = 0. \quad (1.4)$$

This equation naturally appears in many physical and mathematical settings (see, e.g., [22]) and serves as the starting point for our analysis.

Among the many solutions of this equation, a very particular class will draw our attention, namely the *homogenous harmonic polynomials* :

Definition 1.2 (Homogeneous harmonic polynomial). Let $\ell \in \mathbb{N}_{\geq 0}$, the space of homogeneous harmonic polynomials of degree ℓ is defined as:

$$\mathcal{H}_\ell(\mathbb{R}^d) := \{p \in \mathbb{C}[x_1, \dots, x_d] : p(\lambda \mathbf{x}) = \lambda^\ell p(\mathbf{x}) \text{ and } \Delta p(\mathbf{x}) = 0, \forall \mathbf{x} \in \mathbb{R}^d\}. \quad (1.5)$$

The reason these functions are of great interest for us because once restricted to the unit sphere $\mathbb{S}^{d-1}$, they span a *rotation-invariant* space. This property is crucial, as it implies that the representation of the data in this basis is stable under rotations of the data—an essential feature for representing signals on $\mathbb{S}^2$.

This brings us to the definition of *Spherical Harmonics*, which are simply the restrictions of these homogeneous harmonic polynomials to the unit sphere $\mathbb{S}^{d-1}$, the space of Spherical Harmonic of degree ℓ is defined as :

Definition 1.3 (Space of Spherical Harmonics). The space of Spherical Harmonic of degree ℓ reads:

$$\mathcal{H}_\ell(\mathbb{S}^{d-1}) := \{p|_{\mathbb{S}^{d-1}} : p \in \mathcal{H}_\ell(\mathbb{R}^d)\}. \quad (1.6)$$

We can now formalize the rotation invariance property of $\mathcal{H}_\ell(\mathbb{S}^{d-1})$ with the following theorem :

Theorem 1.2 (Dai et al. [23]). *The space $\mathcal{H}_\ell(\mathbb{S}^{d-1})$ is invariant under rotations. That is, for any rotation $\mathbf{Q} \in SO(d)$ and any function $f \in \mathcal{H}_\ell(\mathbb{S}^{d-1})$, the rotated function $f(\mathbf{Q}\cdot)$ also belongs to $\mathcal{H}_\ell(\mathbb{S}^{d-1})$. Moreover, if $\{Y_\ell\}$ is an orthonormal basis of $\mathcal{H}_\ell(\mathbb{S}^{d-1})$, then $\{Y_\ell(\mathbf{Q}\cdot)\}$ is also an orthonormal basis.*

Let $L^2(\mathbb{S}^{d-1})$ the space of square-integrable functions in the sphere $\mathbb{S}^{d-1}$, endowed with the inner product

$$\langle f, g \rangle_{L^2(\mathbb{S}^{d-1})} = \frac{1}{\sigma(\mathbb{S}^{d-1})} \int_{\mathbb{S}^{d-1}} f(\mathbf{x}) \bar{g}(\mathbf{x}) d\sigma(\mathbf{x}), \quad (1.7)$$

where $d\sigma(\cdot)$ is the surface measure on $\mathbb{S}^{d-1}$ and $\sigma(\mathbb{S}^{d-1})$ the total surface area of $\mathbb{S}^{d-1}$. Finally, we state the following fundamental theorem, which ensures that every function in $L^2(\mathbb{S}^{d-1})$ can be expressed as a linear combination of the basis functions from $\mathcal{H}_\ell(\mathbb{S}^{d-1})$ for $\ell = 0, \dots, \infty$:

Theorem 1.3 (Axler et al. [19]). *Let $\{Y_{\ell m}\}_{m=1}^{N(\ell)}$ be an orthonormal basis for $\mathcal{H}_\ell(\mathbb{S}^{d-1})$. Then, the set*

$$\{Y_{\ell m}\}_{\ell \geq 0, 1 \leq m \leq N(\ell)} \quad (1.8)$$

forms an orthonormal basis for $L^2(\mathbb{S}^{d-1})$. Consequently, every function $f \in L^2(\mathbb{S}^{d-1})$ can be expressed as a convergent series

$$f(\mathbf{x}) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(\ell)} a_{\ell m} Y_{\ell m}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^{d-1}, \quad (1.9)$$

that is,

$$L^2(\mathbb{S}^{d-1}) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell(\mathbb{S}^{d-1}). \quad (1.10)$$

To illustrate these ideas, consider the case $d = 2$. On the unit circle $\mathbb{S}^1$, the basis functions take the form $e^{im\theta}$, according to Th. 1.3, any function $f \in L^2(\mathbb{S}^1)$ can be represented as:

$$f(\theta) = \sum_{m \in \mathbb{Z}} a_m e^{im\theta}. \quad (1.11)$$

This is precisely the classical Fourier series on the circle, showing how Th. 1.3 generalizes the Fourier expansion to hyperspheres.

In summary, an orthonormal basis of the spaces $\mathcal{H}_\ell(\mathbb{S}^{d-1})$, for $\ell = 0, \dots, \infty$, provides a natural, rotation-invariant basis for representing functions on the hypersphere.

1.2.2 Derivation

We now focus on the specific case of $\mathbb{S}^2$. Although our previous discussion showed that $L^2(\mathbb{S}^2)$ decomposes orthogonally into the subspaces $\mathcal{H}_\ell(\mathbb{S}^2)$, an explicit construction of an orthonormal basis for these spaces is still needed.

Our first goal is to establish a key property that any function in $\mathcal{H}_\ell(\mathbb{S}^2)$ must satisfy. Since these functions arise as restrictions of homogeneous harmonic polynomials, they naturally obey Laplace's equation. Expressing this equation in spherical

coordinates leads to

$$\begin{aligned}
\Delta f(\mathbf{x}) &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2} \\
&= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \Delta_{\mathbb{S}^2} f(\mathbf{x}) \\
&= 0, \quad \forall \mathbf{x} \in \Omega \subseteq \mathbb{R}^3,
\end{aligned} \tag{1.12}$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator on the sphere.

Let $p_\ell \in \mathcal{H}_\ell(\mathbb{S}^2)$ with $\ell \in \mathbb{N}$. By definition, p_ℓ is both harmonic and homogeneous. Writing a point $\mathbf{x} \in \mathbb{R}^3$ as $\mathbf{x} = r \boldsymbol{\xi}$ with $\boldsymbol{\xi} \in \mathbb{S}^2$, we have

$$p_\ell(\mathbf{x}) = r^\ell p_\ell(\boldsymbol{\xi}).$$

where θ and φ are the colatitude and longitude angles, respectively. Substituting this expression into (1.12) yields the eigenvalue problem for the Laplace-Beltrami operator (see, e.g., [24]):

$$\Delta_{\mathbb{S}^2} p_\ell(\boldsymbol{\xi}) = -\ell(\ell+1) p_\ell(\boldsymbol{\xi}). \tag{1.13}$$

A standard technique for finding these eigenfunctions is separation of variables; which assumes that

$$p_\ell(\boldsymbol{\xi}(\theta, \varphi)) = p_\ell(\theta, \varphi) = \Theta(\theta) \Phi(\varphi).$$

Substituting into (1.13), and developing (see App. D), one can obtain a closed form expression for our basis :

$$p_{\ell m}(\theta, \varphi) = P_\ell^m(\cos \theta) e^{im\varphi}, \quad m = -\ell, \dots, \ell. \tag{1.14}$$

where P_ℓ^m are the *associated Legendre polynomials* (see Fig. 1.4). These polynomial can be derived in several ways; one useful formula is the Rodrigues' one:

$$\begin{aligned}
P_\ell^m(x) &= (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x), \\
P_\ell^{-m}(x) &= \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)
\end{aligned} \quad m \geq 0, \tag{1.15}$$

where $P_\ell(x)$ are the *Legendre polynomials*.

In Eq. (1.14), for a fixed degree ℓ we have $2\ell+1$ basis functions, and we can verify the orthogonality of these functions for fixed ℓ :

$$\begin{aligned}
\langle p_{\ell, m}, p_{\ell, m'} \rangle_{L^2(\mathbb{S}^2)} &= \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} e^{i(m-m')\varphi} P_\ell^m(\cos \theta) P_\ell^{m'}(\cos \theta) \sin \theta d\varphi d\theta \\
&= \frac{1}{(2\ell+1)} \frac{(\ell+m)!}{(\ell-m)!} \delta_{m, m'}.
\end{aligned}$$

From this, we can normalized the functions $p_{\ell, m}$ to obtain an orthonormal basis, which leads to the classical definition of the *spherical harmonics*:

$$Y_{\ell m}(\theta, \varphi) = \sqrt{(2\ell+1) \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\varphi}, \quad m = -\ell, \dots, \ell, \ell = 0, \dots, \infty. \tag{1.16}$$

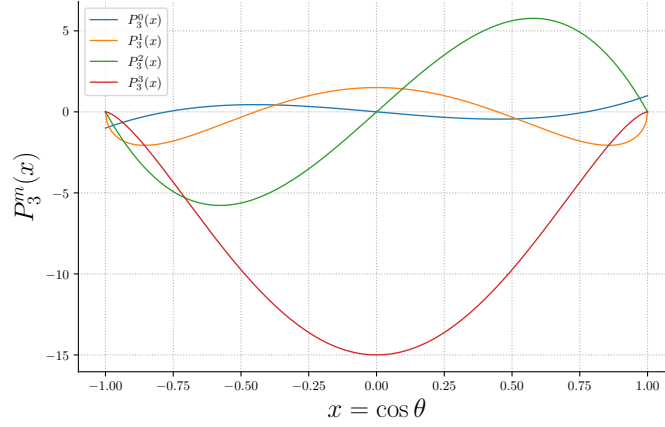


Figure 1.4: Visualization of the associated Legendre polynomials P_ℓ^m for $\ell = 3$ and $m = 0, 1, 2, 3$

In some fields, such as quantum mechanics (see, e.g., [20]), a factor of $(-1)^m$ is introduced; we stick to the definition in (1.16).

We can equivalently define the real spherical harmonics from the complex spherical harmonics defined in (1.16) as follows:

$$Y_{\ell m}^{\text{real}}(\theta, \varphi) = \begin{cases} \sqrt{2} \Im\{Y_{\ell|m|}(\theta, \varphi)\}, & m < 0 \\ Y_{\ell 0}(\theta, \varphi), & m = 0 \\ \sqrt{2} \Re\{Y_{\ell|m|}(\theta, \varphi)\}, & m > 0 \end{cases} = \begin{cases} \bar{P}_{\ell|m|}(\cos \theta) \sin |m|\varphi, & m < 0 \\ \bar{P}_{\ell m}(\cos \theta) \cos m\theta, & m \geq 0 \end{cases} \quad (1.17)$$

where $\bar{P}_{\ell m}(\cos \theta)$ are the *normalized associated Legendre polynomials* defined as follows:

$$\bar{P}_{\ell m}(\cos \theta) = \sqrt{(2 - \delta_{m,0})(2\ell + 1) \frac{(\ell - m)!}{(\ell + m)!}} P_\ell^m(\cos \theta) \quad (1.18)$$

The real spherical harmonics $Y_{\ell m}^{\text{real}}(\theta, \varphi)$ forms an orthogonal basis of $L_{\mathbb{R}}^2(\mathbb{S}^2)$, the space of real-valued square-integrable functions on $\mathbb{S}^2$.

Terminology: Throughout this work, the term *spherical harmonics* refers exclusively to the normalized basis functions $Y_{\ell m}$ defined in (1.16). These functions constitute an orthonormal basis for the spaces of Spherical Harmonics introduced earlier.

Spherical harmonics have been extensively studied, and a wealth of properties and applications have been developed over the years. Readers interested in a more comprehensive treatment of spherical harmonics are encouraged to consult standard references (see, e.g., [25, Chap. IV]) for further details.

1.2.3 Key Results

We briefly recall the essential information about spherical harmonics needed for the remainder of this work. We focus only on the core concepts and therefore omit detailed mathematical derivations.

Spherical harmonics form a set of functions that can represent any function f on the sphere $\mathbb{S}^2$ under mild conditions :

$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m} Y_{\ell m}(\theta, \varphi), \quad (1.19)$$

where $Y_{\ell m}(\theta, \varphi)$ denote the spherical harmonics, and $c_{\ell m} \in \mathbb{C}$ are the associated expansion coefficients.

We can additionally define the real spherical harmonics, from which any real-valued function f on $\mathbb{S}^2$ can expand, under mild conditions, as follows:

$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}^{\text{real}}(\theta, \varphi), \quad (1.20)$$

for $a_{\ell m} \in \mathbb{R}$.

Importantly, spherical harmonics are *homogeneous harmonic polynomials* defined on $\mathbb{R}^3$ and subsequently *restricted* to the sphere $\mathbb{S}^2$. We define the set of *homogeneous harmonic polynomials* of degree ℓ restricted to $\mathbb{S}^2$ as:

$$\mathcal{H}_{\ell}(\mathbb{S}^2) = \{p \in \mathbb{C}[x_1, x_2, x_3] : p(\lambda \mathbf{x}) = \lambda^{\ell} p(\mathbf{x}) \text{ and } \Delta p(\mathbf{x}) = 0, \forall \mathbf{x} \in \mathbb{S}^2\}. \quad (1.21)$$

We obtained that any polynomial $p \in \mathcal{H}_{\ell}(\mathbb{S}^2)$ can be represented as a linear combination exclusively of spherical harmonics of degree ℓ :

$$p(\boldsymbol{\xi}(\theta, \varphi)) = \sum_{m=-\ell}^{\ell} c_{\ell m} Y_{\ell m}(\theta, \varphi), \quad \boldsymbol{\xi} \in \mathbb{S}^2, \quad (1.22)$$

This result, although seemingly technical, is important in the subsequent chapter.

Fig. 1.5 illustrate examples of these real spherical harmonics. The **degree** ℓ controls how fine the pattern is : $\ell = 0$ is a constant signal, $\ell = 1$ splits the sphere into two big halves, $\ell = 2$ into three bands, and so on—higher ℓ means thinner and finer bands. The **moment** m determines how many times the pattern repeats around the equator. For instance, if $m = 0$, the bands run all the way around; if $m \neq 0$ the same shape appears several times, arranged like equal pizza slices. In other words, the degree ℓ sets the overall resolution of the pattern, while the moment m controls how often that detail is copied around the globe.

A convenient—employed in this work—method for summarizing a signal through its spherical harmonic content is illustrated by the spherical harmonic triangle, depicted in Fig. 1.6. In this representation, rows correspond to the degree ℓ , and columns correspond to the moments m . Each cell of the triangle holds the coefficient $c_{\ell m}$ from the spherical harmonic expansion given by (1.19). The triangular structure arises naturally from the properties of spherical harmonics, as each row contains $2\ell + 1$ coefficients, corresponding to moments m ranging from $-\ell$ to ℓ .

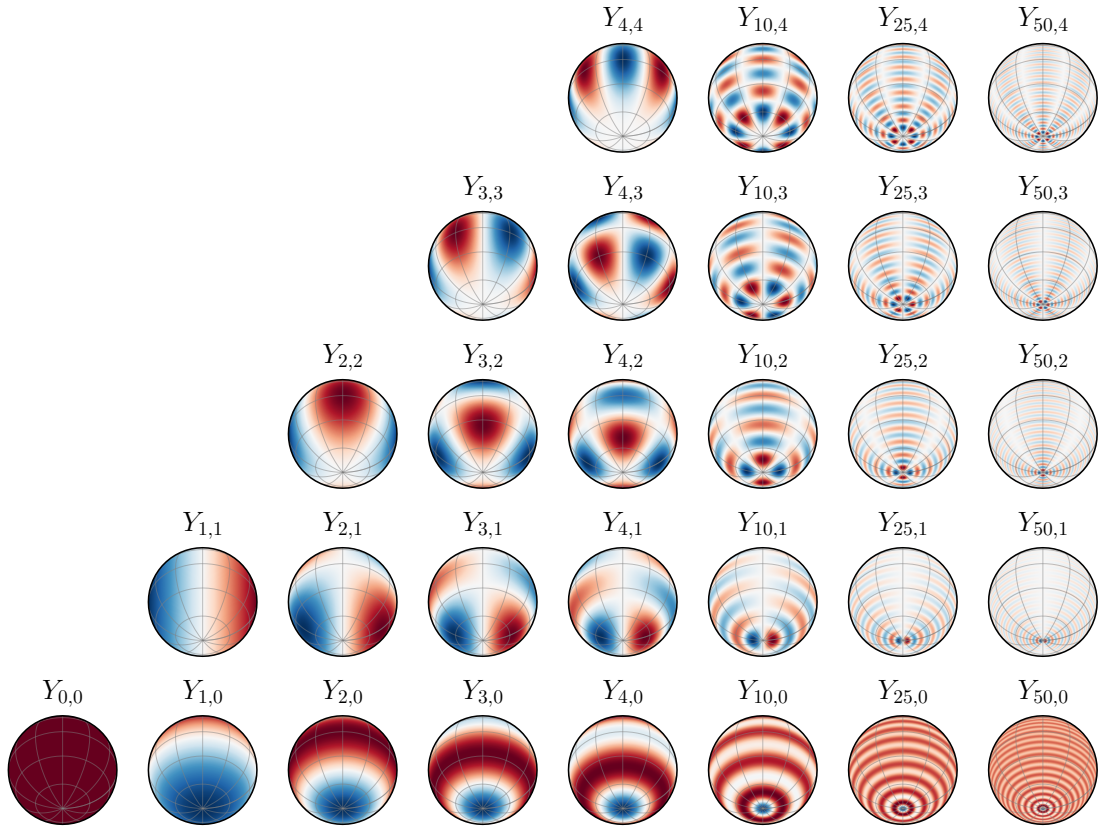


Figure 1.5: Sample of real spherical harmonics

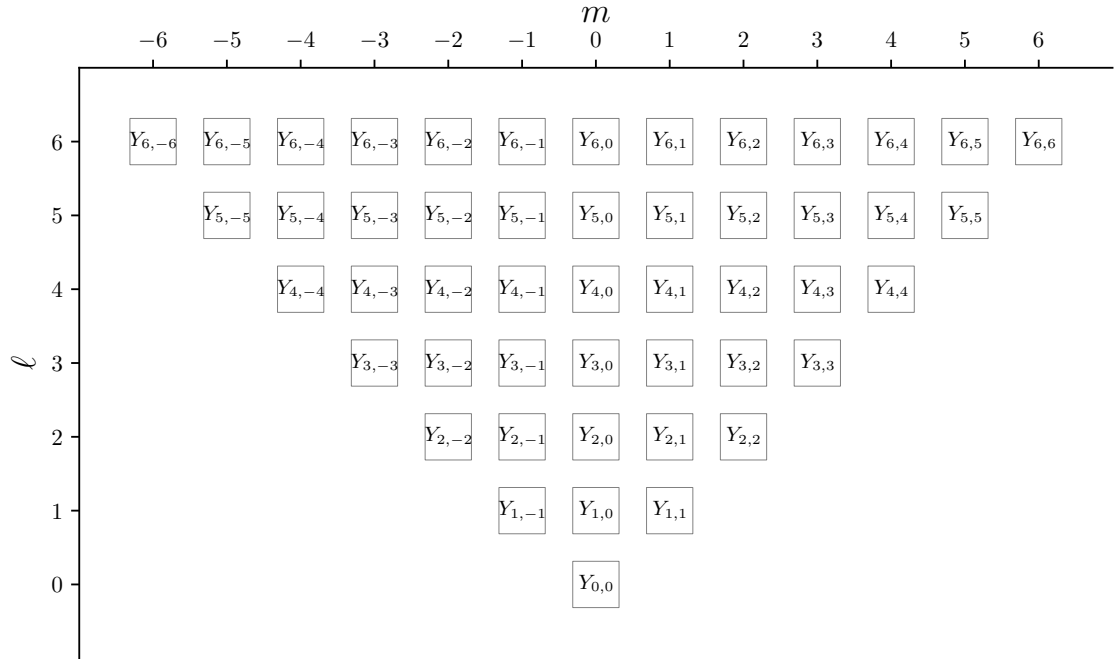


Figure 1.6: Visualization of the spherical harmonic triangle for an expansion of maximum degree $L = 6$.

Chapter 2

Spherical Implicit Neural Representations

This chapter extends the work presented in [26], which has been peer-reviewed and accepted for publication at the SAMPTA 2025 Conference.

Despite the success of INRs in Euclidean settings, extending these architectures to spherical domains poses challenges, as they are prone to generating artifacts, particularly near the poles. The primary difficulty lies in defining an appropriate positional encoding that inherently respects spherical geometry. Such an encoding is crucial, as it subsection ensures the consistency and well-posedness of the entire architecture on the spherical domain.

Interestingly, this challenge has also emerged within the rapidly growing field of geospatial artificial intelligence (GeoAI). Broadly speaking, GeoAI applies machine learning techniques to *geospatial data* across diverse applications such as weather forecasting, air-quality mapping, urban dynamics analysis and image geolocalization [27–30] to name a few. Since geospatial data naturally reside on spherical domains—most commonly Earth—this has motivated the development of specialized architectures and techniques specifically tailored for spherical geometry.

Among the architectures introduced in GeoAI, the concept of *single-point location encoders* closely aligns, from an architectural point of view, with the definition of INRs. Mai et al. [31] defined these encoders as:

$$Enc(\mathbf{x}) = \text{MLP}(\psi(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^3, \quad (2.1)$$

where the positional encoding $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}^m$ and the multilayer perceptron $\text{MLP} : \mathbb{R}^m \rightarrow \mathbb{R}^k$. However, in practice, these encoders are not restricted to INR tasks alone and often serve as input for subsequent machine learning models.

In this context, Mai et al. [32] proposed several variants of location encoders specifically tailored for spherical data. Their approach is based on Double-Fourier Sphere (DFS) [33, 34], which is known as pseudo-spectral method originally developed to overcome computational bottlenecks associated with spherical harmonic transforms. Building upon DFS-based features, they introduced multiple encoders, namely SphereC, SphereC+, SphereM, and SphereM+, demonstrating strong performance

across various GeoAI tasks.

Nonetheless, all these encoders inherently rely on the DFS idea [33, 34], a technique that remaps a function defined on spherical coordinates ($\mathbb{S}^2$) onto a doubly-periodic rectangular domain, enabling the application of a 2-D Fourier series. While conceptually appealing, DFS features are built from the basis functions of a method that inherently does not fully account for the spherical geometry, fundamentally undermining the encoder’s ability to capture true spherical relationships.

To more faithfully respect spherical geometry, Rußwurm et al. [35] replaced DFS with spherical harmonics, resulting in a continuous, differentiable representation on the sphere, referred to as SPH-SIREN. Formally, they introduced the following spherical harmonic positional encoding:

$$(\psi_{\ell,m}(\theta, \varphi))_{\ell=0, |m| \leq \ell}^{\ell=L_{\text{pe}}} = Y_{\ell,m}^{\text{real}}(\theta, \varphi) \quad (2.2)$$

with $m = (L_{\text{pe}} + 1)^2$ encoding dimensions. In [35], SPH-SIREN demonstrated strong performance across INR tasks as well as spherical regression and classification.

However, despite outperforming DFS-based methods, SPH-SIREN suffers from significant computational drawbacks. Explicit evaluations or recursive computations of spherical harmonics become computationally intensive and numerically unstable at higher orders—issues that originally motivated the development of DFS.

To address these limitations, we propose Herglotz-NET (HNET), a novel implicit neural representation specifically designed for spherical data, eliminating the explicit computation of spherical harmonics. HNET introduces a new positional encoding based on the exponential of a Herglotz mapping [36, Sec. VII.7], providing a *harmonic* encoding on the sphere. Further, we generalize aspects of the spectral analysis from Euclidean INRs presented in [14] to the spherical setting, offering new insights into the expressive capabilities of both HNET and SPH-SIREN as functions of network depth. Finally, we present a comprehensive evaluation of HNET through three experiments, demonstrating that it achieves comparable performance to SPH-SIREN while significantly reducing computational costs.

2.1 An Alternative to Spherical Harmonics

In Sec. 1.2, we introduced the concept of spherical harmonics and showed that solving the associated eigenvalue problem yields a closed-form expression for these functions. While this method is widely employed, an alternative, lesser-known approach also exists, which independently recovers spherical harmonics and their properties (see, e.g., [37]). The fundamental idea underlying this method, which we refer to as *Herglotz formalism*¹, serves as a key building block in our proposed Herglotz-NET architecture.

