

École polytechnique de Louvain

Extension of the Hybrid Discrete-Finite Element Method to Dynamic Analyses

**Modal Superposition and Time-stepping
Approaches**

Author: Adrien PIEDBOEUF

**Supervisors: João Saraiva Esteves Pacheco DE ALMEIDA, Michele
GODIO**

**Readers: Igor BOUCKAERT, Giammaria GABBIANELLI, Hadrien RAT-
TEZ**

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Abstract

Earthquakes, like many other natural disasters, continue to this day to cause significant fatalities and economic damages. To mitigate these impacts, the Hybrid Discrete-Finite Element Method (HybriDFEM), a novel approach to structural analysis, has been extended to perform dynamic analysis.

This extension allows HybriDFEM to compute the natural frequencies and mode shapes of various structures, including beams, walls, and arches. Traditional Discrete Element Method codes have never been used for modal analysis, and there is limited information available in the literature on this topic. Therefore, this advancement represents a significant leap in the field.

Furthermore, HybriDFEM now includes dynamic analysis capabilities by implementing various time-stepping algorithms from relevant literature. This enables the modelling of displacements and internal stresses over time for both damped and undamped, linear and non-linear systems, offering users a wide range of options.

All implemented methods have been extensively tested to accurately assess HybriDFEM's performance and identify any potential issues. These advancements enhance the ability to design and analyse structures subjected to dynamic loads, ultimately contributing to greater resilience against earthquakes and other dynamic forces.

Keywords : Discrete Element Method, Modal Superposition, Natural Frequencies, Mode Shapes, Dynamic Analysis, Nonlinear Time Stepping Methods, Linear Time Stepping Methods, Mass Matrix, Damping Matrix, Time Stepping Approaches

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Chapter 1

Introduction and Objectives

1.1 Motivations

Natural disasters—including earthquakes, storms, floods, and droughts—claim the lives of approximately 40,000 to 50,000 people each year. Although this represents a relatively small fraction of all global deaths, the economic impact of such disasters is substantial, often resulting in the destruction of natural environments, private properties, and public infrastructures [16].

Advancements in early warning systems and improved structural design have significantly reduced fatalities from natural disasters over the last century (see Figure 1.1). However, these events can still result in significant tragedies. For instance, in February 2023, the catastrophic Turkey–Syria earthquake, with a magnitude of 7.8, killed 62,013 people. The majority of these deaths were not directly caused by ground movement, but by collapsing buildings and falling debris. Nearly 4 million buildings were affected, with at least 518,009 houses and over 345,000 apartments destroyed, resulting in damages estimated at nearly US\$165 billion [46].

Many of these buildings were designed to withstand their own weight and relatively small loads such as wind or exploitation charges, but were not designed to endure dynamic loads and sudden variation in ground acceleration that typically happens during an earthquake. This underscores the necessity for accurate modelling of dynamic loads.

Dynamic loads are not only limited to rare events like earthquakes; they also occur in more common scenarios. High-rise buildings are affected by strong wind currents, and marine structures, such as offshore wind turbines, face constant currents and