¹Named after Gustav Herglotz, whose unpublished notes inspired the definition of a generating function for spherical harmonics, as highlighted by W. Magnus in an appendix to the book by Courant and Hilbert[36].

2.1.1 Herglotz Formalism

The Herglotz formalism [36, Sec. VII.7] allows one to construct harmonic functions on $\mathbb{R}^3$ from any analytic function, by simply evaluating them along a special linear form. More precisely, let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function and let $\mathbf{a} \in \mathbb{C}^3$ be a vector satisfying the *Herglotz condition*

$$\mathbf{a}^\top \mathbf{a} = 0 \quad (2.3)$$

(for instance, one may take $\mathbf{a} = (1, i, 0)$), and let us define the function

$$g(\mathbf{x}) : \mathbb{R}^3 \rightarrow \mathbb{C} : \mathbf{x} \mapsto f(\mathbf{a}^\top \mathbf{x}). \quad (2.4)$$

From that, a direct calculation shows :

$$\Delta g(\mathbf{x}) = \nabla \cdot \nabla f(\mathbf{a}^\top \mathbf{x}) = f''(\mathbf{a}^\top \mathbf{x}) (\mathbf{a}^\top \mathbf{a}) = 0, \quad (2.5)$$

so that g is harmonic.

Let us investigate the particular function $g_\ell(\mathbf{x}) = (\mathbf{a}^\top \mathbf{x})^\ell$. By construction, this function is : harmonic due to (2.5), a degree ℓ polynomial, and homogeneous of degree ℓ . Thus, g_ℓ lies precisely in the space $\mathcal{H}_\ell(\mathbb{S}^2)$, allowing it to be expanded uniquely in terms of the spherical harmonics of degree ℓ , which form an orthogonal basis of $\mathcal{H}_\ell(\mathbb{S}^2)$:

$$g_\ell(\boldsymbol{\xi}) = \sum_{m=-\ell}^{\ell} c_{\ell m} Y_{\ell m}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \mathbb{S}^2. \quad (2.6)$$

Thus, the simple function g_ℓ implicitly contains all (or at least a subset of) spherical harmonics of degree ℓ . To follow the formalism of Sec. 1.1.2, the *spectral content* of g_ℓ includes (a subset of) the spherical harmonics of degree ℓ .

Building on this observation, we can now extend this property and show that we can define functions whose spectral content contains harmonics from multiple degrees. This is formalized in the following lemma.

Lemma 2.1. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function, such that*

$$f(z) = \sum_{\ell=0}^{\infty} c_\ell z^\ell, \quad \forall z \in \mathbb{C}, \quad (2.7)$$

for some $c_\ell \in \mathbb{C}$. Let $\mathbf{a} \in \mathbb{C}^3$ satisfy $\mathbf{a}^\top \mathbf{a} = 0$, then the function $g(\mathbf{x}) = f(\mathbf{a}^\top \mathbf{x})$ is harmonic on $\mathbb{R}^3$ and, when restricted to the sphere $\mathbb{S}^2$, admits the expansion

$$g(\boldsymbol{\xi}(\theta, \varphi)) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m}(\mathbf{a}) Y_{\ell m}(\theta, \varphi), \quad \boldsymbol{\xi} \in \mathbb{S}^2. \quad (2.8)$$

for some $c_{\ell m}(\mathbf{a}) \in \mathbb{C}$

Proof. see App. B.1 □

Remark, however, that the spectral content of a *single* function $f(\mathbf{a}^\top \cdot)$ is insufficient to faithfully encode all spherical harmonics. For instance, by choosing $\mathbf{a} = (1, i, 0)$, one can verify that all moments with $m \neq 0$ vanish, causing the omission of many spherical harmonics. Similarly, selecting $f(\cdot) = \sin(\cdot)$ excludes all even-degree harmonics from the expansion in (2.8). Consequently, to obtain a complete and diverse spectral content, it is essential to use a constellation of functions $f(\mathbf{a}_1^\top \cdot), f(\mathbf{a}_2^\top \cdot), \dots, f(\mathbf{a}_m^\top \cdot)$, each contributing uniquely to the harmonic information.

This insight forms the basis of the proposed Herglotz-Net. Rather than explicitly relying on spherical harmonics in the positional encoding, as SPH-SIREN does, Herglotz-Net utilizes multiple Herglotz vectors $\{\mathbf{a}_i\}_{i=1}^m$ combined with a suitably chosen analytic function f . By Lem. 2.1, each atom $g(\mathbf{a}_i^\top \cdot)$ implicitly encodes a subset of spherical harmonics. Therefore, increasing the number of atoms—each associated with a distinct Herglotz vector $\mathbf{a}_i$ —enriches the positional encoding:

$$\psi(\boldsymbol{\xi}) = \left(f(\mathbf{a}_i^\top \boldsymbol{\xi}) \right)_{i=1}^m, \quad \boldsymbol{\xi} \in \mathbb{S}^2, \quad (2.9)$$

allowing it to capture spectral content comparable—potentially bigger—than the original positional encoding of SPH-SIREN, and this without ever computing any explicit expression of spherical harmonics.

The two subsequent subsections focus on the construction of the Herglotz vectors and the choice of the analytic function f .

2.1.2 Herglotz Vectors

We now turn our attention to the construction of Herglotz vectors. To do so, let us decompose $\mathbf{a} = \mathbf{a}_{\Re} + i\mathbf{a}_{\Im}$ where $\mathbf{a}_{\Re}, \mathbf{a}_{\Im} \in \mathbb{R}^3$, and in that form, the Herglotz condition reads :

$$\mathbf{a}^T \mathbf{a} = \|\mathbf{a}_{\Re}\|^2 - \|\mathbf{a}_{\Im}\|^2 + i2\mathbf{a}_{\Re}^T \mathbf{a}_{\Im} = 0 \quad (2.10)$$

Thus, imposing $\|\mathbf{a}_{\Re}\|^2 = \|\mathbf{a}_{\Im}\|^2$ and $\mathbf{a}_{\Re}^T \mathbf{a}_{\Im} = 0$ allows us to construct such vectors. For convenience, we will also require that $\|\mathbf{a}\| = 1$. This gives us 3 equations for a total of 6 unknowns (6 coordinates) ; it means we have 3 free parameters that can be randomly drawn. Therefore, we suggest the following process to randomly generate a Herglotz vector :

Algorithm 1 Generate a Herglotz vector

- 1: **procedure** GENERATEHERGLOTZVECTOR
 - 2: $\mathbf{a}_{\Re} \sim \mathcal{N}(0, I_3)$
 - 3: $\mathbf{a}_{\Re} \leftarrow \mathbf{a}_{\Re} / (\sqrt{2} \|\mathbf{a}_{\Re}\|)$
 - 4: $\mathbf{a}_{\Im} \sim \mathcal{N}(0, I_3)$
 - 5: $\mathbf{a}_{\Im} \leftarrow \mathbf{a}_{\Im} - 2(\mathbf{a}_{\Im} \cdot \mathbf{a}_{\Re}) \mathbf{a}_{\Re}$ ▷ Orthogonalize to real part
 - 6: $\mathbf{a}_{\Im} \leftarrow \mathbf{a}_{\Im} / (\sqrt{2} \|\mathbf{a}_{\Im}\|)$
 - 7: **return** $\mathbf{a}_{\Re} + i\mathbf{a}_{\Im}$
 - 8: **end procedure**
-

2.1.3 Herglotz Atoms

We chose the analytic function f based on the known property that the exponential function $\mathbf{x} \mapsto \exp(\mathbf{a}^T \mathbf{x})$ is a *generating* function of all spherical harmonics sometimes referred to as the Herglotz generating function [36]. Motivated by this, we introduce the following class of functions:

$$g(\boldsymbol{\xi}) = \exp\left(\omega_0(\mathbf{a}^T \boldsymbol{\xi})\right), \quad \boldsymbol{\xi} \in \mathbb{S}^2, \quad (2.11)$$

where $\omega_0 > 0$. Although these functions are not the final atoms used in our positional encoding, they serve as a baseline for their construction and enable a qualitative analysis of their properties.

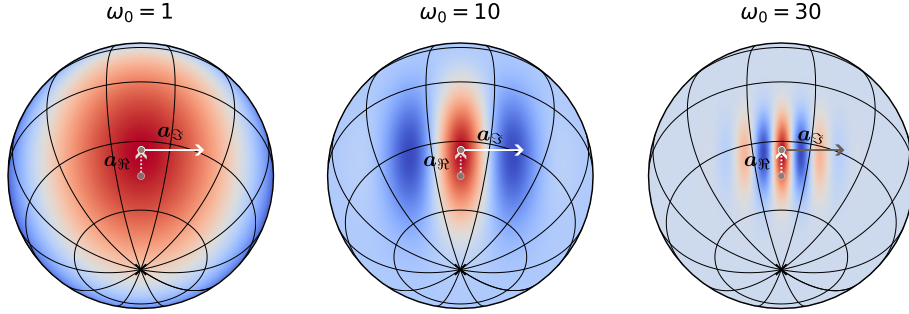


Figure 2.1: Representation of the real part of the Herglotz atom $g(\mathbf{x})$ for $\mathbf{a} = (1, 1, 1)/\sqrt{3} + i(1, -1, 0)/\sqrt{2}$, and different values of ω_0 . The function g is centered on $\mathbf{a}_{\Re} = (1, 1, 1)/\sqrt{3}$ with oscillations locally oriented along the direction $\mathbf{a}_{\Im} = (1, -1, 0)/\sqrt{2}$ and frequency proportional to ω_0 .

Decomposing the complex vector as $\mathbf{a} = \mathbf{a}_{\Re} + i\mathbf{a}_{\Im}$, one can write

$$g(\boldsymbol{\xi}) = \exp(\omega_0 \mathbf{a}_{\Re} \cdot \boldsymbol{\xi}) \cdot \left[\cos(\omega_0 \mathbf{a}_{\Im} \cdot \boldsymbol{\xi}) + i \sin(\omega_0 \mathbf{a}_{\Im} \cdot \boldsymbol{\xi}) \right], \quad \boldsymbol{\xi} \in \mathbb{S}^2. \quad (2.12)$$

From this expression (illustrated in Fig. 2.1), we observe that the function is centered around the direction given by $\mathbf{a}_{\Re}/\|\mathbf{a}_{\Re}\|$, with oscillations locally oriented along the direction of $\mathbf{a}_{\Im}$. Its amplitude reaches a maximum of $|\exp(\omega_0 \mathbf{a}_{\Re}^T \mathbf{x})|$ in directions aligned with $\mathbf{a}_{\Re}$. Additionally, the parameter ω_0 simultaneously modulates the frequency of oscillations and the overall magnitude of the function. Specifically, for points $\mathbf{x} \in \mathbb{S}^2$ oriented opposite to $\mathbf{a}_{\Re}$ (i.e., where $\mathbf{a}_{\Re}^T \mathbf{x} < 0$), the amplitude decreases exponentially. Thus, increasing ω_0 concentrates the function around a specific direction. This observation also shows why it is important to have multiple Herglotz vectors at different locations.

We finally have all the ingredients to define the positional encoding of Herglotz-Net, and it will be presented in the next section.

2.2 Herglotz-NET

In this section, we define Herglotz-Net, an alternative INR for spherical data, that preserves SIREN’s flexibility and scalability while providing a well defined framework

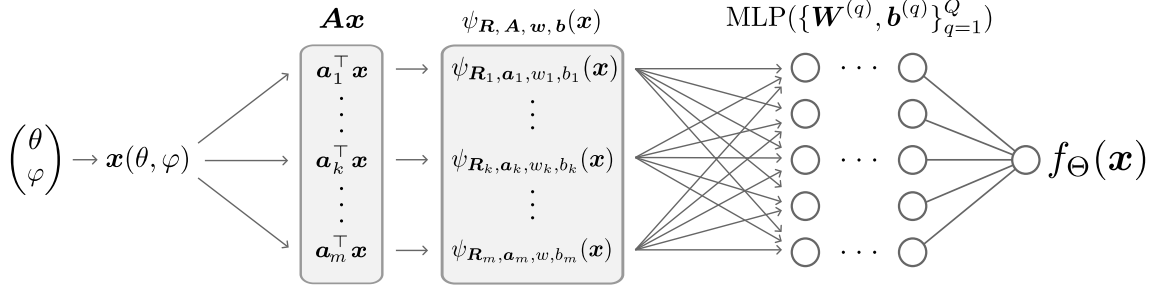


Figure 2.2: The Herglotz-NET architecture using the m -neuron PE defined by (2.13).

on $\mathbb{S}^2$. Let us define the following positional encoding,

$$\psi_{\mathbf{R}, \mathbf{A}, \mathbf{w}, \mathbf{b}}(\mathbf{x}) := \left\{ \psi_{\mathbf{R}_i, \mathbf{a}_i, w_i, b_i}(\mathbf{x}) = \exp\left\{w_i/\varpi_0 \left[(\mathbf{R}_i \mathbf{a}_i)^\top \mathbf{x} + b_i - 1/\sqrt{2}\right]\right\} \right\}_{i=1}^m \quad (2.13)$$

wherein $\mathbf{A} \in \mathbb{C}^{3 \times m} = (\mathbf{a}_1, \dots, \mathbf{a}_m)^T$ are m Herglotz vector, $\mathbf{w} := (w_1, \dots, w_m) \in \mathbb{R}^m$ are trainable weights, $\varpi_0 \in \mathbb{R}$ fixed frequency scale parameter, $\mathbf{b} := (b_1, \dots, b_m) \in \mathbb{C}^m$ are trainable biases and $\mathbf{R} = \text{diag}(\mathbf{R}_1, \dots, \mathbf{R}_m)$ with $\mathbf{R}_i \in \mathbb{R}^{3 \times 3}$ are learnable rotation matrices.

The positional encoding thus encodes any location $\mathbf{x}(\theta, \varphi) \in \mathbb{S}^2$ where $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi)$. Remark that due to Lem. 2.1, the atoms if not restricted to $\mathbb{S}^2$ are harmonic functions on $\mathbb{R}^3$. The positional encoding is subsequently fed to a MLP with periodic activations $\sigma^{(q)}(\cdot) = \sin(\cdot)$ as for SIREN [6] (see Fig. 2.2).

You can find a brief summary of the parameters involved in the proposed positional encoding in Tab. 2.1. In what follows, we will further develop the properties of the positional encoding and discuss practical aspects related to the model's implementation.

2.2.1 Spectral Analysis

We start with a qualitative analysis of the spectral properties of Herglotz atoms and compare them with SIREN and SPH-SIREN. Subsequently, we provide a more detailed characterization of their spectral behavior.

Let us assume without loss of generality that $\mathbf{R}_i = \mathbf{I}_3$ and $b_i = 0$ for all i . Expanding a single atom using its Taylor series gives:

$$\exp\left(\frac{w_i}{\varpi_0} \mathbf{a}_i^\top \mathbf{x}\right) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{w_i^\ell}{\varpi_0^\ell} (\mathbf{a}_i^\top \mathbf{x})^\ell, \quad \mathbf{x} \in \mathbb{S}^2. \quad (2.14)$$

From Eq. (2.6), we know each monomial $(\mathbf{a}_i^\top \mathbf{x})^\ell$ expands exclusively into spherical harmonics of degree ℓ . Consequently, each term in the Taylor series corresponds to a distinct spherical harmonic frequency band. The scaling parameters w_i and ϖ_0 determine which frequency band is emphasized, while the factorial term $1/\ell!$ ensures damping of higher-order terms, effectively limiting the atom's bandwidth.

Let us now briefly compare HNET with the positional encodings used by SIREN and SPH-SIREN. Recall SIREN's positional encoding is given by:

$$\psi_{\sin}(\mathbf{x}) = \left(\sin(\omega_0 \mathbf{w}_i^T \mathbf{x} + b_i) \right)_{i=1}^m, \quad (2.15)$$

Parameter	Description
w_i	Controls the frequency content of the positional encoding. Higher values increase the frequency content by emphasizing higher order spherical harmonics.
ϖ_0	A scaling factor for the frequencies of the positional encoding. Allows to directly control the overall frequency content of the PE; it has the opposite effect on w_i when increased.
$1/\sqrt{2}$	A scaling factor used to prevent excessively large outputs from atoms at high frequencies; optional if frequencies remain low.
$\mathbf{a}_i$	Determines the harmonic properties of the encoding, ensuring it remains harmonic and well posed on the sphere ($\mathbb{S}^2$).
$\mathbf{R}_i$	Rotation matrices that allow the model to adaptively position the encoding atoms, without altering the Herglotz condition. This gives the opportunity for the model to learn the Herglotz vectors that are best suited for the task at hand.
b_i	Bias parameters with two roles: the real part ($b_{i,\Re}$) controls the amplitude of the encoding, while the imaginary part ($b_{i,\Im}$) adjusts the phase of oscillations.

Table 2.1: Overview of the parameters used in the positional encoding.

where increasing frequencies within the positional encoding can be achieved simply by scaling either ω_0 , the vectors $\mathbf{w}_i$, or both. Importantly, we emphasize that adjusting the frequency in this manner introduces *no additional computational cost*, yet it is fundamental to the success of SIREN. In contrast, for SPH-SIREN, increasing the frequency (i.e., raising the frequency ℓ) requires explicitly providing spherical harmonics of the corresponding higher frequency. This leads to a considerable computational burden, as the cost of evaluating spherical harmonics grows rapidly with ℓ . HNET, similar to SIREN, allows efficient adjustment of frequency content by tuning parameters w_i and ϖ_0 at no additional computational expense. However, this flexibility has its limitations: while a desired frequency band can be emphasized, adjacent frequency bands remain inevitably included, and these parameters alone do not provide control over isolating specific moments m .

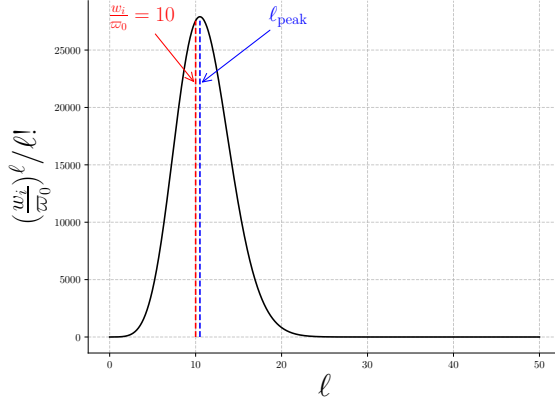


Figure 2.3: Evolution of the coefficients $(\frac{w_i}{\varpi_0})^\ell / \ell!$ in (2.14) for $\frac{w_i}{\varpi_0} = 10$; our approximation make (2.17) slightly underestimates the true ℓ_{peak} value.

We now characterize more precisely the relationship between parameters ϖ_0 , w_i , and the dominant frequency band ℓ_{peak} in the expansion (2.14). We define $\alpha = \frac{w_i}{\varpi_0}$, and we consider the function:

$$\gamma(\ell) = \frac{\alpha^\ell}{\ell!}. \quad (2.16)$$

The dominant band ℓ_{peak} is the one that maximizes $\gamma(\ell)$.

By setting the derivative of $\log \gamma(\ell)$ with respect to ℓ to zero and applying Stirling's approximation, we obtain:

$$\log \alpha = \log \ell_{\text{peak}} + \frac{1}{2\ell_{\text{peak}}} + \mathcal{O}(1/\ell_{\text{peak}}^2).$$

This equation typically requires numerical solutions. However, assuming ℓ_{peak} sufficiently large to neglect the $1/(2\ell_{\text{peak}})$ term, we obtain a simpler and practical approximation of the peak frequency:

$$\ell_{\text{peak}} \approx \frac{w_i}{\varpi_0}. \quad (2.17)$$

Additionally, we can perform a qualitative analysis of the behavior of $\gamma(\ell)$ around

ℓ_{peak} . Specifically, we compute the second derivative of $\log \gamma(\ell)$ evaluated at ℓ_{peak} :

$$\log \gamma(\ell) \approx \log \gamma(\ell_{\text{peak}}) + \frac{1}{2} \frac{d^2}{d\ell^2} \log \gamma(\ell_{\text{peak}}) (\ell - \ell_{\text{peak}})^2, \quad (2.18)$$

Evaluating the second derivative gives:

$$\frac{d^2}{d\ell^2} \log \gamma(\ell_{\text{peak}}) = -\frac{1}{\ell_{\text{peak}}} + \mathcal{O}(1/\ell_{\text{peak}}^2).$$

Neglecting higher-order terms, we have:

$$\frac{\gamma(\ell)}{\gamma(\ell_{\text{peak}})} \approx \exp \left(-\frac{1}{2w_i/\varpi_0} \left(\ell - \frac{w_i}{\varpi_0} \right)^2 \right). \quad (2.19)$$

where we substituted $\ell_{\text{peak}} \approx \frac{w_i}{\varpi_0}$. Thus, $\gamma(\ell)$ approximately follows a Gaussian distribution centered at $\frac{w_i}{\varpi_0}$ with variance $\sigma^2 = \frac{w_i}{\varpi_0}$. The function $\gamma(\ell)$ increases up to ℓ_{peak} and then decreases afterward (see Fig. 2.3), with negligible contributions at the tails of the distribution.

2.2.2 Initialization

Below, we describe the initialization strategies for the different components of our model.

MLP Initialization For the MLP, since we are using sinusoidal activation functions, we adopt the initialization scheme proposed in SIREN [6].

PE Initialization Similar to SPH-SIREN, we aim to activate all spherical harmonic modes up to a specified order L_{pe} . Therefore, given L_{pe} , we initialize $(L_{\text{pe}} + 1)^2$ atoms, matching the total number of spherical harmonics up to that order. Based on our earlier analysis, we concluded that the peak harmonic degree approximately satisfies $\ell_{\text{peak}} \approx w_i/\varpi_0$. Hence, to explicitly target a desired frequencies ℓ_{target} at the i -th atom, we set $w_i = \ell_{\text{target}}$. The parameter ϖ_0 remains user-defined, providing flexible control to adjust the overall frequency content of the positional encoding. The process is formalized on Alg. 2.

Algorithm 2 Initialize exponents for the Herglotz positional encoding

```

1: procedure INITEXPONENTS( $L_{\text{pe}}$ )
2:    $\mathbf{w} \leftarrow []$ 
3:   for  $\ell = 0, \dots, L$  do
4:     for  $m = -\ell, \dots, \ell$  do
5:        $\mathbf{w} \leftarrow \mathbf{w} \cup \{\ell\}$ 
6:     end for
7:   end for
8:   return  $\mathbf{w}$ 
9: end procedure

```

Although many weights initially share identical values, diversity arises from distinct Herglotz vectors. Fig. 2.4 visually compares positional encoding spectra with and without this initialization. Note that the resulting positional encoding contains $\mathcal{O}(L^2)$ atoms and may introduce computational complexity at high L ; fewer atoms can be employed without severely limiting the model.

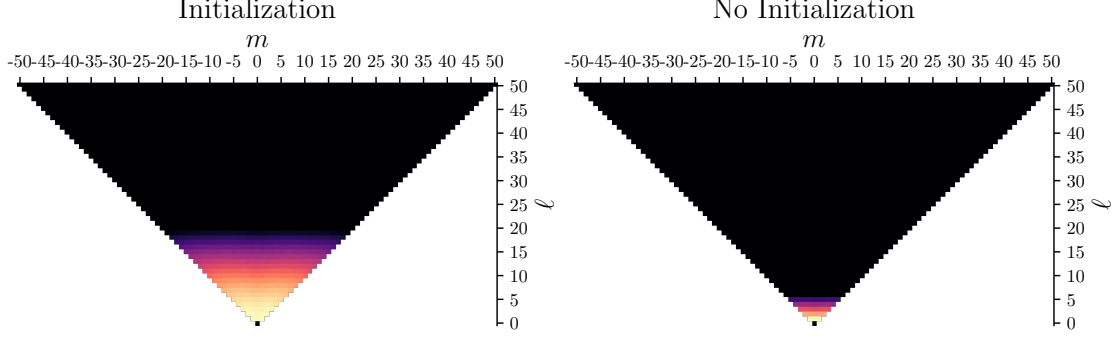


Figure 2.4: The plot compares the *superposed* spectrum of every atom of the positional encoding with (left) and without (right) the initialization for $L = 20$, that is, 21^2 atoms. Right has PE weights $\mathbf{w}$ distributed as $\sim \mathcal{U}(-1/n, 1/n)$ where n is the input dimension. Dark color indicates $|c_{\ell m}| < 10^{-12}$.

Normalization For large w_i value, the exponential function may yield excessively large values in the PE (see Eq. (2.13)) which the network struggles to process effectively. To mitigate this, we introduce a normalization factor of $1/\sqrt{2}$. We therefore have

$$\max_{\mathbf{x} \in \mathbb{S}^2} [\psi_{\mathbf{R}, \mathbf{A}, \mathbf{w}, \mathbf{b}}(\mathbf{x})]_i = \exp(w_i \Re\{b_i\}) \quad (2.20)$$

At initialization, we set $\Re\{b_i\} = 0, \forall i$, such that every atom is initially bounded by 1; during training, the model can adjust biases to control atom magnitudes. For the imaginary part, we set $\Im\{b_i\} \sim \mathcal{U}(0, 2\pi), \forall i$, inducing uniform random phase shifts.

Herglotz Vector Initialization As mentioned previously, initialization diversity primarily emerges from distinct Herglotz vector configurations. Despite the localized nature of each atom (see Fig. 2.1), one can show that, in expectation, no atom favors a specific point or direction. This result is formalized in the following lemma:

Lemma 2.2. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ and define*

$$g : \mathbb{R}^3 \rightarrow \mathbb{C}, \quad g(\mathbf{x}) = \mathbb{E}_{\mathbf{a} \sim D} f(\mathbf{a}^\top \mathbf{x}),$$

where D is the law of the vector returned by Alg. 1. Then g is $\text{SO}(3)$ -invariant: for every $R \in \text{SO}(3)$,

$$g(R\mathbf{x}) = g(\mathbf{x}).$$

Proof. see App. B.2 □

Consequently, vectors are generated according to Alg. 1, ensuring $\text{SO}(3)$ -invariance.

Rotation Initialization Rotations drawn uniformly as $\mathbf{R}_i \sim \mathcal{U}(\text{SO}(3))$ preserve $\text{SO}(3)$ -invariance. Nonetheless, for simplicity in our experiments, we initialize all rotations to the identity matrix.

2.2.3 Practical Implementation

We briefly describe the implementation details of the network. Note that the proposed model, along with SIREN, SPH-SIREN, and various other variants and tools, is available in the dedicated toolbox developed for this work, `spherical-inr`².

In practical scenarios, we typically work with real-valued multilayer perceptrons (MLPs). Hence, we take the imaginary part of (2.13), which yields :

$$\begin{aligned} \Im\{\psi_{\mathbf{R}_i, \mathbf{a}_i, w_i, b_i}(\mathbf{x})\} = & \exp\left\{w_i/\varpi_0 \left[(\mathbf{R}_i \mathbf{a}_{i,\Re})^\top \mathbf{x} - 1/\sqrt{2}\right] + b_{i,\Re}\right\} \\ & \cdot \sin\left\{w_i/\varpi_0 \left[(\mathbf{R}_i \mathbf{a}_{i,\Im})^\top \mathbf{x}\right] + b_{i,\Im}\right\} \end{aligned} \quad (2.21)$$

where $\mathbf{a}_{i,\Re} = \Re\{\mathbf{a}_i\}$, $\mathbf{a}_{i,\Im} = \Im\{\mathbf{a}_i\}$, $\mathbf{b}_{i,\Re} = \Re\{\mathbf{b}_i\}$, and $\mathbf{b}_{i,\Im} = \Im\{\mathbf{b}_i\}$. Note that the initialization for the imaginary bias $b_{i,\Im} \sim \mathcal{U}(0, 2\pi)$, $\forall i$ makes the choice of the real or imaginary part arbitrary.

For rotation matrices, we do not explicitly construct them. Instead, we utilize quaternion rotations, which offer computational and numerical advantages such as reduced computational complexity and better numerical stability. While a full introduction to quaternions is beyond the scope of this thesis, the interested reader is referred to [38]. For our purposes, it is sufficient to note that a quaternion representing a rotation is defined as a four-dimensional vector:

$$\mathbf{q} = (q_0, q_1, q_2, q_3) \in \mathbb{R}^4.$$

A quaternion rotation transforms a vector $\mathbf{v} \in \mathbb{R}^3$ into a rotated vector $\mathbf{v}' \in \mathbb{R}^3$ through the following operation:

$$\mathbf{v}' = \frac{\mathbf{q}}{\|\mathbf{q}\|} \begin{pmatrix} 0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \frac{\mathbf{q}^{-1}}{\|\mathbf{q}\|}, \quad (2.22)$$

where $\mathbf{q}^{-1}$ denotes the quaternion inverse. Using quaternion arithmetic, this operation can be simplified to:

$$\mathbf{v}' = \mathbf{v} + 2(\mathbf{q}_v \times (\mathbf{q}_v \times \mathbf{v}) + q_0(\mathbf{q}_v \times \mathbf{v})), \quad (2.23)$$

where $\mathbf{q}_v = (q_1, q_2, q_3)^\top$. Quaternions are treated as learnable parameters within the network. Note that the identity rotation is given by $\mathbf{q}_I = (1, 0, 0, 0)$, which is the initialization state of all the quaternions.

²<https://pypi.org/project/spherical-inr>

2.3 Spectral Properties

In this section, building upon the work of Yüce et al. [14], we analyze the frequency expansion properties of spherical-based INRs, specifically HNET and SPH-SIREN. Our goal is to generalize the spectral expansion theorem (Th. 1.1), providing theoretical insights into the strong empirical performance observed with SPH-SIREN.

Consistent with the assumptions in Th. 1.1, we assume that both the exponential functions in the HNET positional encoding (2.13) and the subsequent MLP activations in HNET and SPH-SIREN can be approximated accurately by truncated Taylor series. Specifically, we approximate the exponential by

$$\exp(z) \approx \sum_{\ell=0}^{L_0} \frac{1}{\ell!} z^\ell,$$

for a certain truncation order L_0 , and the MLP activation functions as

$$\sigma(z) \approx \sum_{k=1}^K \alpha_k z^k.$$

Under these approximations, the output of the first layer after the positional encoding $\psi(\mathbf{x})$ in both HNET and SPH-SIREN can be represented as a finite spherical harmonic expansion. Specifically, for the j -th neuron, we have:

$$[\mathbf{W}^{(1)}\psi(\mathbf{x}) + \mathbf{b}^{(1)}]_j = \sum_{\ell=0}^{L_{\text{pe}}} \sum_{|m| \leq \ell} c_{j\ell m} Y_{\ell m}(\theta, \varphi) \quad (2.24)$$

for some $c_{j\ell m} \in \mathbb{C}$. This result holds trivially for SPH-SIREN, as its positional encoding explicitly consists of spherical harmonics up to order L_{pe} . For HNET, applying Lem. 2.1 to the truncated exponential series, we similarly have:

$$\psi_{\text{HNET}}(\mathbf{x}(\theta, \varphi)) = \left(\sum_{\ell=0}^{L_0} \sum_{|m| \leq \ell} \alpha_{j\ell m} Y_{\ell m}(\theta, \varphi) \right)_{j=0}^{(L_{\text{pe}}+1)^2} \quad (2.25)$$

from which expression (2.24) directly follows. Thus, each neuron in the first layer of both architectures shares the same spectral form as presented in (2.24).

To generalize the spectral theorem, we require a spectral *expansion property* that ensures frequency content expands when the output of a neuron is raised to a power. This essential ingredient is provided by the *contraction rule* [39, Sec. 5.8] of spherical harmonics, which states that the product of two spherical harmonics can be expanded as a finite linear combination of spherical harmonics:

$$Y_{\ell m}(\theta, \varphi) Y_{\ell' m'}(\theta, \varphi) = \sqrt{\frac{(2\ell+1)(2\ell'+1)}{4\pi}} \sum_{p=|\ell-\ell'|}^{\ell+\ell'} \sum_{k=-p}^p \kappa_{pk} Y_{pk}(\theta, \varphi), \quad (2.26)$$

where the coefficients κ_{pk} are explicitly given by:

$$\kappa_{pk} = (-1)^k \sqrt{2p+1} \begin{pmatrix} \ell & \ell' & p \\ m & m' & -k \end{pmatrix} \begin{pmatrix} \ell & \ell' & p \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.27)$$

The parentheses denote Wigner 3-j symbols (see [40]) and are non-zero only when:

- (i) $m \in \{-\ell, \dots, \ell\}$, $m' \in \{-\ell', \dots, \ell'\}$, $k \in \{-p, \dots, p\}$;
- (ii) $m + m' - k = 0$;
- (iii) $|\ell - \ell'| \leq p \leq \ell + \ell'$;
- (iv) $\ell + \ell' + p$ is even.

Thus, the frequency content of $Y_{\ell,m} Y_{\ell',m'}$ spans harmonics up to degree $\ell + \ell'$, i.e. a strictly larger range than the two original functions. Leveraging this property, we derive the following lemma detailing the spectral expansion:

Lemma 2.3. *Given a complex value polynomial $p_K(z) = \sum_{k=1}^K a_k z^k$ of degree $K \geq 0$ and an order L , for any spherical function $f(\theta, \varphi) := \sum_{\ell=0}^L \sum_{|m| \leq \ell} c_{\ell m} Y_{\ell m}(\theta, \varphi)$ (with $c_{\ell m} \in \mathbb{C}$) with bandlimit L , $p_K \circ f$ has a bandlimit KL , i.e., there exist coefficients $\kappa_{\ell m} \in \mathbb{C}$ such that*

$$p_K(f(\theta, \varphi)) = \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \kappa_{\ell m} Y_{\ell m}(\theta, \varphi). \quad (2.28)$$

Proof. see Appendix □

As an illustrative example, by Lem. 2.3, passing a single-frequency signal $Y_{\ell m}(\theta, \varphi)$ for some ℓ and m through an activation function truncated at order K yields:

$$\sigma^{(q)}(Y_{\ell m}(\theta, \varphi)) = \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \alpha_{\ell m} Y_{\ell m}(\theta, \varphi)$$

Thus, in terms of spectral content, we have expanded from a single mode (ℓ, m) to a set of modes given by $\{(\ell, m) : \ell = 0, \dots, L, |m| \leq \ell\}$. In other words, the spectrum has been *expanded*.

We now present our main theoretical result, generalizing the expressive capability of spherical INRs:

Theorem 2.1. *Let $f_{\Theta} : \mathbb{S}^2 \rightarrow \mathbb{C}$ be an INR defined as in (1.2) with PE $\psi(\mathbf{x})$ satisfying (2.24), $Q \geq 1$ layers and polynomial activation functions $\sigma^{(q)}(z) = \sum_{k=0}^K \alpha_k z^k$. Then, there exist coefficients $\kappa_{\ell m} \in \mathbb{C}$ such that*

$$f_{\Theta}(\theta, \varphi) = \sum_{\ell=0}^{K^{Q-1}L_0} \sum_{|m| \leq \ell} \kappa_{\ell m} Y_{\ell m}(\theta, \varphi) \quad (2.29)$$

Proof. see Appendix □

This theorem shows explicitly how the expressive power of spherical INRs, such as HNET and SPH-SIREN, depends on their positional encoding. Unlike SPH-SIREN, which explicitly encodes one spherical harmonic per atom up to order L , HNET offers a virtually infinite spectrum in its positional encoding, which is adaptable via learnable parameters, making HNET more flexible.

You can find an illustration of the spectral expansion of a HNET model on Fig. 2.5. We correctly observe that a linear activation doesn't modify the frequency content of the model. On the other hand, we can clearly observe the progressive expansion using the monomial x^2 as activation function. The sinus activation despite having a theoretically unbounded expansion has essentially an expansion similar to the degree 2 monomial, further validating our key assumption.

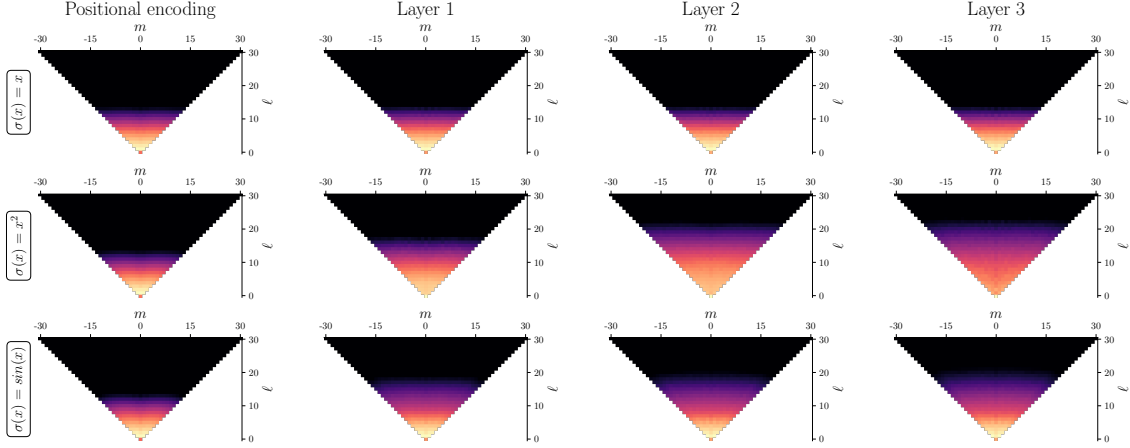


Figure 2.5: Superposed Spherical Harmonics spectrum of the atoms of a given layer (in column) for : (top) $\sigma(x) = x$, (middle) $\sigma(x) = x^2$, (bottom) $\sigma(x) = \sin(x)$. The model is a HNET architecture with $L = 10$, 3 hidden layers. Dark value indicates $|c_{\ell m}| < 10^{-12}$.

Toward a tighter theorem. It is worth noting that, in comparison to the Euclidean INR result of Th. 1.1, our spherical version (Th. 2.1) is inherently looser in one key respect: it only guarantees that all frequencies up to the maximal order appear, but it does not predict which of those frequencies actually have nonzero coefficients. This is because, without loss of generality, our proof relies on a looser *contraction rule*

$$Y_{\ell,m}(\theta, \varphi) Y_{\ell',m'}(\theta, \varphi) = \sum_{p=0}^{2L} \sum_{k=-p}^p \kappa_{pk} Y_{p,k}(\theta, \varphi), \quad (2.30)$$

which simply bounds the output band limit by $2L$ without distinguishing which κ_{pk} is zero. As mentioned above, the Wigner 3j symbols enforce strict selection rules. If one were to restrict every contraction to the true support :

$$\mathcal{H}(Y_{\ell_i m_i} Y_{\ell_j m_j}) = \left\{ (\ell, m) : |\ell_i - \ell_j| \leq \ell \leq \ell_i + \ell_j, \ell + \ell_i + \ell_j \text{ even}, m = m_i + m_j \right\}. \quad (2.31)$$

and then propagate that through each layer, one would obtain a much tighter bound on which spherical-harmonic coefficients can actually appear. However, tracking those index constraints through repeated nonlinear compositions leads to cumbersome notation and combinatorial case-splitting. We therefore decided not to go further in that direction for this work.

2.4 Assessments

In this section, we compare the performance of the three models introduced in this work: HNET, SPH-SIREN, and SIREN. We conduct three distinct experiments described bellow.

Super-resolution experiment Storing high-resolution signals is a challenging task, due to hardware limitations. Thus, having an algorithm that can accurately reconstruct high-resolution signals from low-resolution data is of great interest. INRs have shown to be a promising approach to this problem [41]; thus we propose to evaluate the performance of the three models in a spherical high-resolution context.

Representation efficiency experiment We measure how efficiently each model captures a high-frequency spherical-harmonic signal from ground-truth data. Efficiency means reproducing the signal—and its derivatives—with fewer parameters than the reference harmonic expansion. By analysing the root-mean-square (RMS) power spectrum of the reconstruction error, we expose each model’s strengths and weaknesses across the spherical-harmonic degrees, i.e., at every spatial resolution.

Inference time experiment Computational efficiency is crucial for practical deployment of INR-based methods, particularly in high-resolution settings. We therefore measure and compare the computational performance of each model by evaluating their respective inference times.

The datasets used in these experiments are sourced from the [SHTOOLS](#) library [42]. Specifically, we employ Earth2014 (Earth), MOLA_shape (Mars), NLR_shape (Eros), LOLA_shape (Moon), USGS_SPG_shape (Mercury), JPL_SPC_shape (Enceladus), and DLR_SPG_shape (Ceres).

Note that while the Earth2014 dataset describes topography, that is, altitude relative to sea level; the other datasets characterize planet or asteroid shapes, represented by their absolute radii measured from the body’s center.

The metrics used in this section are standard and are defined in App. A.

2.4.1 Super-Resolution on Planet Shapes

Data In this experiment, we use only the shape dataset (excluding the Earth). The training data consist of spherical harmonic representations of each planet and asteroid, truncated at order $L = 150$. The corresponding training grid is selected as a Gauss-Legendre grid [43] consistent with this truncation order. The testing data similarly comprise spherical harmonic representations truncated at a higher resolution, $L = 300$, with their Gauss-Legendre grid chosen accordingly. Both training and testing data are scaled using a standard scaler, with the scaling parameters derived solely from the training set. The metrics are reported on the scaled data.

Models All three models share a closely aligned architecture. Specifically, SPH-SIREN is initialized using spherical harmonics up to order $L_{\text{pe}} = 15$, resulting in $(L_{\text{pe}} + 1)^2$ atoms. Similarly, SIREN and HNET architectures use an identical number of atoms, $(L_{\text{pe}} + 1)^2$. Each model employs an MLP with three hidden layers, each containing 150 neurons. Further details regarding model architectures can be found in Tab. C.1.

Training Training is performed with the Adam optimizer, using a learning rate of $\text{lr} = 10^{-4}$, for a total of 5000 epochs, and a batch size of 2048.

Results Fig. 2.6 shows standardized pointwise residuals for each body; a value of about 1 corresponds to one standard deviation of shape height in the training set. The Moon and Mercury, both rich in craters, are the hardest cases for the three models, with residual peaks ranging from 1.5–1.8 (Moon) and 2.1–2.3 (Mercury). Mars and Enceladus follow, with smaller peak residuals around 0.5–0.6. The easiest cases are the asteroid Eros and the dwarf planet Ceres, with residuals between 0.12 and 0.25 across all models.

Unsurprisingly, residuals tend to occur at craters or other high-frequency surface details, which are difficult to capture accurately from lower-resolution training data. The residual patterns of the models are generally similar, though we observe a difference in Eros: HNET shows higher residual values at locations of the asteroid that look smooth, and that are not present for the other models.

An initial look at SNR suggests HNET performs best overall. Table 2.2 give additional metrics confirming this. On five out of six bodies (Mars, Moon, Mercury, Enceladus, and Ceres), HNET achieves the lowest RMSE and MAE values, and the highest test SNR. Compared to SIREN (excluding Eros), HNET significantly improves performance, reducing RMSE by 20–35% and improving SNR by 2.0–3.8 dB. Compared to SPH-SIREN, HNET remains better but by a smaller margin—RMSE is lower by 15–30% and SNR is higher by 2–4 dB. The asteroid Eros is the only exception, where SPH-SIREN leads by about 1 dB.

Importantly, none of the networks were fine-tuned per planet, so the absolute performance of each model could change with extra tuning; the takeaway is that, under the same settings, HNET can perform on par with established baselines such as SIREN and SPH-SIREN.

2.4.2 Representation Efficiency on Earth Topography

Data The training and testing datasets are identical, consisting of Earth’s topography represented via spherical harmonics, evaluated on a Gauss-Legendre grid corresponding to $L = 500$. Data are standardized using a standard scaler.

Models All models share a similar neural network architecture, detailed in Tab. C.2. Notably, each model contains approximately 105,000 parameters, significantly fewer than the 250,000 required by the full spherical harmonics representation, achieving a compression ratio of roughly 2.4.

Training Training is conducted over 1000 epochs, using the Adam optimizer with a learning rate of $\text{lr} = 10^{-4}$ and a batch size of 2048.

Results Reconstruction residuals are shown in Fig. 2.7. All three models have similar patterns of residual localization. However, upon closer inspection, SIREN

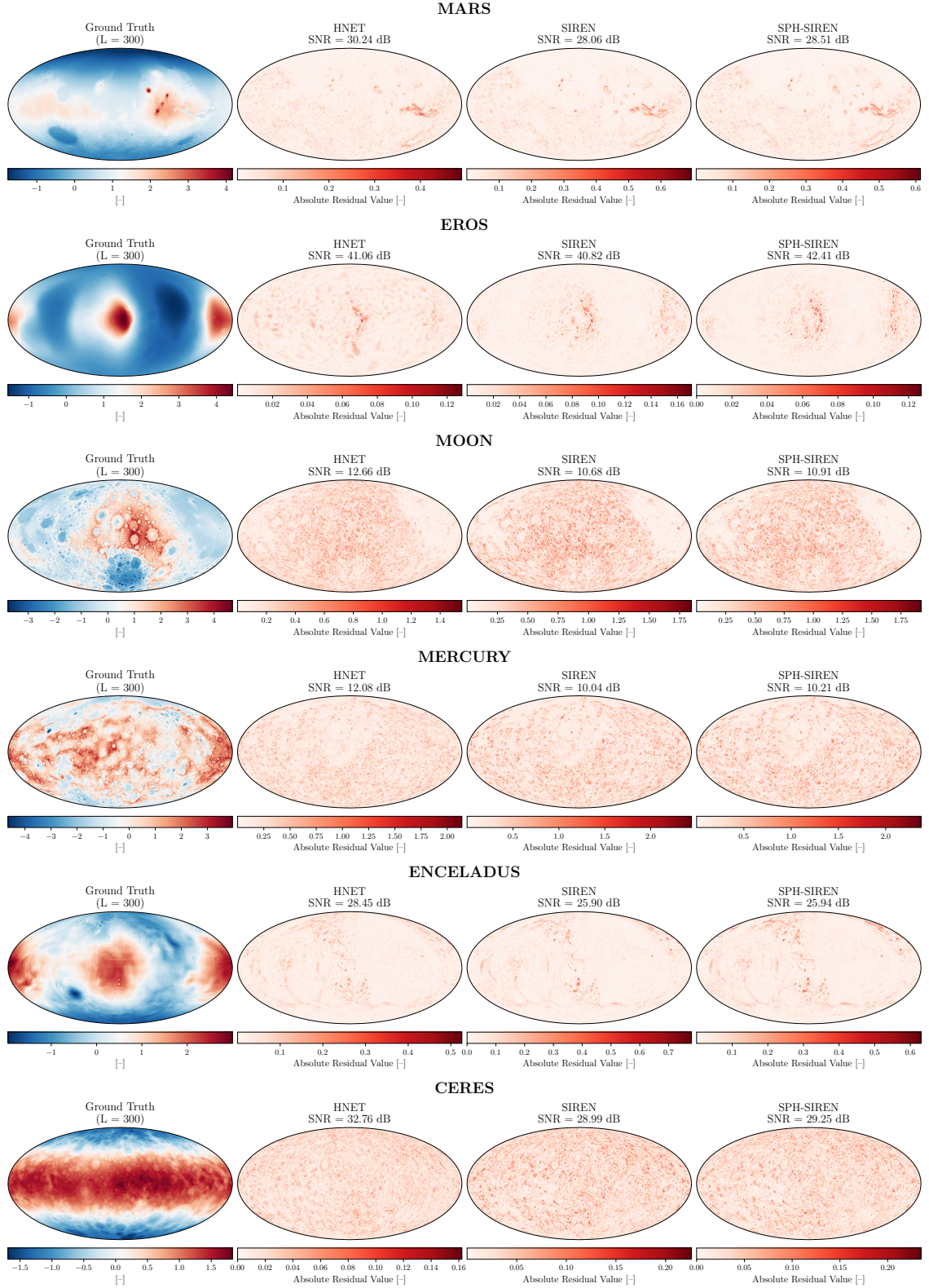


Figure 2.6: Residual maps for the shape-reconstruction experiment. Each row corresponds to one Solar-System body (Mars, Eros, Moon, Mercury, Enceladus, Ceres). The leftmost panel shows the ground-truth shape at the test resolution ($L = 300$); the three panels to the right give the pointwise residuals (ground truth – prediction) for HNET, SIREN, and SPH-SIREN, respectively.

Planet	Model	Training			Test		
		RMSE ($\downarrow$)	MAE ($\downarrow$)	SNR(dB) ($\uparrow$)	RMSE ($\downarrow$)	MAE ($\downarrow$)	SNR(dB) ($\uparrow$)
Mars	HNET (ours)	1.592×10^{-2}	1.150×10^{-2}	35.960	3.078×10^{-2}	1.894×10^{-2}	30.24
	SIREN	2.974×10^{-2}	1.921×10^{-2}	30.530	3.954×10^{-2}	2.355×10^{-2}	28.060
	SPH-SIREN	2.715×10^{-2}	1.756×10^{-2}	31.320	3.757×10^{-2}	2.210×10^{-2}	28.510
Eros	HNET (ours)	8.193×10^{-3}	6.222×10^{-3}	41.730	8.848×10^{-3}	6.550×10^{-3}	41.060
	SIREN	8.508×10^{-3}	5.493×10^{-3}	41.400	9.091×10^{-3}	5.703×10^{-3}	40.820
	SPH-SIREN	6.881×10^{-3}	4.482×10^{-3}	43.250	7.571×10^{-3}	4.734×10^{-3}	42.41
Moon	HNET (ours)	8.492×10^{-2}	6.083×10^{-2}	21.420	2.382×10^{-1}	1.736×10^{-1}	12.66
	SIREN	2.018×10^{-1}	1.450×10^{-1}	13.900	2.989×10^{-1}	2.139×10^{-1}	10.680
	SPH-SIREN	1.921×10^{-1}	1.371×10^{-1}	14.330	2.912×10^{-1}	2.081×10^{-1}	10.910
Mercury	HNET (ours)	8.384×10^{-2}	6.210×10^{-2}	21.530	2.556×10^{-1}	1.855×10^{-1}	12.08
	SIREN	2.176×10^{-1}	1.583×10^{-1}	13.250	3.235×10^{-1}	2.304×10^{-1}	10.040
	SPH-SIREN	2.095×10^{-1}	1.536×10^{-1}	13.580	3.171×10^{-1}	2.253×10^{-1}	10.210
Enceladus	HNET (ours)	2.044×10^{-2}	1.466×10^{-2}	33.790	3.784×10^{-2}	2.557×10^{-2}	28.45
	SIREN	3.970×10^{-2}	2.648×10^{-2}	28.020	5.072×10^{-2}	3.279×10^{-2}	25.900
	SPH-SIREN	3.929×10^{-2}	2.580×10^{-2}	28.110	5.054×10^{-2}	3.223×10^{-2}	25.940
Ceres	HNET (ours)	1.456×10^{-2}	1.109×10^{-2}	36.740	2.302×10^{-2}	1.755×10^{-2}	32.76
	SIREN	3.077×10^{-2}	2.373×10^{-2}	30.240	3.554×10^{-2}	2.713×10^{-2}	28.990
	SPH-SIREN	2.974×10^{-2}	2.295×10^{-2}	30.530	3.450×10^{-2}	2.621×10^{-2}	29.250

Table 2.2: Training and test performance of the three implicit-network models (HNET, SIREN, and SPH-SIREN) on six Solar-System bodies. For each body we report root-mean-square error (RMSE), mean-absolute error (MAE) and signal-to-noise ratio (SNR, dB) computed on the training grid ($L = 150$) and on the higher-resolution test grid ($L = 300$).

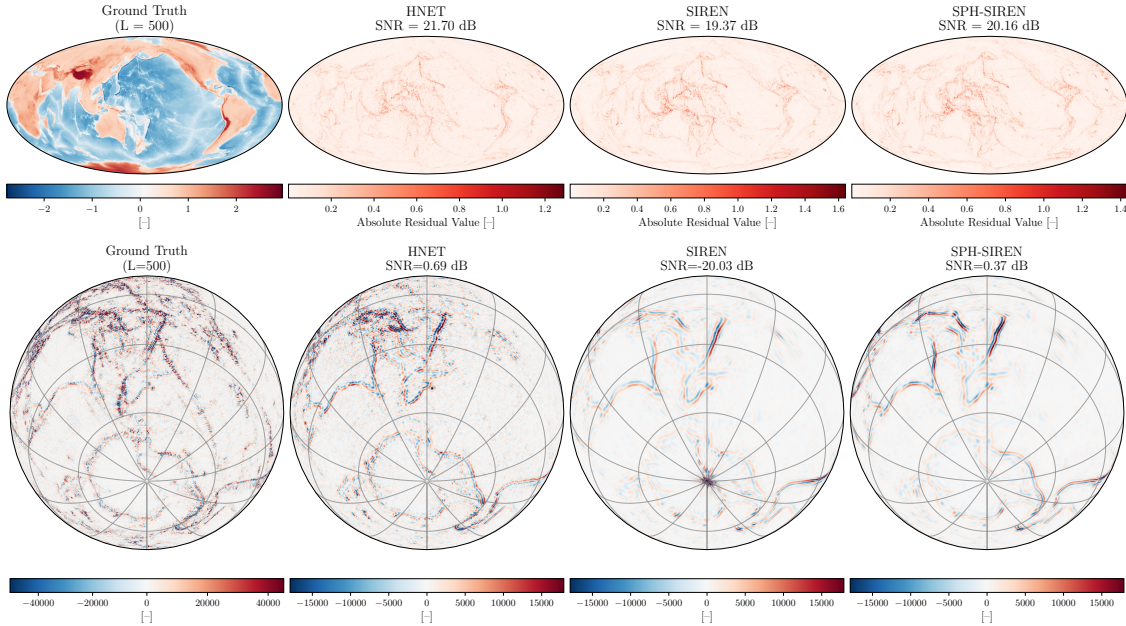


Figure 2.7: Top row : Ground-truth Earth topography at $L = 500$ (top leftmost column) and absolute point wise residuals for HNET, SIREN and SPH-SIREN (top row). Bottom row: Ground-truth laplacian and reconstructed laplacian for HNET, SIREN and SPH-SIREN.

and SPH-SIREN show more difficulty than HNET in capturing fine-scale details. HNET achieves the best SNR, surpassing SPH-SIREN by approximately 1.5 dB and SIREN by about 2.4 dB.

Nevertheless, despite full availability of ground truth data, these SNR values are not high enough to fully replace spherical harmonics representations. Peak residual magnitudes across the three models range from 1.2 to 1.6. Still, considering the lower number of parameters ($\times 2.4$ less), the models manage to capture a substantial portion of the signal but inevitably struggle with high-frequency details.

The reconstructed Laplacians (Fig. 2.7) further illustrate the difficulty of the task, with modest SNR values: 0.69 dB (HNET), -20.03 dB (SIREN), and 0.37 dB (SPH-SIREN). SIREN particularly struggles, producing noticeable artifacts at the poles due to its inherent Euclidean structure, which is ill-suited for spherical domain, a problem not observed with the spherical INRs, HNET and SPH-SIREN.

Fig. 2.8 (left) displays the root-mean-square power spectrum of the reconstruction error (RMSE power spectrum), giving deeper insights into each model’s performance across different spherical harmonic orders. We observe distinct differences between SPH-SIREN and HNET. SPH-SIREN achieves lower error energy than HNET between orders $L = 10$ and $L = 100$. In contrast, HNET struggles at lower harmonic orders but gradually improves, ultimately achieving lower error energy than SPH-SIREN in the range of $L = 100$ to 300 . For $L \geq 300$ both models show the same energy distribution. This difference highlights how each model implicitly encodes spatial information: SPH-SIREN appears more effective at capturing global (coarser) details, whereas HNET excels at finer-scale features but struggles more with global structure.

The RMSE spectra for the laplacian (Fig. 2.8, right) show a marked shift of error energy towards higher harmonic orders. This occurs because errors in the original signal at order ℓ are amplified by approximately a factor $\mathcal{O}(\ell^2)$ when computing the laplacian. Thus, even tiniest errors in the signal at high orders significantly impact the laplacian approximation. Consequently, both SPH-SIREN and HNET, which do not fully capture the finest details of the original signal, show relatively poor Laplacian approximations, reflected by modest SNR values (0.69 dB for HNET and 0.37 dB for SPH-SIREN). Nevertheless, visually, HNET’s reconstruction is significantly more faithful and detailed compared to that of SPH-SIREN.

In summary, at a compression ratio of approximately 2.4, we achieve satisfactory signal reconstruction performance, though the models struggle to accurately capture the Laplacian, suggesting the potential benefit of including an additional Laplacian fitting term in the loss. The experiment highlights distinct error characteristics of SPH-SIREN and HNET and clarifies that SIREN is fundamentally unsuitable for spherical domains, notably failing at the poles.

Due to extensive training times, particularly for SPH-SIREN, optimal hyperparameter tuning was limited. Thus, this experiment primarily sheds light on the distinctive capabilities and limitations of these models.

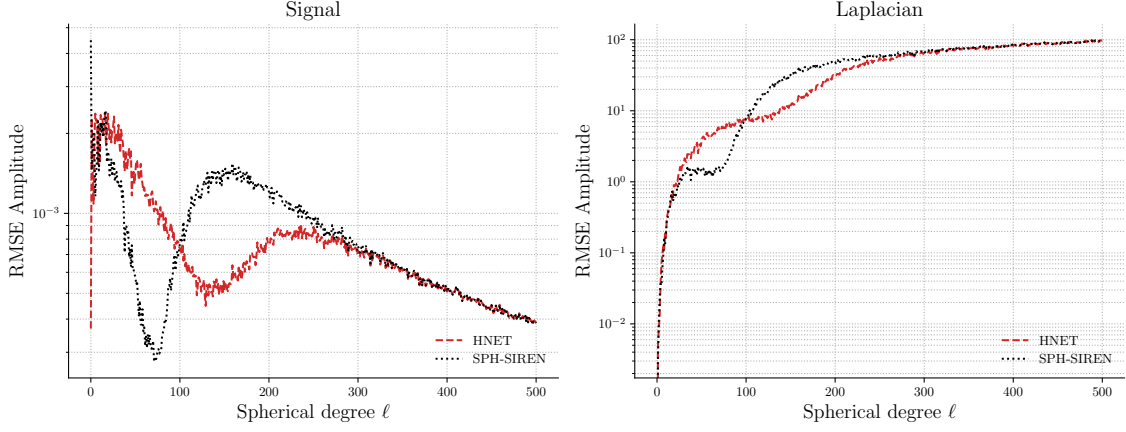


Figure 2.8: Root Mean Square power spectra of the error at $L = 500$ for the three models. Left panel: signal RMSE spectrum; right panel: Laplacian RMSE spectrum.

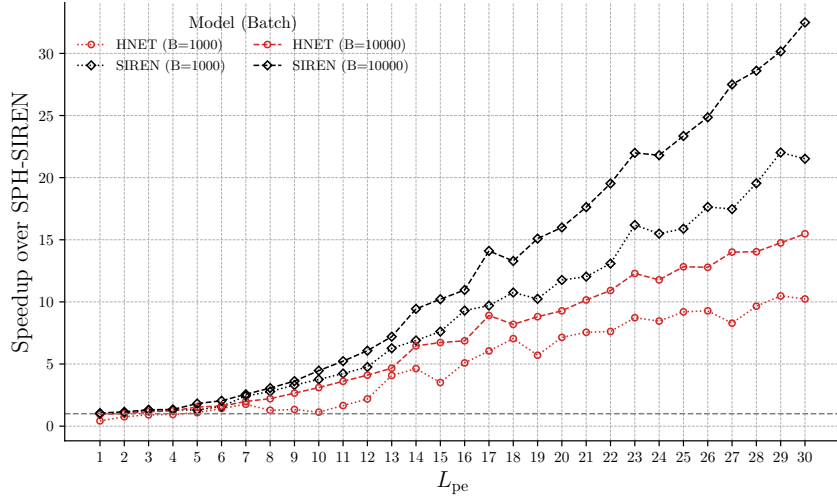


Figure 2.9: Speed-up relative to SPH-SIREN for SIREN and HNET as a function of the positional-encoding L_{pe} parameter for batch size of $B = 10^3, 10^4$. Number of atoms in the positional encodings for each model is given by $(L_{pe} + 1)^2$. Data points are averages of 10 runs.

2.4.3 Time Benchmark

Process We measured the duration of a single forward pass while varying the batch size $B \in \{10^3, 10^4\}$ and the positional encoding parameter L_{pe} . This parameter controls the number of atoms in the positional encoding, which is given by $(L_{pe} + 1)^2$.

Models All three evaluated networks use the identical sinusoidal MLP architecture, comprising three hidden layers with 250 neurons each. Therefore, the performance differences shown in Fig. 2.9 solely reflect variations in the computational cost associated with the encoding step.

Results Fig. 2.9 shows the speedup over SPH-SIREN for both batch sizes, defined as the ratio of inference times between SPH-SIREN and the other two models (SIREN and HNET). For $L_{pe} \leq 10$, at $B = 10^3$, performance between HNET and SPH-SIREN is comparable, with SPH-SIREN slightly outperforming HNET at $L_{pe} \in [1, 4]$. However, for $B = 10^4$, HNET increasingly outperforms SPH-SIREN, reaching approximately a $2\times$ speedup at $L_{pe} = 10$. In this range, SIREN consistently outperforms the others.

For $L_{pe} \in [10, 20]$, the performance gap widens significantly. With $B = 10^3$, HNET ranges from $1\times$ to $8\times$ faster than SPH-SIREN, while SIREN achieves speedups of about $5\text{--}12\times$. For $B = 10^4$, these speedups drastically increase, reaching roughly $4\text{--}10\times$ for HNET and $6\text{--}16\times$ for SIREN, respectively.

The performance difference between HNET and SIREN arises primarily due to HNET’s encoding involving additional matrix-vector multiplications.

It should be noted that our SPH-SIREN implementation uses the closed-form expression of spherical harmonics, as proposed by Rußwurm et al. [35]. A recursion-based C++/CUDA kernel could potentially improve SPH-SIREN’s runtime performance, but such low-level optimization is beyond the scope of this work. In contrast, HNET’s encoding involves only an additional matrix multiplication, which benefits from highly optimized routines available in PyTorch.

2.5 Conclusion

In this chapter, we introduced and analyzed HNET, a novel implicit neural representation specifically designed for spherical data. Through a combination of theoretical and experimental insights, we highlighted the effectiveness and practical benefits of HNET compared to existing methods. Our key findings are:

- We introduced HNET, which leverages the Herglotz formalism to achieve harmonic positional encoding without explicit computation of spherical harmonics.
- Additionally, under a polynomial approximation of activation functions, we showed that the spectral expressivity of both HNET and SPH-SIREN scales with network depth, enabling these models to represent a rich—and potentially infinite—range of spherical frequencies.
- Experimental results demonstrated that HNET achieves performance comparable to the SPH-SIREN architecture, while offering significant practical advantages including simpler implementation and faster inference speed.
- We highlighted that HNET exhibits limitations in capturing low-frequency spherical harmonic contributions but excels at higher-frequency components. Conversely, SPH-SIREN struggles with high-frequency components, favoring global, low-frequency features instead.

These findings highlight the potential of HNET as a versatile and efficient architecture for spherical implicit neural representations.

Chapter 3

Gravimetry Problem

The gravitational field of celestial bodies provides valuable insights into their internal structure and density distribution. Understanding these characteristics is essential for investigating their formation and evolution, as well as evaluating their potential to host life.

To reconstruct the gravitational potential of celestial bodies, we start with tracking data that only provide partial information about spacecraft trajectories as they orbit or pass nearby. Since a spacecraft’s trajectory is influenced directly by the gravitational field as well as by external forces, the process involves *simultaneously* estimating both the gravitational field and the spacecraft’s path. This iterative estimation, known as *orbit determination* (OD), leverages various kinds of spacecraft tracking data. Typical examples include Doppler measurements, Satellite Laser Ranging (SLR), altimeter data, onboard camera images, gradiometer data, and accelerometer data [44–49].

Conventionally, gravitational potentials are modeled using spherical harmonics, with the primary objective of accurately estimating the coefficients of this expansion. While spherical harmonics are particularly effective for quasi-spherical bodies, they struggle to accurately represent smaller bodies, such as asteroids and comets, which often have highly irregular shapes [50]. Popular alternative approaches include Mascons (mass-concentrations) representations [51], which assume discrete mass distributions, polyhedral gravity models [52], which approximate the body as a polyhedron defined by vertices and faces, and ellipsoidal harmonic expansions [50], generalizing spherical harmonics to ellipsoidal shapes (see, e.g., [25]). A notable limitation of these models is their specificity; the optimal model often depends heavily on the particular characteristics of the celestial body being studied. Ideally, one would have a unified modeling framework effective for both small, irregularly shaped bodies and large, quasi-spherical planets.

Recently, Implicit Neural Representations (INRs) have demonstrated impressive capabilities in modeling complex shapes and continuous signals, including time series, images, and 3D geometries [6, 53, 54]. These advancements naturally open opportunities for leveraging INRs to model gravitational fields.

Given that gravitational fields are harmonic functions, the underlying INR must satisfy the Laplace equation, i.e., $\Delta f = 0$. This constraint can be enforced

by augmenting the neural network’s loss function with a Laplacian penalty term, resulting in what is known as a Physics-Informed Neural Network (PINN) [55, 56]. PINNs have been successfully applied to various PDE-governed problems, including heat transfer, Euler equations, Burgers’ equation, and Schrödinger’s equation [57–59]. However, relatively few studies have addressed gravitational field reconstruction using neural networks (NN), with notable exceptions including [60–63]. In particular, Martin et al. [61] compared several gravitational NNs, including GeodesyNets [60], PINN-GM-I [62], PINN-GM-II [63], and PINN-GM-III [61]. Their best-performing model, PINN-GM-III, demonstrated strong results, comparable to or surpassing Mascon and polyhedral models. However, these conclusions were drawn from a specific scenario and may not generalize to less favorable conditions.

In this chapter, we propose to adapt the HNET architecture, first introduced in [26] and further developed in Chap. 2, to the gravitational potential reconstruction problem. In this initial investigation, we will formulate a natural generalization of HNET and highlight key properties desirable for gravitational modeling. We then evaluate its performance in simplified orbit-determination scenarios, comparing our results with classical least-squares (LS) inversion methods used for estimating spherical harmonic coefficients.

3.1 Gravitational Models

In this section, we briefly review 2 classical models of the gravitational potential, that is, the spherical harmonics and the polyhedral model. The former will be the baseline to compare with our model, while the latter will be used to generate synthetic data.

3.1.1 Spherical Harmonic Model

The most common model used to describe gravitational potential is the *spherical harmonics* representation, which extends the classical point-mass approximation by adding a spherical harmonics series to capture higher-order perturbations of the gravitational field. The gravitational potential Φ is expressed as ¹:

$$\Phi(r, \theta, \varphi) = \frac{\mu}{r} \sum_{\ell=0}^{\infty} \left(\frac{a_e}{r} \right)^{\ell} \sum_{m=0}^{\ell} \bar{P}_{\ell m}(\cos \theta) \left[C_{\ell m} \cos m\varphi + S_{\ell m} \sin m\varphi \right], \quad (3.1)$$

where $\mu = Gm$ is the gravitational parameter (gravitational constant multiplied by the mass of the celestial body), and a_e is the semi-major axis of the reference ellipsoid for the body. The angles $\theta \in (0, \pi)$ and $\varphi \in (0, 2\pi)$ represent colatitude and longitude, respectively. The functions $\bar{P}_{\ell m}(\cdot)$ are the normalized associated Legendre polynomial (see Eq. (1.18)), while $C_{\ell m}$ and $S_{\ell m}$, the Stokes’ coefficients, are to be determined.

¹Eq. (3.1) differs slightly from the real spherical harmonic expansion of Eq. (1.20). Eq. (3.1) can be recovered by explicitly expanding the real spherical harmonics defined in Eq. (1.17) and appropriately converting negative indices m into positive ones.

The gravitational potential can be conveniently separated into three distinct contributions:

$$\begin{aligned}\Phi(r, \theta, \varphi) = & \frac{\mu}{r} \\ & + \frac{\mu}{r} \sum_{\ell=1}^{\infty} \left(\frac{a_e}{r}\right)^{\ell} \bar{P}_{\ell,0}(\cos \theta) C_{\ell,0} \\ & + \frac{\mu}{r} \sum_{\ell=1}^{\infty} \left(\frac{a_e}{r}\right)^{\ell} \sum_{m=1}^{\ell} \bar{P}_{\ell m}(\cos \theta) \left[C_{\ell m} \cos m\varphi + S_{\ell m} \sin m\varphi \right].\end{aligned}\tag{3.2}$$

The first term corresponds to the classical Newtonian gravitational potential, treating the celestial body as a point mass; this represents the dominant component. The second term, termed the *zonal contribution*, accounts specifically for the body's oblateness—reflecting the flattening or elongation of the celestial body due primarily to its rotation. Finally, the third term captures additional variations in gravitational potential arising from non-uniform mass distribution within the body.

3.1.2 Polyhedral Model

Another commonly used alternative to the classical spherical harmonics approach is the polyhedral gravity model. In this method, the celestial body is approximated as a polyhedral shape defined by a mesh composed of vertices and polygonal faces. With this representation, the gravitational potential can be calculated as a sum of contributions from individual geometric elements (edges and faces) of the mesh. Specifically, the gravitational potential and its gradient are given by the following expressions [64]:

$$\begin{aligned}\Phi = & \frac{1}{2}G\sigma \sum_{e \in \text{edges}} \mathbf{r}_e \cdot \mathbf{E}_e \cdot \mathbf{r}_e \cdot L_e - \frac{1}{2}G\sigma \sum_{f \in \text{faces}} \mathbf{r}_f \cdot \mathbf{F}_f \cdot \mathbf{r}_f \cdot \omega_f \\ \nabla \Phi = & -G\sigma \sum_{e \in \text{edges}} \mathbf{E}_e \cdot \mathbf{r}_e + G\sigma \sum_{f \in \text{faces}} \mathbf{F}_f \cdot \mathbf{r}_f \cdot \omega_f\end{aligned}\tag{3.3}$$

Here, G denotes the gravitational constant, and σ represents the body's density. The vectors $\mathbf{r}_e$ and $\mathbf{r}_f$ indicate the positions of edges and faces, respectively, relative to a point of evaluation. The dyadic terms $\mathbf{E}_e$ and $\mathbf{F}_f$ represent geometric factors associated with edges and faces, respectively, defined in detail in [64]. Additionally, L_e and ω_f are terms quantifying the potential contributions of edges and faces.

The polyhedral gravity model is highly versatile and particularly advantageous for modeling the gravitational fields of irregularly shaped bodies, provided that the mesh is detailed enough to accurately represent the body's geometry. Its main limitation, however, lies in computational complexity; evaluating the potential at a point involves summations over numerous geometric elements, making the method computationally intensive, particularly for large or high-resolution meshes. Furthermore, the accuracy of this approach relies on prior knowledge of the body's density distribution, an assumption that is often replaced by a uniform density distribution.

3.2 Problem Formulation

As previously stated, we will consider a simplified orbit determination scenario, primarily assuming that the gravitational force is the only force acting on the spacecraft. Under this assumption, the acceleration of the spacecraft is directly related to the gravitational potential through the following relationship, expressed explicitly in spherical coordinates:

$$\begin{aligned}\mathbf{a} &= -\nabla\Phi(\mathbf{x}(r, \theta, \varphi)). \\ &= -\left(\frac{\partial\Phi}{\partial r}(r, \theta, \varphi) \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial\Phi}{\partial\theta}(r, \theta, \varphi) \hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta} \frac{\partial\Phi}{\partial\varphi}(r, \theta, \varphi) \hat{\boldsymbol{\varphi}}\right),\end{aligned}\quad (3.4)$$

where $\mathbf{x} \in \mathbb{R}^3$ represents the position vector, $\mathbf{a}(\mathbf{x}) \in \mathbb{R}^3$ is the corresponding acceleration vector at position $\mathbf{x}$, and $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\varphi}}$ denote the unit vectors in the radial, polar, and azimuthal directions, respectively.

We further assume that measurements of the spacecraft's position and corresponding acceleration are readily available. We denote this measurement dataset as $\mathcal{D} = \{(\mathbf{x}_i, \mathbf{a}_i)\}_{i=1}^N$, where $\mathbf{x}_i$ and $\mathbf{a}_i$ represent, respectively, the position and acceleration at the i -th measurement.

Therefore, the problem of interest can be concisely formulated as follows:

Problem Statement

Given a measurement dataset :

$$\mathcal{D} = \{(\mathbf{x}_i, \mathbf{a}_i)\}_{i=1}^N,$$

consisting of positions $\mathbf{x}_i$ and their corresponding accelerations $\mathbf{a}_i$, the objective is to reconstruct the gravitational potential Φ satisfying the fundamental relationship:

$$\mathbf{a}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}).$$

Hereafter, we will develop the Least Square (LS) method, which will allow us to compute the spherical harmonic coefficients in this simple OD scenario, and then we will briefly discuss the limitations and discrepancies with a realistic scenario.

3.2.1 Least-Square Inversion

Given the problem at hand, a standard approach to reconstruct the potential is to postulate a spherical harmonic expansion as introduced previously and to perform a least-squares inversion to estimate the coefficients. Let us assume the spherical harmonic expansion of the gravitational potential truncated at degree L . First, we compute the partial derivatives of the potential. In spherical coordinates, these

derivatives are:

$$\begin{aligned}
\frac{\partial \Phi}{\partial r} &= -\frac{\mu}{r^2} \sum_{\ell=0}^L \sum_{m=0}^{\ell} (\ell+1) \left(\frac{a_e}{r}\right)^{\ell} P_{\ell m}(\cos \theta) \left[C_{\ell m} \cos(m\varphi) + S_{\ell m} \sin(m\varphi) \right], \\
\frac{1}{r} \frac{\partial \Phi}{\partial \theta} &= \frac{\mu}{r^2} \sum_{\ell=0}^L \sum_{m=0}^{\ell} \left(\frac{a_e}{r}\right)^{\ell} \frac{\partial P_{\ell m}(\cos \theta)}{\partial \theta} \left[C_{\ell m} \cos(m\varphi) + S_{\ell m} \sin(m\varphi) \right], \\
\frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \varphi} &= \frac{\mu}{r^2} \sum_{\ell=0}^L \sum_{m=0}^{\ell} \left(\frac{a_e}{r}\right)^{\ell} m \frac{P_{\ell m}(\cos \theta)}{\sin \theta} \left[-C_{\ell m} \sin(m\varphi) + S_{\ell m} \cos(m\varphi) \right].
\end{aligned} \tag{3.5}$$

These expressions allow us to relate the measured accelerations to the spherical harmonic coefficients via a linear system, compactly written as $\mathbf{z} = \mathbf{H}\mathbf{x} + \boldsymbol{\nu}$. Here, $\mathbf{z}$ is the vector of measured acceleration components arranged as

$$\left(a_{1,r}, a_{1,\theta}, a_{1,\varphi}, a_{2,r}, a_{2,\theta}, a_{2,\varphi}, \dots, a_{N,r}, a_{N,\theta}, a_{N,\varphi} \right)^T,$$

$\mathbf{H}$ is the design matrix constructed by evaluating the above partial derivatives at the measurement locations, $\mathbf{x}$ is the vector of unknown coefficients consisting of the $C_{\ell m}$ and $S_{\ell m}$, and $\boldsymbol{\nu}$ represents centered Gaussian noise assumed to have a diagonal covariance matrix $\mathbf{W}$, i.e., $\boldsymbol{\nu} \sim \mathcal{N}(0, \mathbf{W})$. The estimate $\hat{\mathbf{x}}$ is obtained by minimizing the ℓ_2 -norm of the residuals, or by equivalently, solving the maximum likelihood problem:

$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x}} p(\mathbf{z}|\mathbf{x}) = \arg \min_{\mathbf{x}} \left\| \mathbf{z} - \mathbf{H}\mathbf{x} \right\|_{\mathbf{W}^{-1}}^2. \tag{3.6}$$

which leads to the normal equations:

$$\mathbf{H}^T \mathbf{W} \mathbf{H} \hat{\mathbf{x}} = \mathbf{H}^T \mathbf{W} \mathbf{z}. \tag{3.7}$$

This least square system can then be solved using standard numerical methods. In order to further simplify the problem, we assume that the noise is isotropic, i.e., $\mathbf{W} = \sigma^2 \mathbf{I}$, where σ^2 is the noise variance. In this case, the normal equations reduce to:

$$\mathbf{H}^T \mathbf{H} \hat{\mathbf{x}} = \mathbf{H}^T \mathbf{z}. \tag{3.8}$$

3.2.2 Limitations

As previously mentioned, this OD scenario is far from realistic; thus, it is important to understand the limitations of the experiments described in this chapter.

In practical scenarios, gravitational accelerations themselves are not directly measurable because the spacecraft is effectively in free fall. Instead, onboard instruments only record the non-gravitational forces acting on the spacecraft, which we have disregarded in our simplified problem formulation.

Furthermore, real-world measurements typically consist of one-dimensional observables, such as Doppler or range measurements, providing only line-of-sight information between a spacecraft and an Earth-based tracking station. In contrast, the methodology adopted here assumes direct availability of complete three-dimensional

acceleration vectors at specific locations. This assumption implicitly relies on having already reconstructed the gravitational potential, which allows to retrieve the accelerations from positions.

Finally, a realistic OD involves simultaneously estimating both the spacecraft’s trajectory and the gravitational parameters. Starting from an *a priori* initial trajectory based on a dynamical model, the process iteratively refines both trajectory and gravitational parameters by comparing predicted values to raw measurements. The iterative corrections take into account measurement correlations over time, uncertainties in parameters, and inaccuracies in the dynamical model. In contrast, we bypass trajectory reconstruction entirely and estimate the spherical harmonic coefficients directly via a straightforward least square fit. Therefore assuming both position and acceleration measurements are readily available and not correlated over time.

3.3 INR for Gravimetry

The goal of this section is to generalize the HNET architecture for the gravimetry problem by extending its domain of definition from the unit sphere $\mathbb{S}^2$ to the entire space $\mathbb{R}^3$. To achieve this, we first introduce the concept of *solid harmonics*, which naturally extend spherical harmonics to $\mathbb{R}^3$. Next, we demonstrate how the Herglotz formalism provides an elegant way to construct atoms implicitly encoding these solid harmonics. Finally, we describe the preprocessing steps and the loss function employed to train our model.

3.3.1 Solid Harmonics & Herglotz Formalism

As we have seen in Sec. 1.2, spherical harmonics are functions defined on the unit sphere $\mathbb{S}^2$, but it can be naturally extended to $\mathbb{R}^3$ as functions known as *solid harmonics*. These solid harmonics fall into two categories: the *regular solid harmonics* and the *irregular solid harmonics*.

Both types of solid harmonics can be derived via separation of variables applied to the Laplace equation expressed in spherical coordinates. However, for brevity, we will omit the explicit derivation details here (see, e.g., [25]), instead directly defining these harmonics and highlighting their relationship with spherical harmonics.

Recall that spherical harmonics correspond to harmonic homogeneous polynomials *restricted* to the unit sphere. The regular solid harmonics is the simple extension of these polynomials to the entire $\mathbb{R}^3$. More explicitly, let $Y_{\ell m}(\mathbf{x})$ denote the homogeneous polynomial associated with $Y_{\ell m}(\theta, \varphi)$. Then the regular solid harmonic can be obtained as:

$$\begin{aligned} R_{\ell m}(r, \theta, \varphi) &= Y_{\ell m}(\mathbf{x}(r, \theta, \varphi)) \\ &= Y_{\ell m}(\mathbf{x}/|\mathbf{x}|)|\mathbf{x}|^\ell \\ &= r^\ell Y_{\ell m}(\theta, \varphi). \end{aligned} \tag{3.9}$$

where the second equality follows from the homogeneity of $Y_{\ell m}(\mathbf{x})$. Note that we purposely omit the scaling factor, which is conceptually not important in our context.

The regular solid harmonics are harmonic throughout $\mathbb{R}^3$ and remain well-defined at the origin. In contrast, irregular solid harmonics, defined by

$$I_{\ell m}(r, \theta, \varphi) = r^{-(\ell+1)} Y_{\ell m}(\theta, \varphi), \quad (3.10)$$

are harmonic everywhere in $\mathbb{R}^3$ except at the origin.

This naturally suggests an extension of SPH-SIREN where the positional encodings is based on solid harmonics. Indeed, we could define two distinct positional encodings, each a natural extension of SPH-SIREN's PE:

$$\left(\psi_{\ell m}^{\text{REG}}(r, \theta, \varphi) \right)_{\ell, |m| \leq \ell}^{\ell=L} = R_{\ell m}(r, \theta, \varphi) = r^\ell Y_{\ell m}(\theta, \varphi) \quad (3.11)$$

and,

$$\left(\psi_{\ell m}^{\text{IRR}}(r, \theta, \varphi) \right)_{\ell, |m| \leq \ell}^{\ell=L} = I_{\ell m}(r, \theta, \varphi) = r^{-(\ell+1)} Y_{\ell m}(\theta, \varphi). \quad (3.12)$$

The choice between these two positional encodings depends on the specific application at hand. For instance, the irregular encoding suits the application in this chapter, as it matches a potential that is singular at the origin and vanishes at infinity.

However, as previously noted, explicitly evaluating spherical harmonics imposes significant computational costs. For that reason, we again utilize the Herglotz formalism, which will implicitly encode solid harmonics, and this with low computational costs. This approach is formalized in the following theorem.

Lemma 3.1. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function, such that*

$$f(z) = \sum_{\ell=0}^{\infty} c_\ell z^\ell, \quad \forall z \in \mathbb{C},$$

and let $\mathbf{a} \in \mathbb{C}^3$ satisfy $\mathbf{a}^T \mathbf{a} = 0$. We define

$$g_{\text{reg}}(\mathbf{x}) = f(\mathbf{a}^T \mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^3, \quad g_{\text{irr}}(\mathbf{x}) = \frac{1}{|\mathbf{x}|} f\left(\frac{\mathbf{a}^T \mathbf{x}}{|\mathbf{x}|^2}\right) \quad \mathbf{x} \in \mathbb{R}_0^3,$$

then g_{reg} is harmonic on $\mathbb{R}^3$ and g_{irr} is harmonic on $\mathbb{R}_0^3$; moreover, they admit a solid-harmonic expansion given by

$$\begin{aligned} g_{\text{reg}}(\mathbf{x}) &= \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} \alpha_{\ell m}(\mathbf{a}) r^\ell Y_{\ell, m}(\boldsymbol{\xi}) = \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} \alpha_{\ell m}(\mathbf{a}) R_{\ell m}(\mathbf{x}) \\ g_{\text{irr}}(\mathbf{x}) &= \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} \alpha_{\ell m}(\mathbf{a}) r^{-(\ell+1)} Y_{\ell, m}(\boldsymbol{\xi}) = \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} \alpha_{\ell m}(\mathbf{a}) I_{\ell m}(\mathbf{x}), \end{aligned}$$

where $\mathbf{x} = r\boldsymbol{\xi} \in \mathbb{R}^3$, $\boldsymbol{\xi} \in \mathbb{S}^2$ and for some coefficients $\alpha_{\ell m}(\mathbf{a}) \in \mathbb{C}$.

Proof. see App. B.4 □

This theorem is the continuity of Lem. 3.1, and validates the possibility of extending our approach to $\mathbb{R}^3$. Importantly, it extends even to the irregular case, which is less straightforward but essential for our purposes. With this theorem in hand, we can now generalize the HNET framework to $\mathbb{R}^3$, as detailed in the following section.

3.3.2 Solid Herglotz-NET

In light of Lem. 3.1, the most natural extension of the HNET’s positional encoding is therefore given by the following definition:

$$\begin{aligned} & \tilde{\psi}_{\mathbf{R}, \mathbf{A}, \mathbf{w}, \mathbf{b}}(\mathbf{x}) \\ & := \left\{ \psi_{\mathbf{R}_i, \mathbf{a}_i, w_i, b_i}(\mathbf{x}) = \frac{1}{|\mathbf{x}|} \exp \left(\frac{w_i}{\varpi_0} [(\mathbf{R}_i \mathbf{a}_i)^\top \frac{\mathbf{x}}{|\mathbf{x}|} \frac{r_{\text{ref}}}{|\mathbf{x}|} - \frac{1}{\sqrt{2}}] + b_i \right) \right\}_{i=1}^m, \end{aligned} \quad (3.13)$$

where $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_m)^T \in \mathbb{C}^{3 \times m}$ are the Herglotz vectors, $\mathbf{w} = (w_1, \dots, w_m) \in \mathbb{R}^m$ and $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{C}^m$ are trainable weights and biases respectively, ϖ_0 is a fixed frequency scale parameter, $\mathbf{R} = \text{diag}(\mathbf{R}_1, \dots, \mathbf{R}_m)$ with $\mathbf{R}_i \in \mathbb{R}^{3 \times 3}$ are learnable rotation matrices, and r_{ref} is a reference radius. Aside from r_{ref} , Tab. 2.1 summarizes the parameters involved in the positional encoding.

This positional encoding thus encodes any location $\mathbf{x}(r, \theta, \varphi) \in \mathbb{R}^3$, with coordinates $(r, \theta, \varphi) \in (0, +\infty) \times [0, \pi] \times [0, 2\pi)$. According to Lem. 3.1, these atoms are harmonic functions on $\mathbb{R}_0^3$ and admit an irregular solid harmonic expansion.

In practice though, we work with a real-valued positional encoding, which we decided to be the imaginary part of the complex positional encoding:

$$\psi_{\mathbf{R}, \mathbf{A}, \mathbf{w}, \mathbf{b}}(\mathbf{x}) = \Im \{ \tilde{\psi}_{\mathbf{R}, \mathbf{A}, \mathbf{w}, \mathbf{b}}(\mathbf{x}) \}, \quad (3.14)$$

we don’t develop the expression for readability.

The final architecture, Solid Herglotz-NET, is then defined as follows:

$$\begin{aligned} f_{\Theta}(\mathbf{x}) &= \beta_1 / |\mathbf{x}| \\ &+ \beta_2 \cdot \mathbf{w}_{\text{skip}}^\top \psi_{\mathbf{R}, \mathbf{A}, \mathbf{w}, \mathbf{b}}(\mathbf{x}) \\ &+ \beta_3 \cdot \text{SineMLP}(\psi_{\mathbf{R}, \mathbf{A}, \mathbf{w}, \mathbf{b}}(\mathbf{x})) \end{aligned} \quad (3.15)$$

wherein, SineMLP is a multi-layer perceptron (MLP) with periodic activation functions $\sigma^{(q)}(\cdot) = \sin(\cdot)$ and *no biases*, $\mathbf{w}_{\text{skip}} \in \mathbb{R}^m$ is a trainable vector, and $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ are fixed constants, which depend on the physical unit of the input and target.

In what follows, we describe the motivations and properties underlying our INR design. We cover aspects and parameters distinct from those already presented in the HNET framework and refer the reader to Chap. 2 for details about them.

3.3.3 Properties of Solid Herglotz-NET

With Solid Herglotz-NET, we aim to learn a potential field, a physical quantity that must inherently satisfy certain fundamental constraints. Ideally, the model should satisfy the following properties:

- The potential should be singular at the origin, and vanish at infinity, i.e., $f_{\Theta} \xrightarrow{r \rightarrow \infty} 0$.
- The rate of decay should be $\mathcal{O}(1/r)$, i.e., $f_{\Theta}(\mathbf{x}) \propto \frac{\mu}{|\mathbf{x}|}$ as $|\mathbf{x}| \rightarrow +\infty$.

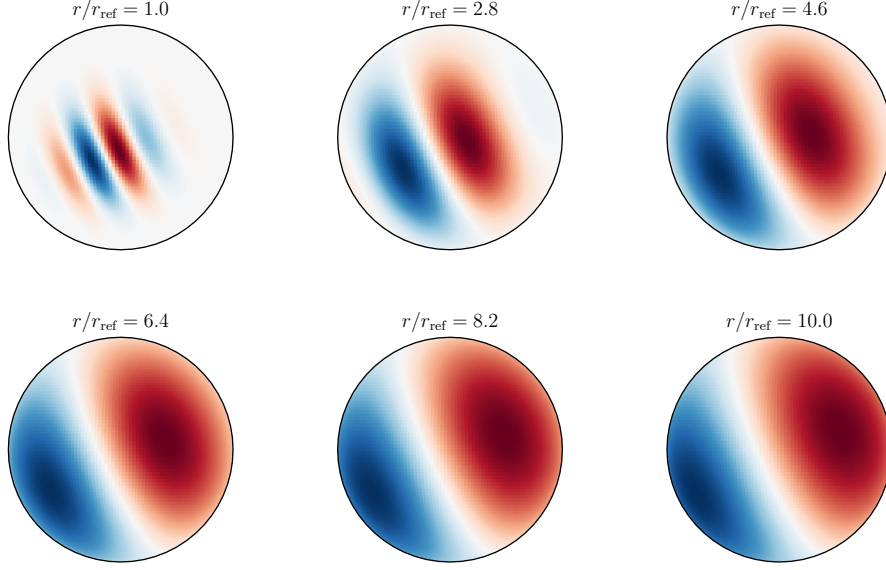


Figure 3.1: Atoms Evolution with the radius r .

- The potential should be harmonic everywhere in $\mathbb{R}^3$ except at the origin, i.e., $\Delta f_{\Theta}(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$.

These properties are nontrivial to satisfy, particularly the harmonicity condition. As a starting point, it is essential that at least the positional encoding itself respects these three properties.

Lem. 3.1 directly addresses these requirements. Indeed, the atoms are harmonic functions on $\mathbb{R}_0^3$, and their real solid harmonic expansions explicitly show the desired decay properties:

$$\psi_{\mathbf{R}_i, \mathbf{a}_i, w_i, b_i}(\mathbf{x}(r, \theta, \varphi)) = \sum_{\ell=0}^{\infty} \left(\frac{r_{\text{ref}}}{r} \right)^{\ell+1} \sum_{|m| \leq \ell} c_{\ell m} \left(\frac{w_i}{\varpi_0} \right)^{\ell} Y_{\ell m}^{\text{real}}(\theta, \varphi) \quad (3.16)$$

for some coefficients $c_{\ell m} \in \mathbb{R}$. For which we have indeed that

$$\psi_{\mathbf{R}_i, \mathbf{a}_i, w_i, b_i}(\mathbf{x}(r, \theta, \varphi)) = \frac{c_{00}}{r} + \mathcal{O}(1/r^2). \quad (3.17)$$

Hence, the positional encoding naturally satisfies the required decay at infinity with a dominant $\mathcal{O}(1/r)$ term. Moreover, due to the Herglotz formalism, each atom implicitly encodes the spherical geometry and has a spherical harmonic expansion exactly the same form as classical potentials. Although theoretically infinite bandwidth, the rapid decay of coefficients $c_{\ell m}$ ensures the atoms are effectively band-limited in practice.

We previously analyzed how frequency parameters w_i, ϖ_0 influence the positional encoding (Fig. 2.1), and this relationship remains valid for a fixed radius. Eq. (3.16) highlights that varying the radius r produces a similar effect (Fig. 3.1): increasing r diminishes the contribution of high-frequency components, smoothing the potential and reducing localization, while smaller radii favor localization. Furthermore, for

$r \geq r_{\text{ref}}$, the atoms are bounded as:

$$\max_{\mathbf{x} \in \mathbb{R}^3} \psi_{\mathbf{R}_i, \mathbf{a}_i, w_i, b_i}(\mathbf{x}(r, \theta, \varphi)) = \frac{1}{r_{\text{ref}}} \exp\{b_i, \mathfrak{R}\}. \quad (3.18)$$

Considering the full model defined in (3.19), we notably select sinusoidal activation functions and exclude biases. This choice is driven by two factors: sinusoidal activation has shown effectiveness in the original HNET architecture, and importantly, bias-free layers and sinusoidal activation preserve the dominant $\mathcal{O}(1/r)$ decay required physically. Introducing biases would instead shift the potential by a constant, disrupting the desired decay behavior.

Note that, while the positional encoding comprises harmonic atoms, harmonicity is not maintained after propagation through the MLP.

Finally, recall that we have broken down the model explicitly into three components as:

$$\begin{aligned} f_{\Theta}(\mathbf{x}) &= \beta_1 / |\mathbf{x}| \\ &+ \beta_2 \cdot \mathbf{w}_{\text{skip}}^\top \psi_{\mathbf{R}, \mathbf{A}, \mathbf{w}, \mathbf{b}}(\mathbf{x}) \\ &+ \beta_3 \cdot \text{SineMLP}(\psi_{\mathbf{R}, \mathbf{A}, \mathbf{w}, \mathbf{b}}(\mathbf{x})). \end{aligned} \quad (3.19)$$

The first component is fixed and therefore non-trainable, since the leading-order term of the potential is usually known and given by $\beta_1 = G \cdot m$ (where $|x|$ has distance units). We do not require the model to learn this component explicitly, but instead focus the learning on higher-frequency contributions. The second component, a harmonic skip connection directly from the positional encoding, captures low-frequency variations in the potential. Finally, the third component, produced by the sinusoidal MLP, is not harmonic but aims to learn the high-frequency features of the potential. The constants β_2 and β_3 are chosen to have the correct physical units, and will be developed in the next section.

3.4 Preprocessing & Loss

A common practice in deep learning is to scale input and output data, which stabilizes gradient updates during training [65]. This scaling is particularly crucial in our context since we deal with large-scale phenomena, such as modeling Mars, involving radii of order 10^6 m and potential fields of order $10^6 \text{ m}^2/\text{s}^2$. The significant difference in magnitudes between input and target complicates the training further. Additionally, careful scaling is required since inputs, targets, and gradients all have distinct physical units linked by physical relationships that must remain consistent under scaling.

Preprocessing We propose the following scaling factors for radius, acceleration, and potential:

$$\begin{aligned} r^* &= r_0 \\ a^* &= \frac{1}{N} \sum_{i=1}^N \|\mathbf{a}_i(\mathbf{x}_i) - (-\mu/r_i^2, 0, 0)\| \\ \Phi^* &= a^* \cdot r^* \end{aligned} \quad (3.20)$$

where r_0 is a user-defined reference radius (typically the body's semi-major axis). Given a training dataset $\mathcal{D} = \{(\mathbf{x}(r_i, \theta_i, \varphi_i), \mathbf{a}_i)\}_{i=1}^N$, we define the following dimensionless quantities:

$$\begin{aligned}\mathbf{x}'_i &= (r_i/r^*, \theta_i, \varphi_i), \quad \forall i = 1, \dots, N \\ \mathbf{a}'_{\text{res},i} &= \frac{\mathbf{a}_i(\mathbf{x}_i) - (-\mu/r_i^2, 0, 0)}{a^*}, \quad \forall i = 1, \dots, N\end{aligned}\tag{3.21}$$

where θ_i and φ_i denote the colatitude and longitude angles, respectively. Notice we explicitly remove the first-order term of the potential since it is not learned by the model. Hence, the model used during training is:

$$\begin{aligned}g_{\Theta}(\mathbf{x}') &= \left(f_{\Theta}(\mathbf{x}') - \frac{\mu}{r^*|\mathbf{x}'|}\right)/\Phi^* \\ &= \mathbf{w}_{\text{skip}}^\top \psi_{\mathbf{R}, \mathbf{A}, \mathbf{w}, b}(\mathbf{x}') + \text{SineMLP}(\psi_{\mathbf{R}, \mathbf{A}, \mathbf{w}, b}(\mathbf{x}'))\end{aligned}\tag{3.22}$$

where we set $\beta_1 = \mu/r^* = G \cdot m/r^*$, $\beta_2 = \beta_3 = \Phi^*$; thus, the model $g_{\Theta}(\mathbf{x}')$ is dimensionless.

Loss function To compare our approach consistently against the LS fit, we adopt the same loss function formulation. Recall it assumes the relationship:

$$\mathbf{a}'_{\text{res},i} = \nabla_{\mathbf{x}'} g_{\Theta}(\mathbf{x}'_i) + \boldsymbol{\nu}_i, \quad \boldsymbol{\nu}_i \sim \mathcal{N}(0, \sigma^2 \mathbf{I}_3)\tag{3.23}$$

for all $i = 1, \dots, N$, where $\boldsymbol{\nu}_i$'s are i.i.d. isotropic noise terms. The maximum likelihood estimate of the model parameters Θ is therefore:

$$\hat{\Theta} = \arg \max_{\Theta} \prod_{i=1}^N p(\mathbf{a}_{\text{res},i} | \Theta) = \arg \min_{\Theta} \sum_{i=1}^N \frac{1}{\sigma^2} \left\| \nabla_{\mathbf{x}'} g_{\Theta}(\mathbf{x}'_i) - \mathbf{a}'_{\text{res},i} \right\|^2\tag{3.24}$$

where σ^2 is the noise variance. Given our simplifying assumptions, the exact value of σ^2 is not crucial, thus we set $\sigma^2 = N$. The loss function is then expressed as:

$$\mathcal{L}_{\Theta} = \frac{1}{N} \sum_{i=1}^N \left\| \nabla_{\mathbf{x}'} g_{\Theta}(\mathbf{x}'_i) - \mathbf{a}'_{\text{res},i} \right\|^2.\tag{3.25}$$

Moreover, we recall the model is generally not harmonic. Hence, a regularization term penalizing non-harmonicity must be introduced:

$$\mathcal{R}_{\Theta} = \frac{1}{M} \sum_{i=1}^M \left\| \Delta g_{\Theta}(\mathbf{x}'_i) \right\|^2, \quad \mathbf{x}'_i \sim \mathcal{U}(\mathbb{R}^3 | r' \geq 1).\tag{3.26}$$

where M is the number of regularization samples. Notice that points are sampled with $r' = r/r_0 \geq 1$, since we are interested in the potential outside the body of interest. The final loss function used during training is thus:

$$\mathcal{L}_{\Theta} = \frac{1}{N} \sum_{i=1}^N \left\| \nabla_{\mathbf{x}'} g_{\Theta}(\mathbf{x}'_i) - \mathbf{a}'_{\text{res},i} \right\|^2 + \lambda \cdot \frac{1}{M} \sum_{i=1}^M \left\| \Delta g_{\Theta}(\mathbf{x}'_i) \right\|^2,\tag{3.27}$$

where λ is a hyperparameter controlling the influence of the regularization term.

3.5 Datasets

We are going to study the performance of the model across multiple datasets. These datasets have been selected to cover various scenarios, including both realistic and synthetic cases. The goal is to examine how the model behaves under different conditions, some more favorable than others. The specifications of each dataset are summarized in Tab. 3.2, and the details of each scenario are described further below.

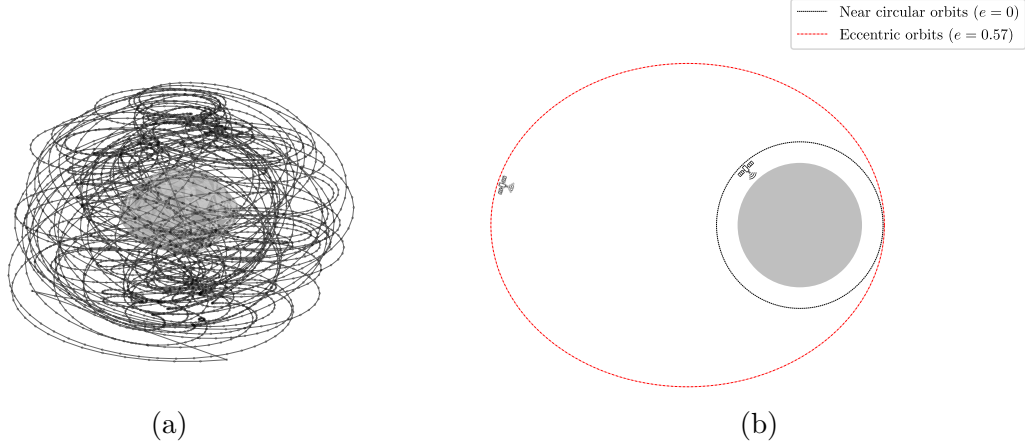


Figure 3.2: Orbital trajectories used in our experiments: (a) Particle trajectories around Bennu and (b) synthetic near-circular (MRO-like) and eccentric (MEX-like) orbits.

Mars The ground truth gravitational field for Mars is represented by a spherical harmonics (SH) model of order $L = 120$, estimated by Konopliv et al. [44]. We consider two distinct synthetic trajectory types: a near-circular orbit resembling that of the Mars Reconnaissance Orbiter (MRO) [66], and an eccentric orbit similar to the Mars Express (MEX) mission [67]. Both synthetic trajectories are generated as purely Keplerian orbits with an added constant rate on the right-ascension of the ascending node. The NAIF² SPICE [68] toolkit is used for conversions between Keplerian elements and Cartesian coordinates. Tab. 3.1 lists the Keplerian elements for the two cases we consider. Representations of the trajectories are shown in Fig. 3.2 (b). The training set consists of accelerations computed from the ground truth SH model evaluated along these synthetic trajectories.

Bennu Asteroid We use publicly available position data from NASA’s OSIRIS-REx mission SPICE kernels³. These data provide trajectories of small particles orbiting Bennu [69]. For each selected particle, we sample their path every $\Delta t = 300$ s, and computes the local gravitational acceleration $\mathbf{a}(t_k)$ with the ESA *polyhedral-gravity* [70] toolkit applied to the Bennu mesh model, assuming a uniform density

²NASA’s Navigation and Ancillary Information Facility (NAIF)

³https://naif.jpl.nasa.gov/pub/naif/pds/pds4/orex/orex_spice/document/spicedsv001.html

Orbit	a (km)	e	i (deg)	ω (deg)	Ω_0 (deg)	$\dot{\Omega}$ (deg day $^{-1}$)	M_0 (deg)
Near-circular (“MRO-like”)	3 630	0.00	92.6	270	0	0.988	0
Eccentric (“MEX-like”)	8 590	0.57	92.6	270	0	0.988	0

Table 3.1: Keplerian elements used to generate the synthetic Mars orbits. The eccentric set reproduces the *shape* of the current Mars Express (MEX) orbit, while the circular set matches the altitude of the Mars Reconnaissance Orbiter (MRO). A constant nodal drift rate $\dot{\Omega}$ is imposed on the trajectory to achieve full longitudinal coverage.

of $1\,200\text{ kg m}^{-3}$. The resulting pairs $\{\mathbf{r}(t_k), \mathbf{a}(t_k)\}_{k=1}^K$ constitute the synthetic measurements; no spherical-harmonic field is involved in their generation. You can see the particles trajectories on Fig. 3.2 (a). The ground truth is a spherical harmonics model of degree $L = 10$, computed from the polyhedral mesh assuming homogeneous density, using the line-integral approach proposed by Jamet and Tsoulis [71].

Synthetic Asteroid We consider a custom-designed synthetic asteroid with a spherical shape and a single cross-shaped cavity on its surface (see Fig. 3.3). A uniform density distribution is assumed, computed directly from the asteroid’s polyhedral volume. The asteroid’s radius matches the semi-major axis of Mars ($r_0 = 3396\text{ km}$), and the gravitational parameter is also set to match Mars $\mu = 4.282837 \times 10^{13}\text{ m}^3\text{ s}^{-2}$. The ground truth is the polyhedron model, which is computed using the ESA *polyhedral-gravity* toolkit [70]. Training measurements are obtained by evaluating accelerations on the asteroid’s surface at $r = r_0$, sampled using a Gaus-Legendre (GL) grid.

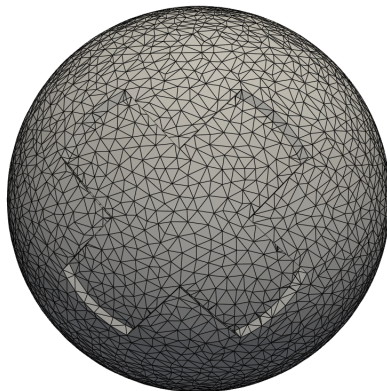


Figure 3.3: Synthetic Asteroid Mesh

Testing Sets In all cases, the testing sets are constructed by evaluating the respective ground truth gravitational fields on Gauss-Legendre (GL) grids placed at ten distinct radii, evenly spaced between $r/r_0 = 1$ and $r/r_0 = 3$. For the Mars and synthetic asteroid scenarios, GL grids of resolution $L = 100$ are used, while we use $L = 10$ for the Bennu experiment.

Dataset	Training Set		Testing Set		Ground Truth
	Measurements	# Pts	Measurements	# Pts	
Mars (near-circular)	Near-circular orbits	87 600	GL grids	203 010	SH model
Mars (eccentric)	Eccentric orbits	87 600	GL grids	203 010	SH model
Bennu Asteroid	Particle trajectories	4 289	GL grids	2 310	SH model
Synthetic Asteroid	GL grid	20 301	GL grids	203 010	Polyhedron model

Table 3.2: Overview of training and testing datasets, measurement types, and ground-truth models.

3.6 Assessments

In this section, we present the results of our experiments. The detailed specifications of all models are summarized in Tab. C.3.

Training and Testing Setup All models were trained using the AdamW optimizer with a learning rate of $1r = 2 \times 10^{-3}$ over 500 epochs. Except for the Bennu experiment, which utilized a batch size of 256, all other experiments employed a batch size of 1024. The regularization parameter was set to $\lambda = 10^{-1}$, unless stated otherwise, with $M = 1000$ regularization samples drawn at each epoch. Each training dataset was split into 90% training and 10% validation sets, and the model with the lowest validation loss was selected for testing. Training was terminated early if the validation loss did not improve for 300 epochs. The longest training duration occurred for the Mars experiments, taking approximately—per experiment—3 hours on an M3 chip using only the CPU.

In addition to the four primary cases (Bennu, Mars with MRO orbits, Mars with MEX orbits, and a synthetic asteroid), we also considered sub-cases in which we independently removed either the initialization scheme (described in Alg. 2), the rotation of the atoms, or the regularization term. These specific scenarios are detailed further in their respective sections below.

The raw results for all cases are summarized in Tab. 3.3, and the different metrics are defined in App.A. For simplicity, we will refer to Solid Herglotz-NET as HNET in the following.

3.6.1 Mars Gravity Fields

The near-circular (MRO-like) and eccentric (MEX-like) orbits share the same ground truth, allowing direct comparison between cases. Additionally, we introduced a sub-case for MEX orbits without the regularization term, which enforces harmonicity, to assess the model’s generalization capability without it.

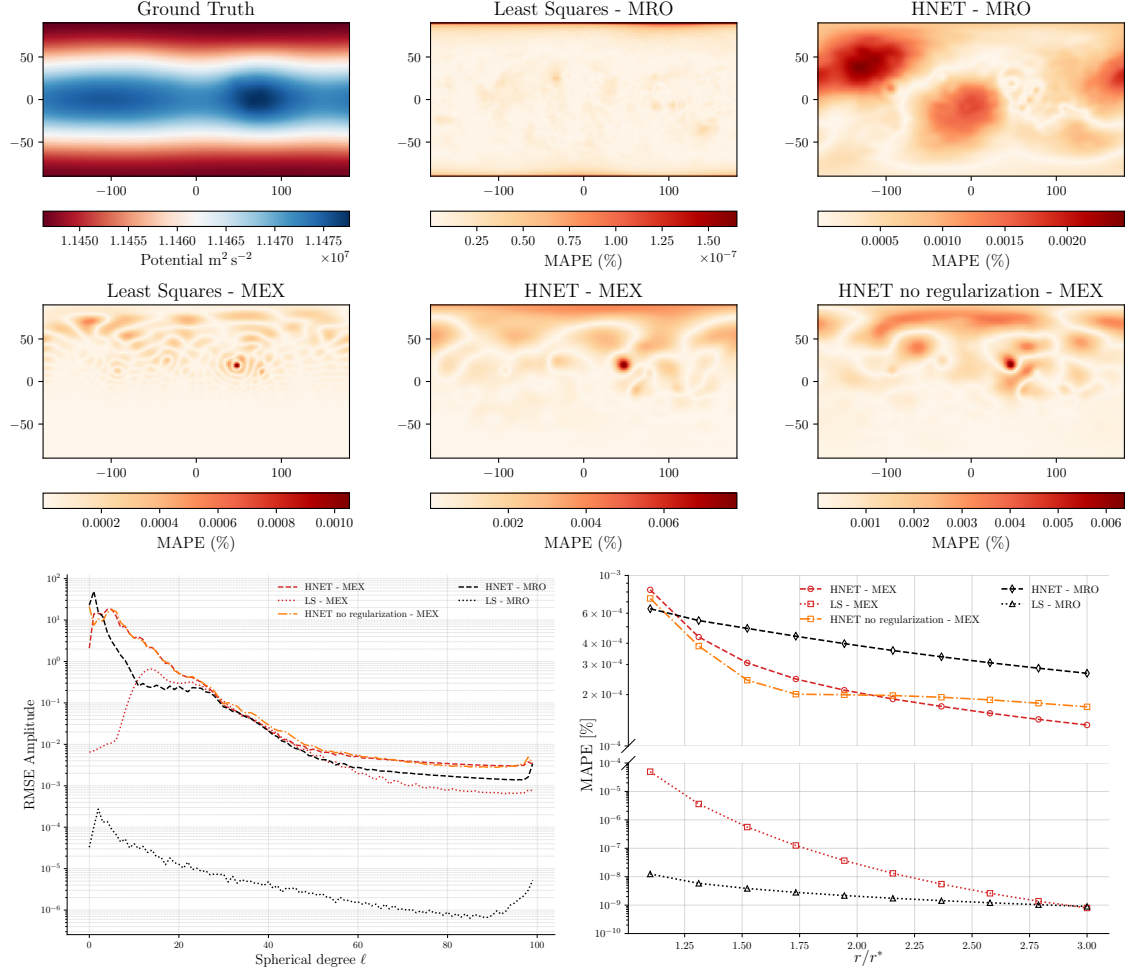


Figure 3.4: Comparison of model performance for gravitational potential reconstruction around Mars using near-circular (MRO-like) and eccentric (MEX-like) orbits. The top row shows the ground truth potential field at $r/r^* = 1.1$ (left) and spatial distributions of the Mean Absolute Percent Error (MAPE) for Least Squares (LS) and HNET models under the different orbital scenarios at $r/r^* = 1.1$. The bottom-left plot presents the RMSE power spectrum against the spherical harmonic degree ℓ , while the bottom-right plots illustrate the radial evolution of MAPE as a function of normalized radius r/r^* .

Experiment	Model	Acceleration (Train)		Potential (Test)			
		MAE($\downarrow$) [m s^{-2}]	RMSE($\downarrow$) [m s^{-2}]	MAE($\downarrow$) [$\text{m}^2 \text{s}^{-2}$]	MAPE ($\downarrow$) [%]	RMSE($\downarrow$) [$\text{m}^2 \text{s}^{-2}$]	SNR ($\uparrow$) [dB]
Bennu	HNET	4.531×10^{-9}	1.219×10^{-8}	1.583×10^{-6}	1.30×10^{-2}	3.739×10^{-6}	67.1
	LS	1.747×10^{-9}	5.590×10^{-9}	4.863×10^{-7}	3.51×10^{-3}	1.552×10^{-6}	71.5
Mars MEX Orbits	HNET	1.898×10^{-6}	3.870×10^{-6}	2.331×10^1	2.82×10^{-4}	5.554×10^1	102.2
	LS	1.010×10^{-10}	3.583×10^{-10}	6.094×10^{-1}	5.40×10^{-6}	3.367	126.5
Mars MEX Orbits (no regularization)	HNET	1.514×10^{-6}	3.448×10^{-6}	2.149×10^1	2.69×10^{-4}	4.645×10^1	103.8
	LS	1.010×10^{-10}	3.583×10^{-10}	6.094×10^{-1}	5.40×10^{-6}	3.367	126.5
Mars MRO Orbits	HNET	5.762×10^{-5}	7.898×10^{-5}	3.019×10^1	4.07×10^{-4}	4.455×10^1	104.1
	LS	7.065×10^{-8}	9.140×10^{-8}	2.975×10^{-4}	3.35×10^{-9}	9.146×10^{-4}	197.9
Synthetic Asteroid	HNET	2.461×10^{-3}	4.264×10^{-3}	5.040×10^3	7.09×10^{-2}	6.805×10^3	60.4
	LS	4.515×10^{-3}	7.784×10^{-3}	8.955×10^3	1.46×10^{-1}	1.009×10^4	57.0
Synthetic Asteroid (no initialization)	HNET	2.344×10^{-3}	4.442×10^{-3}	5.445×10^3	7.29×10^{-2}	7.549×10^3	59.5
	LS	4.515×10^{-3}	7.784×10^{-3}	8.955×10^3	1.46×10^{-1}	1.009×10^4	57.0
Synthetic Asteroid (no rotation)	HNET	2.331×10^{-3}	4.169×10^{-3}	5.067×10^3	7.28×10^{-2}	6.841×10^3	60.4
	LS	4.515×10^{-3}	7.784×10^{-3}	8.955×10^3	1.46×10^{-1}	1.009×10^4	57.0

Table 3.3: Training and testing performance of the Herglotz-NET (HNET) and Least-Square (LS) models on the 4 datasets : Bennu, Mars MEX Orbits, Mars MRO Orbits and Synthetic Asteroids. For each case, we report the training mean absolute error (MAE), root-mean-square error (RMSE) and the testing MAE, RMSE, mean absolute percentage error (MAPE) and signal-to-noise ratio (SNR). The best performance for each subcases is highlighted in bold.

Fig. 3.4 (bottom right) present the evolution of mean absolute percent error (MAPE) versus the normalized radius r/r^* . Least squares (LS) fits clearly outperform the HNET models: their MAPE stays between $10^{-9}\%$ and $10^{-4}\%$ whereas HNET models range from $10^{-4}\%$ to $10^{-3}\%$. Within the LS group, the MEX least square (MEX-LS) struggles at low altitude with a MAPE of $3 \times 10^{-4}\%$ while the MRO least square fit is already down to $10^{-8}\%$, but the performance gap narrows at larger radii.

The same pattern appears in Tab. 3.3, the MRO orbit attains mean absolute error (MAE) of $3 \times 10^{-4} \text{ m}^2 \text{s}^{-2}$, compared to $6.0 \times 10^{-1} \text{ m}^2 \text{s}^{-2}$ for MEX-LS. The HNET models' MAE are much higher, the lowest case being $2.1 \times 10^1 \text{ m}^2 \text{s}^{-2}$, about 2 orders of magnitude worse than the MEX-LS case, and 5 orders of magnitude worse than the MRO-LS case.

This difference in performance between MEX and MRO cases is expected due to the highly eccentric MEX orbits, making gravitational contributions from one side of the planet harder to capture accurately (see Fig. 3.2). This challenge is clearly visible at the top of Fig. 3.4, where MEX cases show larger errors concentrated at the north pole, in contrast to the evenly distributed errors for MRO cases.

Turning to the power spectrum (bottom left of Fig. 3.4), the MRO-LS shows the lowest RMSE amplitude across all spherical harmonic degrees. All the HNET variants struggle to capture the low-frequency spherical harmonic contributions, particularly for $L \in [0, 30]$ where a large part of the power resides. In this band, MEX-LS shows a lower RMSE amplitude, and above $L \geq 30$, it performs comparably to the HNET models.

Removing the regularization term has minimal impact on accuracy. The non-regularized HNET records a test MAPE of $2.69 \times 10^{-4} \%$ and an SNR of 103.8dB, essentially matching the regularized variant's $2.82 \times 10^{-4} \%$ and 102.2dB.

3.6.2 Bennu Gravity Field

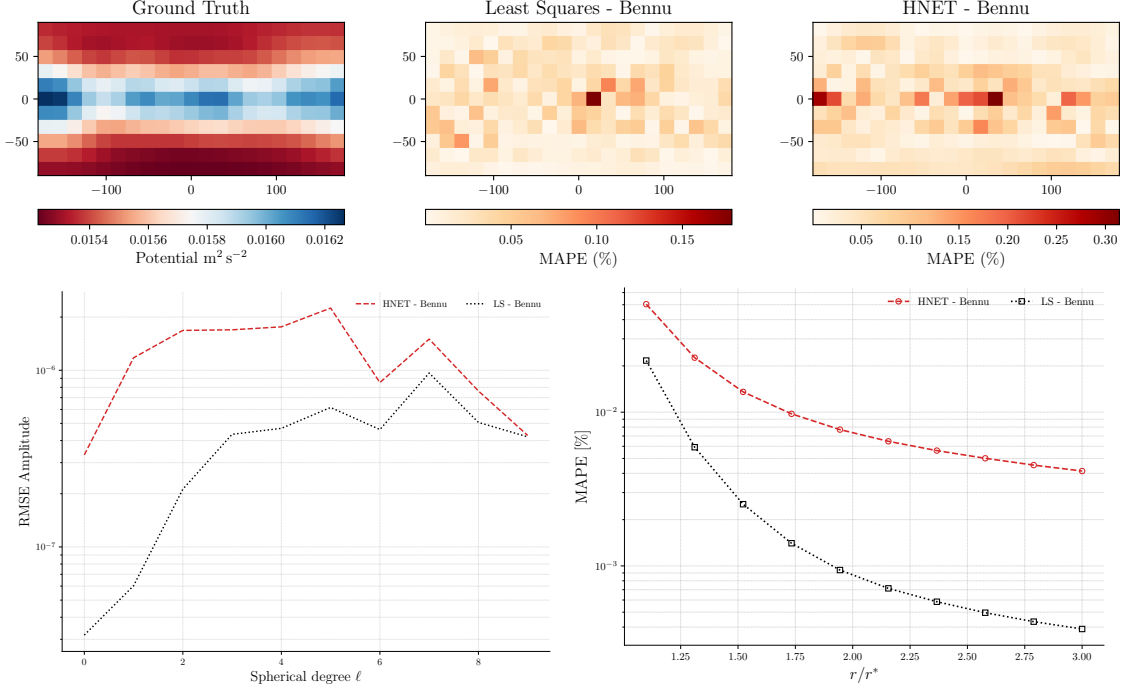


Figure 3.5: Performance comparison between HNET and Least Squares (LS) models for gravitational potential reconstruction on the Bennu asteroid. The top row displays the ground truth potential field at $r/r^* = 1.1$ (left) and spatial distributions of the Mean Absolute Percent Error (MAPE) for both LS (center) and HNET (right) models at $r/r^* = 1.1$. The bottom-left panel illustrates the RMSE power spectrum as a function of spherical harmonic degree ℓ , while the bottom-right panel shows how MAPE varies radially as a function of normalized radius r/r^* .

For the Bennu asteroid, the performance gap between HNET and the LS fit is significantly narrower. Spatial MAPE distributions (Fig. 3.5, top right) reveal comparable accuracy, with HNET exhibiting slightly higher errors near the equator (maximum pointwise MAPE $\approx 3.0 \times 10^{-1}\%$ vs. LS's $2.0 \times 10^{-1}\%$).

The radial evolution of MAPE (Fig. 3.5, bottom right) indicates better generalization by LS, achieving a MAPE of $3.0 \times 10^{-4}\%$ at $r/r^* \approx 3.0$, which is about $10\times$ lower than HNET's $3.0 \times 10^{-3}\%$.

Examining the power spectrum (Fig. 3.5, bottom left) further confirms HNET's challenge in capturing low-frequency spherical harmonic terms. However, the RMSE amplitudes of HNET and LS become comparable at higher spherical harmonic degrees.

Table 3.3 highlights similar training and testing performances between the two models, although the LS fit consistently achieves lower errors. Specifically, the test-set RMSE for LS is $1.55 \times 10^{-6} \text{ m}^2 \text{ s}^{-2}$ for LS, about $2.5\times$ lower than HNET's $3.7 \times 10^{-6} \text{ m}^2 \text{ s}^{-2}$. This reduction corresponds to an improvement in signal-to-noise ratio (SNR) from 67.1 dB (HNET) to 71.5 dB (LS), roughly a 6% improvement.

3.6.3 Synthetic Gravity Field

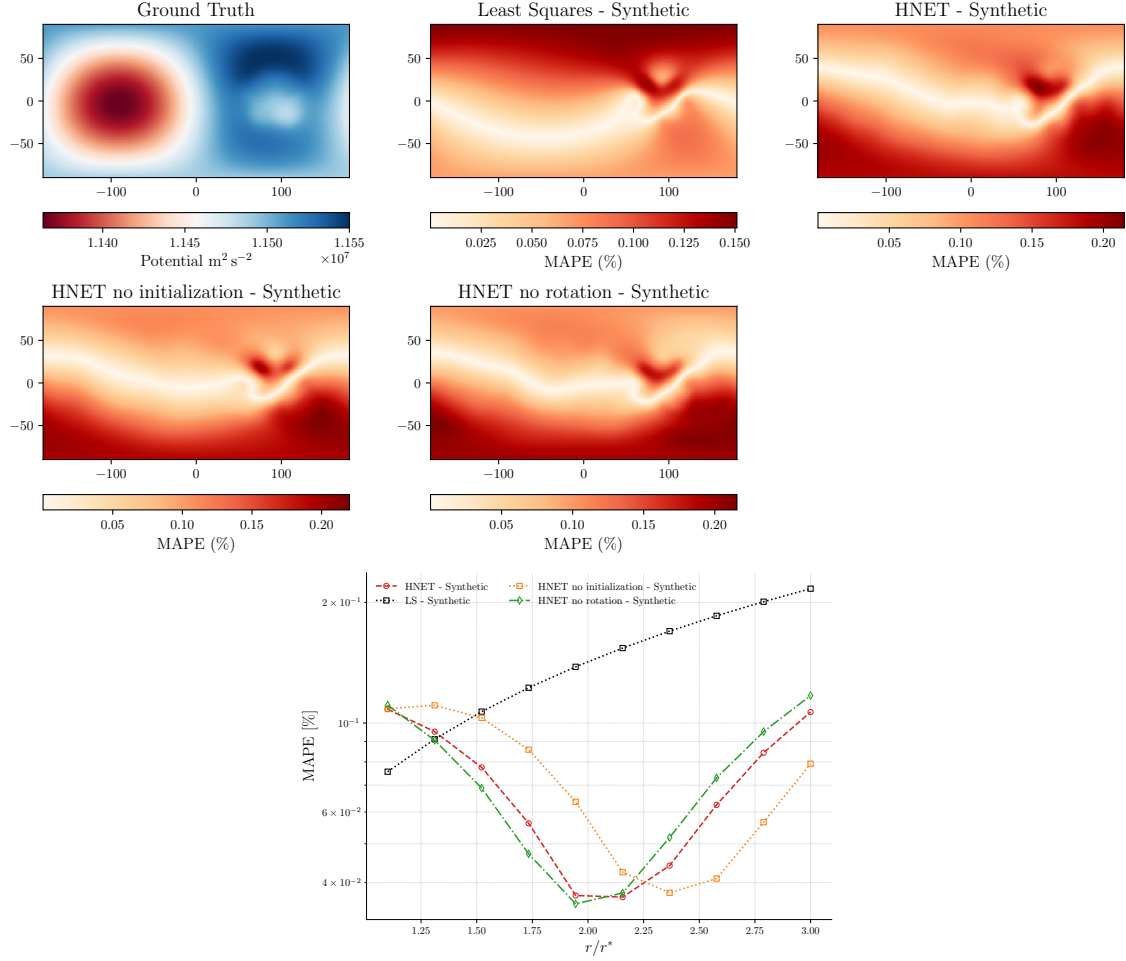


Figure 3.6: Performance comparison of HNET and Least Squares (LS) models for gravitational potential reconstruction of a synthetic asteroid. The top row presents the ground truth potential field at $r/r^* = 1.1$ (left), along with spatial distributions of the Mean Absolute Percent Error (MAPE) at $r/r^* = 1.1$ for the LS fit (center) and the standard HNET model (right). The middle row shows the spatial MAPE distributions for the HNET model variants: without initialization (left) and without atom rotation (right). The bottom panel illustrates the radial evolution of MAPE as a function of the normalized radius r/r^* for all tested configurations.

Given the geometry of the synthetic asteroid experiment (Fig. 3.3), it is clear that the cavity region presents a greater reconstruction challenge compared to the rest of the asteroid, as it requires higher-order spherical harmonic expansions for accurate representation. Due to HNET’s implicit frequency spectrum expansion, we anticipate better performance in reconstructing the cavity region.

To further explore the model’s capabilities, we introduced two additional sub-cases: one without the initialization scheme (Alg. 2) and another without atom rotation. The characteristics of these features are briefly recalled as follows: removing the initialization scheme significantly reduces the initial frequency content available,

potentially hindering the convergence speed and capability of learning high-frequency target. Atom rotation, on the other hand, allows the model to align atoms along dominant signal directions (see Fig. 3.1 and Fig. 2.1), thereby potentially improving its ability to effectively address challenging regions, particularly the cavity.

Figure 3.6 (bottom panel) shows the radial evolution of MAPE. At low altitudes ($r/r^* \approx 1$), the LS fit performs slightly better ($7 \times 10^{-2}\%$ MAPE for LS versus $1 \times 10^{-1}\%$ for HNET models), but as the altitude increases, the LS fit generalizes poorly, with errors continually rising. Conversely, HNET variants closely match the LS performance at low altitudes but significantly outperform it at higher altitudes, reaching minimum MAPEs of approximately $4 \times 10^{-2}\%$ around $r/r^* \approx 2.0$ before errors begin to increase again.

Residual plots in Fig. 3.6 illustrate spatial differences in performance. The LS fit experiences higher errors near the north pole, while HNET models primarily struggle at the south pole.

Numerically, in Table 3.3, HNET models consistently perform slightly better than the LS fit across all metrics in both training and testing phases. HNET achieves training RMSEs of $(2.3\text{--}2.4) \times 10^{-3} \text{ m s}^{-2}$, whereas the LS fit reaches $4.5 \times 10^{-3} \text{ m s}^{-2}$. For testing SNR, HNET attains 59.5–60.4 dB—approximately a 5% improvement over the LS fit’s 57dB.

The rotation of atoms has minimal performance impact. Both variants give an SNR of 60.4 dB and a test MAPE within $0.3 \times 10^{-2}\%$.

On the other hand, the initialization scheme does impact the training convergence. Although the final numerical metrics remain similar (SNR of 59.5 dB without initialization compared to 60.4 dB with initialization), training convergence differs. As illustrated in Fig. 3.7, the initialized model converges sharply within approximately 70 iterations, whereas the non-initialized model requires about 200 iterations to achieve similar accuracy.

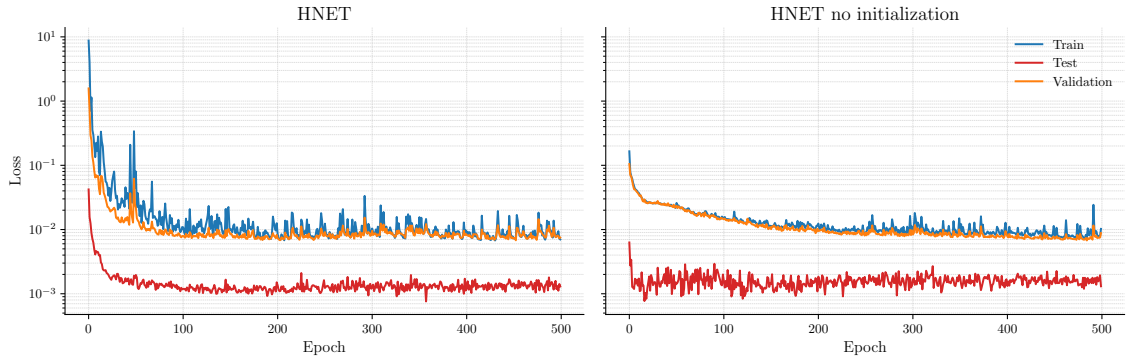


Figure 3.7: Comparison of the training, validation and testing loss between the HNET model with and without initialization scheme.

3.6.4 Discussions

Overall, HNET demonstrates encouraging performance across various datasets and sampling strategies. Yet, it faces a fundamental challenge: small inaccuracies in

predicting the acceleration field propagate into significant errors in the gravitational potential.

Mars This sensitivity is particularly pronounced for Mars. Although HNET achieves notable training precision, its reconstruction of the potential field remains limited to tenths of m^2s^{-2} , whereas the least squares (LS) fit reaches precision down to $10^{-4} \text{m}^2\text{s}^{-2}$. One critical limitation for HNET comes from the use of 32-bit floating-point precision at training, limiting the achievable scaled acceleration error to approximately 10^{-7} . For instance, assuming this scaled error ($\|\delta a\| = 10^{-7}$), the theoretical minimum potential error is estimated at about $1\text{m}^2\text{s}^{-2}$:

$$\|\delta a\| \cdot a^* = 10^{-7} \cdot a^* \approx 3.71 \times 10^{-7} \text{m s}^{-2} \quad \implies \quad \delta\Phi \approx \|\delta a\| \cdot a^* \cdot r^* \approx 1\text{m}^2\text{s}^{-2}.$$

Our experimental results (Tab. 3.3)⁴, approach this theoretical limit. Although training at 64-bit precision could potentially overcome this constraint, it was too computationally expensive for us. Though, preliminary tests with 64-bit precision did not reveal significant performance improvements, yet this definitely has to be further investigated. For Mars, numerical precision potentially limited our results; however, for Bennu, the 32-bit precision is sufficient due to its lower gravitational potential magnitude.

Additionally, removing the regularization term had negligible impact. This may suggest that the model’s non-harmonicity is minimal impact and do not significantly hinder generalization. Additionally, this result may also indicate that the training was suboptimal due to an overly dominant regularization term. Indeed, the regularization coefficient λ is difficult to tune, and can hinder training if not properly set; indicating that the model may be improved on that point.

Bennu In the Mars scenario, HNET did not match the performance of the LS fit, but the results on Bennu were promising. The performance gap is narrow enough to consider HNET as a viable alternative; and additional model runs or further tuning could potentially close the gap. Importantly, the ground truth used here is a spherical harmonic representation, which is inherently advantageous to the LS fit due to its similar implicit representation. A fairer assessment would involve comparing both models against a polygonal representation of the Bennu potential field, as HNET may have captured higher-order contributions not represented in the low-order ($L = 10$) spherical harmonic expansion, thereby potentially understating HNET’s true capabilities.

Synthetic asteroid This case shows that HNET can perform better than LS fits, but only in specific situations, such as bodies with complicated internal shapes. The observed testing errors are significantly larger than those in the Mars scenario, likely due to the synthetic asteroid dataset being four times smaller. Atom rotation did not noticeably affect performance, as variants achieved similar signal-to-noise ratios

⁴The table reports metrics with explicit units, in contrast to the dimensionless training loss in Eq. (3.27).

(SNR) and MAPE values. This indicates the model’s ability to adapt to the data distribution was sufficient without the need for rotation. However, we hypothesize that atom rotation might become relevant when domain sampling is incomplete, enabling atoms to better focus on regions with sparse or uneven data—an aspect reserved for future investigation.

It is not surprising that HNET performs better on irregular objects like Bennu and our synthetic asteroid. Unlike spherical harmonics, HNET is not limited to global features, and it can better capture local details. However, as a result, it performs worse on large regular shapes like Mars, where spherical harmonics are more suitable.

Therefore, our results show that the current version of HNET, as defined and tested in this work, is not a universal solution for every type of gravitational potential—this was clearly shown in the Mars case, where the LS fit performed significantly better. However, for irregularly shaped bodies, this version of HNET can still be a viable alternative. Further tests on more complex shapes are needed to fully explore its potential.

Finally, we must not forget the numerous simplifications we made in our problem setup, and consequently, the results presented here are likely far from representative enough to fully assess HNET’s capabilities in a realistic orbit determination (OD) scenario. This study is a first step, and substantial work remains to be done towards achieving this objective.

3.7 Conclusion

In conclusion, we presented Solid Herglotz-NET as a natural extension of Herglotz-NET, specifically designed for reconstructing gravitational fields of celestial bodies. Solid Herglotz-NET leverages a positional encoding that implicitly encodes irregular solid harmonics, inheriting the correct $1/r$ monopole and multipole structures from spherical-harmonic gravity models, and ensures the learned potential preserves the physical $1/r$ decay.

Through experimental assessments, we demonstrated that Solid Herglotz-NET achieves promising performance, notably in scenarios involving small and irregularly shaped bodies such as the asteroid Bennu and our synthetic asteroid with a cross shaped cavity. However, continued development and validation under realistic operational conditions remain essential.

Further research directions on that topic include:

- Investigate Solid HNET’s performance on smaller asteroids gravitational reconstruction, and incorporating direct comparisons with PINN-GM-III [61] and GeodesyNets [60].
- Explore the model capability in under-sampled scenarios, characterized by poor domain coverage, for which we believe atom rotation could play a key role.
- Applying and validating the HNET architecture in more realistic OD contexts,

extending the experiment with other measurements and more realistic than acceleration data.

Conclusion & Future Work

This thesis introduced **Herglotz-NET (HNET)**, an implicit neural representation framework that encodes spherical—and, by natural extension, volumetric—geometry using the Herglotz formalism instead of explicit spherical harmonics. We have demonstrated that:

- HNET provides a well-posed, efficient, and flexible implicit neural representation specifically tailored for spherical data.
- A depth-dependent spectral analysis generalizes previous Euclidean analyses of INRs [14], highlighting the expressive capabilities of both HNET and SPH-SIREN.
- The framework was successfully extended to $\mathbb{R}^3$, resulting in Solid Herglotz-NET. Its positional encoding implicitly embeds irregular spherical harmonics, thus enabling gravitational potential reconstruction without additional regularization or constraints to enforce decay at infinity.
- Experimental evaluations confirmed: (i) competitive accuracy compared to SPH-SIREN for spherical super-resolution tasks, (ii) significantly faster inference and training, (iii) promising results for gravitational potential reconstruction in small-body gravimetry, and (iv) identification of strength (resp. weaknesses) of HNET at capturing high (resp. low) frequency components.

Future work could explore several directions:

- Given HNET’s difficulty with low-frequency components, future approaches might integrate positional encodings combining lower-degree spherical harmonics—such as those employed in SPH-SIREN—and Herglotz atoms. This integration would allow spherical harmonics to handle lower-frequency components while using Herglotz atoms for higher frequencies.
- The flexibility of the Herglotz formalism permits encoding spherical geometry using any analytic function f . While this thesis employed $f(z) = \exp(z)$, future investigations could explore other analytic functions.
- The Herglotz formalism allows for the development of harmonic neural networks in $\mathbb{R}^3$, beyond the harmonicity on $\mathbb{R}^2$ achieved by Gosh et al. [72] in $\mathbb{R}^2$. Specifically, one could consider architectures of the form: $f_{\Theta}(\mathbf{x}) = \sum_{i=1}^m f_{\Theta_i}(\mathbf{a}_i^{\top} \mathbf{x})$, with parameters Θ and $\mathbf{a}_i$ as Herglotz vectors. A potential challenge here is

performance degradation due to the need to divide the network into multiple branches, which increases the complexity and number of parameters.

Appendix A

Metrics

Hereafter, we describe the different metrics that would be used to evaluate the performance of our model. Bold symbols indicate vectors, while non-bold symbols represent scalars.

Root Mean Square Error (RMSE) The RMSE measures the average magnitude of prediction errors, giving higher weight to larger errors:

$$\text{RMSE} = \sqrt{\frac{1}{N} \sum_{i=1}^N \|\mathbf{y}_i - \hat{\mathbf{y}}_i\|_2^2}, \quad (\text{A.1})$$

with $\mathbf{y}_i$ the true value, and $\hat{\mathbf{y}}_i$ the prediction. A lower RMSE indicates higher accuracy but is sensitive to large errors.

Mean Absolute Error (MAE) The MAE evaluates prediction accuracy by averaging absolute errors uniformly:

$$\text{MAE} = \frac{1}{N} \sum_{i=1}^N \|\mathbf{y}_i - \hat{\mathbf{y}}_i\|_1. \quad (\text{A.2})$$

It is robust against outliers, providing a straightforward measure of average error magnitude.

Mean Absolute Percentage Error (MAPE) The MAPE normalizes errors as percentages of true values:

$$\text{MAPE} = \frac{1}{N} \sum_{i=1}^N \left| \frac{y_i - \hat{y}_i}{y_i} \right| \times 100, \quad (\text{A.3})$$

offering intuitive interpretation regardless of absolute scale.

Signal-to-Noise Ratio (SNR) The SNR quantifies prediction quality by comparing the norm of the true signal to that of the error:

$$\text{SNR} = 10 \log_{10} \left(\frac{\sum_{i=1}^N |y_i|^2}{\sum_{i=1}^N |y_i - \hat{y}_i|^2} \right). \quad (\text{A.4})$$

Intuitively, the SNR indicates how much stronger the true signal is compared to the errors made by the model. Expressed in decibels (dB), higher SNR values indicate stronger signals relative to prediction noise.

Root Mean Square Error Power Spectrum Specific to spherical signals, the Root Mean Squared Error (RMSE) power spectrum highlights how prediction errors distribute across spherical harmonic degrees:

$$S_\ell = \sqrt{\frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |c_{\ell,m} - \hat{c}_{\ell,m}|^2}, \quad (\text{A.5})$$

with $c_{\ell,m}$ and $\hat{c}_{\ell,m}$ the ground truth and predicted spherical harmonic coefficients. It identifies precisely the spectral degrees at which the model succeeds or struggles.

Appendix B

Proofs of Theorem

Lemma B.1. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function, such that*

$$f(z) = \sum_{\ell=0}^{\infty} c_{\ell} z^{\ell}, \quad \forall z \in \mathbb{C}, \quad (\text{B.1})$$

for some $c_{\ell} \in \mathbb{C}$. Let $\mathbf{a} \in \mathbb{C}^3$ satisfy $\mathbf{a}^{\top} \mathbf{a} = 0$, then the function $g(\mathbf{x}) = f(\mathbf{a}^{\top} \mathbf{x})$ is harmonic on $\mathbb{R}^3$ and, when restricted to the sphere $\mathbb{S}^2$, admits the expansion

$$g(\boldsymbol{\xi}(\theta, \varphi)) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m}(\mathbf{a}) Y_{\ell m}(\theta, \varphi), \quad \boldsymbol{\xi} \in \mathbb{S}^2. \quad (\text{B.2})$$

for some $c_{\ell m}(\mathbf{a}) \in \mathbb{C}$

Proof. First, note that for any $\mathbf{x} \in \mathbb{R}^3$,

$$\Delta g(\mathbf{x}) = f''(\mathbf{a}^{\top} \mathbf{x}) (\mathbf{a}^{\top} \mathbf{a}) = 0,$$

such that g is harmonic. Next, observe that for each ℓ and $\boldsymbol{\xi} \in \mathbb{S}^2$, we have $(\mathbf{a}^{\top} \boldsymbol{\xi})^{\ell} \in \mathcal{H}_{\ell}(\mathbb{S}^2)$. Therefore it exists $\tilde{c}_{\ell, m}$ such that

$$(\mathbf{a}^{\top} \boldsymbol{\xi})^{\ell} = \sum_{m=-\ell}^{\ell} \tilde{c}_{\ell, m} Y_{\ell m}(\theta, \varphi), \quad \boldsymbol{\xi} \in \mathbb{S}^2. \quad (\text{B.3})$$

Since

$$g(\boldsymbol{\xi}) = f(\mathbf{a}^{\top} \boldsymbol{\xi}) = \sum_{\ell=0}^{\infty} c_{\ell} (\mathbf{a}^{\top} \boldsymbol{\xi})^{\ell},$$

we can use the expansion above to obtain

$$g(\boldsymbol{\xi}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell} \tilde{c}_{\ell m} Y_{\ell m}(\theta, \varphi),$$

which proves the claim. □

Lemma B.2. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ and define*

$$g : \mathbb{R}^3 \rightarrow \mathbb{C}, \quad g(\mathbf{x}) = \mathbb{E}_{\mathbf{a} \sim D} f(\mathbf{a}^\top \mathbf{x}),$$

where D is the law of the vector returned by Alg. 1. Then g is $\text{SO}(3)$ -invariant: for every $R \in \text{SO}(3)$,

$$g(R\mathbf{x}) = g(\mathbf{x}).$$

Proof. Let $\mathbf{a}_0 = \frac{1}{\sqrt{2}}(1, i, 0)^\top = \frac{1}{\sqrt{2}}\mathbf{e}_1 + i\frac{1}{\sqrt{2}}\mathbf{e}_2$ and set

$$\Phi : \text{SO}(3) \longrightarrow \left\{ \text{vectors satisfying Eq. 2.10} \right\}, \quad \Phi(\mathbf{R}) = \mathbf{R}\mathbf{a}_0.$$

We shall first prove that the mapping Φ is bijective. If $\Phi(\mathbf{R}) = \Phi(\mathbf{S})$, i.e. $\mathbf{R}\mathbf{a}_0 = \mathbf{S}\mathbf{a}_0$, then

$$\mathbf{R}\mathbf{e}_1 = \mathbf{S}\mathbf{e}_1, \quad \mathbf{R}\mathbf{e}_2 = \mathbf{S}\mathbf{e}_2,$$

because $\mathbf{e}_1$ and $\mathbf{e}_2$ are respectively the real and imaginary parts of $\mathbf{a}_0$. Since $(\sqrt{2}\mathbf{e}_1, \sqrt{2}\mathbf{e}_2)$ is an orthonormal 2-frame and both $\mathbf{R}, \mathbf{S}$ are rotations (they preserve orientation and lengths), agreement on that frame forces agreement on the whole space: $\mathbf{R} = \mathbf{S}$. Thus Φ is injective.

Conversely, given any admissible vector $\mathbf{a} = \mathbf{a}_{\Re} + i\mathbf{a}_{\Im}$ with $\|\mathbf{a}_{\Re}\| = \|\mathbf{a}_{\Im}\| = \frac{1}{\sqrt{2}}$ and $\mathbf{a}_{\Re} \perp \mathbf{a}_{\Im}$, the pair $(\sqrt{2}\mathbf{a}_{\Re}, \sqrt{2}\mathbf{a}_{\Im})$ is an orthonormal 2-frame. There exists a unique rotation $\mathbf{R}$ sending $(\mathbf{e}_1, \mathbf{e}_2)$ to that frame; then $\mathbf{a} = \mathbf{R}\mathbf{a}_0$, so Φ is surjective.

Hence Φ is bijective. Drawing $\mathbf{R} \sim \mathcal{U}(\text{SO}(3))$ and setting $\mathbf{a} = \Phi(\mathbf{R})$ reproduces exactly the distribution generated by Alg. 1, i.e. $\mathbf{a} \sim D$. Therefore, for any $\mathbf{S} \in \text{SO}(3)$,

$$\begin{aligned} g(\mathbf{S}\mathbf{x}) &= \mathbb{E}_{\mathbf{a} \sim D} f(\mathbf{a}^\top \mathbf{S}\mathbf{x}) \\ &= \mathbb{E}_{\mathbf{R} \sim \mathcal{U}(\text{SO}(3))} f(\mathbf{a}_0^\top \mathbf{R}^\top \mathbf{S}\mathbf{x}). \end{aligned}$$

With $\mathbf{T} := \mathbf{R}^\top \mathbf{S}$ we have $\mathbf{T} \sim \mathcal{U}(\text{SO}(3))$ because left multiplication by the fixed $\mathbf{S}$ leaves the uniform law unchanged; hence

$$\begin{aligned} g(\mathbf{S}\mathbf{x}) &= \mathbb{E}_{\mathbf{T} \sim \mathcal{U}(\text{SO}(3))} f(\mathbf{a}_0^\top \mathbf{T}\mathbf{x}) \\ &= \mathbb{E}_{\mathbf{a}' \sim D} f(\mathbf{a}'^\top \mathbf{x}) \\ &= g(\mathbf{x}) \end{aligned}$$

□

Lemma B.3. *Given a complex value polynomial $p_K(z) = \sum_{k=1}^K a_k z^k$ of degree $K \geq 0$ and an order L , for any spherical function $f(\theta, \varphi) := \sum_{\ell=0}^L \sum_{|m| \leq \ell} c_{\ell m} Y_{\ell m}(\theta, \varphi)$ (with $c_{\ell m} \in \mathbb{C}$) with bandlimit L , $p_K \circ f$ has a bandlimit KL , i.e., there exist coefficients $\kappa_{\ell m} \in \mathbb{C}$ such that*

$$p_K(f(\theta, \varphi)) = \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \kappa_{\ell m} Y_{\ell m}(\theta, \varphi).$$

Proof. By induction, we start by observing that $K = 0$ is trivial since $Y_{0,0}(\theta, \varphi)$ is constant. Next, for general $K > 0$, the proof only needs to consider monomials of degree $K + 1$ since $p_{K+1}(z) = a_{K+1}z^{K+1} + p_K(z)$. We can develop

$$\begin{aligned}
f(\theta, \varphi)^{K+1} &= \sum_{(\ell, m)}^{\llbracket KL \rrbracket} \sum_{(\ell', m')}^{\llbracket L \rrbracket} \kappa_{\ell m} c_{\ell' m'} Y_{\ell m}(\theta, \varphi) Y_{\ell' m'}(\theta, \varphi) \\
&= \sum_{(\ell, m)}^{\llbracket KL \rrbracket} \sum_{(\ell', m')}^{\llbracket L \rrbracket} \kappa_{\ell m} c_{\ell' m'} \sum_{(k, p)}^{\llbracket \ell + \ell' \rrbracket} \alpha_{kp} Y_{kp}(\theta, \varphi) \\
&= \sum_{(\ell, m)}^{\llbracket (K+1)L \rrbracket} \hat{\kappa}_{\ell m} Y_{\ell m}(\theta, \varphi),
\end{aligned}$$

where the second line uses the contraction rule of SHs [39, Sec. 5.8], the coefficients $\alpha_{kp} \in \mathbb{C}$ involve Wigner 3- j symbols, and $\hat{\kappa}_{\ell m} \in \mathbb{C}$. $\square$

Theorem B.1. Let $f_{\Theta} : \mathbb{S}^2 \rightarrow \mathbb{C}$ be an INR defined as in (1.2) with PE $\psi(\mathbf{x})$ satisfying (2.24), $Q \geq 1$ layers and polynomial activation functions $\sigma^{(q)}(z) = \sum_{k=0}^K \alpha_k z^k$. Then, there exist coefficients $\kappa_{\ell m} \in \mathbb{C}$ such that

$$f_{\Theta}(\theta, \varphi) = \sum_{\ell=0}^{KQ-1} \sum_{|m| \leq \ell} \kappa_{\ell m} Y_{\ell m}(\theta, \varphi) \quad (\text{B.4})$$

Proof. We first consider a bias-free scenario ($\mathbf{b}^{(q)} = \mathbf{0}$ for all $q = 1, \dots, Q-1$). The i -th component of the pre-activation input of the second layer reads

$$\begin{aligned}
z_i^{(2)} &= \sum_{j=1}^{d^{(2)}} W_{ij}^{(2)} \sigma^{(1)} \left([\mathbf{W}^{(1)} \psi(\mathbf{x}) + \mathbf{b}^{(1)}]_j \right) \\
&= \sum_{j=1}^{d^{(2)}} W_{ij}^{(2)} \sigma^{(1)} \left(\sum_{(\ell, m)}^{\llbracket L_0 \rrbracket} c_{j\ell m} Y_{\ell m}(\theta, \varphi) \right) \\
&= \sum_{(\ell, m)}^{\llbracket KL_0 \rrbracket} \left(\sum_{j=1}^{d^{(2)}} W_{ij}^{(2)} \kappa_{j\ell m} \right) Y_{\ell m}(\theta, \varphi) \\
&= \sum_{(\ell, m)}^{\llbracket KL_0 \rrbracket} \hat{\kappa}_{i\ell m} Y_{\ell m}(\theta, \varphi),
\end{aligned}$$

with $\hat{\kappa}_{i\ell m} \in \mathbb{C}$. The second line comes from Eq. 2.24, and the third one from Lem. B.3 for some $\kappa_{j\ell m} \in \mathbb{C}$. The result of Th. 2.1 follows by applying Lem. 2.3 for each layer, that is, $(Q-1)$ times. Adding biases shifts the constant term of the polynomial activation function. Consequently, biases do not alter the above reasoning, which concludes the proof. $\square$

Lemma B.4. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be an analytic function, such that*

$$f(z) = \sum_{\ell=0}^{\infty} c_{\ell} z^{\ell}, \quad \forall z \in \mathbb{C},$$

and let $\mathbf{a} \in \mathbb{C}^3$ satisfy $\mathbf{a}^T \mathbf{a} = 0$. We define

$$g_{\text{reg}}(\mathbf{x}) = f(\mathbf{a}^T \mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^3, \quad g_{\text{irr}}(\mathbf{x}) = \frac{1}{|\mathbf{x}|} f\left(\frac{\mathbf{a}^T \mathbf{x}}{|\mathbf{x}|^2}\right) \quad \mathbf{x} \in \mathbb{R}_0^3,$$

Then g_{reg} is harmonic on $\mathbb{R}^3$ and g_{irr} is harmonic on $\mathbb{R}_0^3$; moreover, they admit a solid-harmonic expansions given by

$$\begin{aligned} g_{\text{reg}}(\mathbf{x}) &= \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} \alpha_{\ell m}(\mathbf{a}) r^{\ell} Y_{\ell, m}(\boldsymbol{\xi}) = \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} \alpha_{\ell m}(\mathbf{a}) R_{\ell, m}(\mathbf{x}) \\ g_{\text{irr}}(\mathbf{x}) &= \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} \alpha_{\ell m}(\mathbf{a}) r^{-(\ell+1)} Y_{\ell, m}(\boldsymbol{\xi}) = \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} \alpha_{\ell m}(\mathbf{a}) I_{\ell, m}(\mathbf{x}), \end{aligned}$$

where $\mathbf{x} = r\boldsymbol{\xi} \in \mathbb{R}^3$, $\boldsymbol{\xi} \in \mathbb{S}^2$ and for some coefficients $\alpha_{\ell m}(\mathbf{a}) \in \mathbb{C}$.

Proof. Let us expand both g_{reg} and g_{irr} using the analyticity of f :

$$\begin{aligned} g_{\text{reg}}(\mathbf{x}) &= f(\mathbf{a}^T \mathbf{x}) = \sum_{\ell=0}^{\infty} c_{\ell} (\mathbf{a}^T \mathbf{x})^{\ell}, \\ g_{\text{irr}}(\mathbf{x}) &= \frac{1}{|\mathbf{x}|} f\left(\frac{\mathbf{a}^T \mathbf{x}}{|\mathbf{x}|^2}\right) = \sum_{\ell=0}^{\infty} c_{\ell} \frac{(\mathbf{a}^T \frac{\mathbf{x}}{|\mathbf{x}|})^{\ell}}{|\mathbf{x}|^{\ell+1}}. \end{aligned}$$

Let $\mathbf{x} = r\boldsymbol{\xi}$, with $r = |\mathbf{x}|$, we can rewrite the relations as:

$$\begin{aligned} g_{\text{reg}}(\mathbf{x}) &= \sum_{\ell=0}^{\infty} c_{\ell} r^{\ell} (\mathbf{a}^T \boldsymbol{\xi})^{\ell}, \\ g_{\text{irr}}(\mathbf{x}) &= \sum_{\ell=0}^{\infty} c_{\ell} r^{-(\ell+1)} (\mathbf{a}^T \boldsymbol{\xi})^{\ell}. \end{aligned}$$

Applying (B.3) yields :

$$\begin{aligned} g_{\text{reg}}(\mathbf{x}(r, \theta, \varphi)) &= \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} c_{\ell} \tilde{c}_{\ell m}(\mathbf{a}) r^{\ell} Y_{\ell m}(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} \alpha_{\ell m}(\mathbf{a}) R_{\ell m}(r, \theta, \varphi), \\ g_{\text{irr}}(\mathbf{x}(r, \theta, \varphi)) &= \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} c_{\ell} \tilde{c}_{\ell m}(\mathbf{a}) r^{-(\ell+1)} Y_{\ell m}(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{|m| \leq \ell} \alpha_{\ell m}(\mathbf{a}) r^{(\ell+1)} Y_{\ell m}(\theta, \varphi), \end{aligned}$$

for some $\alpha_{\ell m}(\mathbf{a}) \in \mathbb{C}$ and $\tilde{c}_{\ell m}(\mathbf{a}) \in \mathbb{C}$, which gives the desired expansions for g_{reg} and g_{irr} . Harmonicity follows, from the definition of the solid harmonics. $\square$

Lemma B.5. *Given a complex-valued polynomial*

$$p_K(z) = \sum_{k=1}^K a_k z^k,$$

of degree $K \geq 0$, for any function

$$f(r, \theta, \varphi) := \sum_{\ell=0}^L \sum_{|m| \leq \ell} c_{\ell m} R_{\ell m}(r, \theta, \varphi) + \sum_{\ell=0}^L \sum_{|m| \leq \ell} \sum_{q=0}^{\ell-1} \kappa_{q\ell m} r^\ell Y_{qm}(\theta, \varphi),$$

with $c_{\ell m}, \kappa_{q\ell m} \in \mathbb{C}$ and bandlimit L , there exist coefficients $\beta_{\ell m}, \gamma_{q\ell m} \in \mathbb{C}$ such that

$$p_K(f(r, \theta, \varphi)) = \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \beta_{\ell m} R_{\ell m}(r, \theta, \varphi) + \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \sum_{q=0}^{\ell-1} \gamma_{q\ell m} r^\ell Y_{qm}(\theta, \varphi).$$

Proof. We prove the lemma by induction on the degree K of the polynomial

$$p_K(z) = \sum_{k=1}^K a_k z^k.$$

Base Case ($K = 1$): For $p_1(z) = a_1 z$, by the new definition we have

$$p_1(f(r, \theta, \varphi)) = a_1 f(r, \theta, \varphi),$$

by letting $\beta_{\ell m} = a_1 c_{\ell m}$ and $\gamma_{q\ell m} = a_1 \kappa_{q\ell m}$, we have the desired expansion.

Inductive Step: Assume that for some $K \geq 2$ the expansion

$$f(r, \theta, \varphi)^K = \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \kappa_{\ell m}^{(K)} R_{\ell m}(r, \theta, \varphi) + \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \sum_{q=0}^{\ell-1} \kappa_{q\ell m}^{(K)} r^\ell Y_{qm}(\theta, \varphi)$$

holds. We have that :

$$\begin{aligned} f(r, \theta, \varphi)^{K+1} &= \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \sum_{\ell'=0}^L \sum_{|m'| \leq \ell'} \kappa_{\ell m}^{(K)} c_{\ell' m'} r^{\ell+\ell'} Y_{\ell m}(\theta, \varphi) Y_{\ell' m'}(\theta, \varphi) \\ &+ \sum_{\ell=0}^L \sum_{|m| \leq \ell} \sum_{\ell'=0}^{KL} \sum_{|m'| \leq \ell'} \sum_{q'=0}^{\ell'-1} \kappa_{q' \ell' m'}^{(K)} c_{\ell m} r^{\ell'+\ell} Y_{\ell m}(\theta, \varphi) Y_{q' m'}(\theta, \varphi) \\ &+ \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \sum_{\ell'=0}^L \sum_{|m'| \leq \ell'} \sum_{q'=0}^{\ell'-1} \kappa_{q' \ell' m'}^{(K)} \kappa_{\ell m}^{(K)} r^{\ell'+\ell} Y_{\ell' m'}(\theta, \varphi) Y_{\ell m}(\theta, \varphi) \\ &+ \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \sum_{q=0}^{\ell-1} \sum_{\ell'=0}^L \sum_{|m'| \leq \ell'} \sum_{q'=0}^{\ell'-1} \kappa_{q\ell m} \kappa_{q' \ell' m'}^{(K)} r^{\ell+\ell'} Y_{qm}(\theta, \varphi) Y_{\ell' m'}(\theta, \varphi) \end{aligned}$$

Applying the contraction rule on each term term yields:

$$\begin{aligned}
f(r, \theta, \varphi)^{K+1} &= \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \sum_{\ell'=0}^L \sum_{|m'| \leq \ell'} \kappa_{\ell m}^{(K)} c_{\ell' m'} r^{\ell+\ell'} \sum_{c=0}^{\ell+\ell'} \tilde{\kappa}_{c m+m'} Y_{c m+m'}(\theta, \varphi) \\
&+ \sum_{\ell=0}^L \sum_{|m| \leq \ell} \sum_{\ell'=0}^{KL} \sum_{|m'| \leq \ell'} \sum_{q'=0}^{\ell'-1} \kappa_{q' \ell' m'}^{(K)} c_{\ell m} r^{\ell'+\ell} \sum_{c=0}^{\ell+\ell'} \tilde{\kappa}_{c m+m'} Y_{c m+m'}(\theta, \varphi) \\
&+ \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \sum_{\ell'=0}^L \sum_{|m'| \leq \ell'} \sum_{q'=0}^{\ell'-1} \kappa_{q' \ell' m'}^{(K)} \kappa_{\ell m}^{(K)} r^{\ell'+\ell} \sum_{c=0}^{\ell+\ell'} \tilde{\kappa}_{c m+m'} Y_{c m+m'}(\theta, \varphi) \\
&+ \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \sum_{q=0}^{\ell-1} \sum_{\ell'=0}^L \sum_{|m'| \leq \ell'} \sum_{q'=0}^{\ell'-1} \kappa_{q \ell m} \kappa_{q' \ell' m'}^{(K)} r^{\ell'+\ell} \sum_{c=0}^{\ell+\ell'} \tilde{\kappa}_{c m+m'} Y_{c m+m'}(\theta, \varphi)
\end{aligned}$$

Which we further simplify to:

$$\begin{aligned}
f(r, \theta, \varphi)^{K+1} &= \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \sum_{\ell'=0}^L \sum_{|m'| \leq \ell'} \kappa_{\ell m}^{(K)} c_{\ell' m'} \left[\tilde{\kappa}_{\ell+\ell' m+m'} R_{\ell+\ell' m+m'} + \sum_{c=0}^{\ell+\ell'-1} \tilde{\kappa}_{c m+m'} r^{\ell+\ell'} Y_{c m+m'} \right] \\
&+ \sum_{\ell=0}^L \sum_{|m| \leq \ell} \sum_{\ell'=0}^{KL} \sum_{|m'| \leq \ell'} \sum_{q'=0}^{\ell'-1} \kappa_{q' \ell' m'}^{(K)} c_{\ell m} \left[\tilde{\kappa}_{\ell+\ell' m+m'} R_{\ell+\ell' m+m'} + \sum_{c=0}^{\ell+\ell'-1} \tilde{\kappa}_{c m+m'} r^{\ell+\ell'} Y_{c m+m'} \right] \\
&+ \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \sum_{\ell'=0}^L \sum_{|m'| \leq \ell'} \sum_{q'=0}^{\ell'-1} \kappa_{q' \ell' m'}^{(K)} \kappa_{\ell m}^{(K)} \left[\tilde{\kappa}_{\ell+\ell' m+m'} R_{\ell+\ell' m+m'} + \sum_{c=0}^{\ell+\ell'-1} \tilde{\kappa}_{c m+m'} r^{\ell+\ell'} Y_{c m+m'} \right] \\
&+ \sum_{\ell=0}^{KL} \sum_{|m| \leq \ell} \sum_{q=0}^{\ell-1} \sum_{\ell'=0}^L \sum_{|m'| \leq \ell'} \sum_{q'=0}^{\ell'-1} \kappa_{q \ell m} \kappa_{q' \ell' m'}^{(K)} \left[\tilde{\kappa}_{\ell+\ell' m+m'} R_{\ell+\ell' m+m'} + \sum_{c=0}^{\ell+\ell'-1} \tilde{\kappa}_{c m+m'} r^{\ell+\ell'} Y_{c m+m'} \right]
\end{aligned}$$

We can simplified out the different terms and obtain the final expression :

$$f(r, \theta, \varphi)^{K+1} = \sum_{\ell=0}^{(K+1)L} \sum_{|m| \leq \ell} \kappa_{\ell m}^{(K+1)} R_{\ell m}(r, \theta, \varphi) + \sum_{\ell=0}^{(K+1)L} \sum_{|m| \leq \ell} \sum_{q=0}^{\ell-1} \kappa_{q \ell m}^{(K+1)} r^{\ell} Y_{q m}(\theta, \varphi)$$

where the coefficients $\kappa_{\ell m}^{(K+1)}$ and $\kappa_{q \ell m}^{(K+1)}$ are chosen conveniently. Finally, since

$$p_{K+1}(f(r, \theta, \varphi)) = p_K(f(r, \theta, \varphi)) + a_{K+1} f(r, \theta, \varphi)^{K+1},$$

and both $p_K(f(r, \theta, \varphi))$ and $f(r, \theta, \varphi)^{K+1}$ admit expansions of the stated form, it follows that

$$p_{K+1}(f(r, \theta, \varphi))$$

admits an expansion with bandlimit $(K+1)L$. □

Theorem B.2. Let $f_{\Theta} : \mathbb{R}^3 \rightarrow \mathbb{R}$ be an INR defined as in (3.19), $Q \geq 1$ layers, polynomial activation functions $\sigma^{(q)}(z) = \sum_{k=0}^K \alpha_k z^k$ and the atoms in (2.13) (where $\mathbf{x} \in \mathbb{R}^3$) are truncated at order L_0 in their Taylor expansions. Then, there exist coefficients $c_{\ell m}, \kappa_{q\ell m} \in \mathbb{C}$ such that

$$\begin{aligned} f_{\Theta}(r, \theta, \varphi) = & \sum_{\ell=0}^{K^{Q-1}L_0} \sum_{|m| \leq \ell} c_{\ell m} R_{\ell m}(r, \theta, \varphi) \\ & + \sum_{\ell=0}^{K^{Q-1}L} \sum_{|m| \leq \ell} \sum_{q=0}^{\ell-1} \kappa_{q\ell m} r^{\ell} Y_{q,m}(\theta, \varphi). \end{aligned} \quad (\text{B.5})$$

Proof. We first consider a bias-free scenario ($\mathbf{b}^{(q)} = \mathbf{0}$ for all $q = 1, \dots, Q-1$). The i -th component of the pre-activation input of the second layer reads

$$\begin{aligned} z_i^{(2)} &= \sum_{j=1}^{d^{(2)}} W_{ij}^{(2)} \sigma^{(1)}([\mathbf{W}^{(1)}\psi(\mathbf{x}) + \mathbf{b}^{(1)}]_j) \\ &= \sum_{j=1}^{d^{(2)}} W_{ij}^{(2)} \sigma^{(1)}\left(\sum_{(\ell,m)}^{[L_0]} c_{j\ell m} R_{\ell m}(r, \theta, \varphi)\right) \\ &= \sum_{(\ell,m)}^{[KL_0]} \left(\sum_{j=1}^{d^{(2)}} W_{ij}^{(2)} \hat{c}_{j\ell m}\right) R_{\ell m}(r, \theta, \varphi) + \sum_{(\ell,m)}^{[KL_0]} \sum_{q=0}^{\ell-1} \left(\sum_{j=1}^{d^{(2)}} W_{ij}^{(2)} \kappa_{jq\ell m}\right) r^{\ell} Y_{qm}(\theta, \varphi) \\ &= \sum_{(\ell,m)}^{[KL_0]} \gamma_{i\ell m} R_{\ell m}(r, \theta, \varphi) + \sum_{(\ell,m)}^{[KL_0]} \sum_{q=0}^{\ell-1} \alpha_{iq\ell m} r^{\ell} Y_{qm}(\theta, \varphi), \end{aligned}$$

with $\gamma_{i\ell m}, \alpha_{iq\ell m} \in \mathbb{C}$. The second line comes from Eq. 2.24, and the third one from Lem. B.5. The result follows by applying Lem. B.5 for each layer, that is, $(Q-1)$ times. Adding biases shifts the constant term of the polynomial activation function. Consequently, biases do not alter the above reasoning, which concludes the proof. $\square$

Appendix C

Models Specifications

Model	Positional Encoding					Sinus MLP			# parameters
	L_{pe}	m	ω_0	ϖ_0	init	Layers	Units	ω_0	
HNET	15	256	—	1.0	✓	3	150	1	63143
SIREN	—	256	10.0	—	—	3	150	1	62119
SPH-SIREN	15	256	—	—	—	3	150	1	61351

Table C.1: Models’ specifications (✓ = used, “—” = not applicable) for super-resolution experiment.

Model	Positional Encoding					Sinus MLP			# parameters
	L_{pe}	m	ω_0	ϖ_0	init	Layers	Units	ω_0	
HNET	17	324	—	1.0	✓	3	200	1	107669
SIREN	—	324	10.0	—	—	3	200	1	106373
SPH-SIREN	17	324	—	—	—	3	200	1	105401

Table C.2: Models’ specifications (✓ = used, “—” = not applicable) for representation efficiency experiment.

Model	HNET								Spherical Harmonics model
	Positional Encoding					Sinus MLP		# parameters	L
	L_{pe}	ϖ_0	r_{ref}	init	rotation	Layers	Units		
Bennu	10	1.5	1.0	✓	✓	2	100	13047	10
Mars (all cases)	15	1.5	1.0	✓	✓	3	125	49542	100
Synthetic	15	1.5	1.0	✓	✓	3	125	49542	100
Synthetic (no init)	15	1.5	1.0	✗	✓	3	125	49542	100
Synthetic (no rotation)	15	1.5	1.0	✓	✗	3	125	49542	100

Table C.3: Model specifications (✓ = used, ✗ = not used) for the gravimetry experiments.

Appendix D

Derivation of Spherical Harmonics

Our first goal is to establish a key property that any function in $\mathcal{H}_\ell(\mathbb{S}^2)$ must satisfy. Since these functions arise as restrictions of homogeneous harmonic polynomials, they naturally obey Laplace's equation. Expressing this equation in spherical coordinates leads to

$$\begin{aligned}\Delta f(\mathbf{x}) &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2} \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \Delta_{\mathbb{S}^2} f(\mathbf{x}) \\ &= 0, \quad \forall \mathbf{x} \in \Omega \subseteq \mathbb{R}^3,\end{aligned}\tag{D.1}$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator on the sphere.

Let $p_\ell \in \mathcal{H}_\ell(\mathbb{S}^2)$ with $\ell \in \mathbb{N}$. By definition, p_ℓ is both harmonic and homogeneous. Writing a point $\mathbf{x} \in \mathbb{R}^3$ as $\mathbf{x} = r \boldsymbol{\xi}$ with $\boldsymbol{\xi} \in \mathbb{S}^2$, we have

$$p_\ell(\mathbf{x}) = r^\ell p_\ell(\boldsymbol{\xi}).$$

Substituting this expression into (D.1) yields the eigenvalue problem for the Laplace-Beltrami operator (see, e.g., [24]):

$$\Delta_{\mathbb{S}^2} p_\ell(\boldsymbol{\xi}) = -\ell(\ell + 1) p_\ell(\boldsymbol{\xi}).\tag{D.2}$$

A standard technique for finding these eigenfunctions is separation of variables; assuming that

$$p_\ell(\boldsymbol{\xi}(\theta, \varphi)) = p_\ell(\theta, \varphi) = \Theta(\theta) \Phi(\varphi),$$

and substituting into (D.2), one obtains after expanding $\Delta_{\mathbb{S}^2}$ and rearranging:

$$\frac{\sin \theta}{\Theta(\theta)} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta(\theta)}{d\theta} \right) + \ell(\ell + 1) \sin^2 \theta = \frac{1}{\Phi(\varphi)} \frac{d^2 \Phi(\varphi)}{d\varphi^2}.$$

Since the left-hand side depends solely on θ and the right-hand side solely on φ , both must be equal to a constant. We denote this constant by $-m^2$ (with $m^2 \in \mathbb{R}_+$) so as

to ensure that the solution in φ is periodic. This yields the following pair of ordinary differential equations:

$$\frac{d^2\Phi}{d\varphi^2} = -m^2 \Phi, \quad (\text{D.3})$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2 \theta} \right] \Theta = 0. \quad (\text{D.4})$$

The solution to (D.3) is given by

$$\Phi(\varphi) = e^{im\varphi}, \quad m \in \mathbb{Z}, \quad (\text{D.5})$$

where the periodicity condition $\Phi(\varphi) = \Phi(\varphi + 2\pi)$ forces m to be an integer.

In Eq. (D.4), a change of variables $x = \cos \theta$ and the definition $y(x) = \Theta(\theta)$ recast the equation as

$$\frac{d}{dx} \left[(1-x^2) \frac{dy}{dx} \right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2} \right] y = 0. \quad (\text{D.6})$$

This is the general Legendre equation, whose solutions are the *associated Legendre polynomials*:

$$y(x) = P_\ell^m(x), \quad m = -\ell, -\ell+1, \dots, \ell. \quad (\text{D.7})$$

Combining the φ -dependent solution (D.5) with the θ -dependent solution (D.7), we obtain an expression for the Spherical Harmonics' basis:

$$p_{\ell,m}(\theta, \varphi) = P_\ell^m(\cos \theta) e^{im\varphi}, \quad m = -\ell, \dots, \ell. \quad (\text{D.8})$$

In Eq. (D.8), for a fixed degree ℓ we have $2\ell+1$ basis functions, and we can verify the orthogonality of these functions for fixed ℓ :

$$\begin{aligned} \langle p_{\ell,m}, p_{\ell,m'} \rangle_{L^2(\mathbb{S}^2)} &= \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} e^{i(m-m')\varphi} P_\ell^m(\cos \theta) P_\ell^{m'}(\cos \theta) \sin \theta d\varphi d\theta \\ &= \frac{1}{(2\ell+1)} \frac{(\ell+m)!}{(\ell-m)!} \delta_{m,m'}. \end{aligned}$$

From this, we can normalize the functions $p_{\ell,m}$ to obtain an orthonormal basis, which leads to the classical definition of the *spherical harmonics*:

$$Y_{\ell m}(\theta, \varphi) = \sqrt{(2\ell+1) \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\varphi}, \quad m = -\ell, \dots, \ell, \ell = 0, \dots, \infty. \quad (\text{D.9})$$

Use of AI

The work used AI tools to improve the writing quality of the text. However, the content itself is entirely original and was not generated by AI from scratch. AI tools were also used in the coding process, primarily to be more efficient on redundant tasks, such as writing code for the figures, and to assist with debugging.

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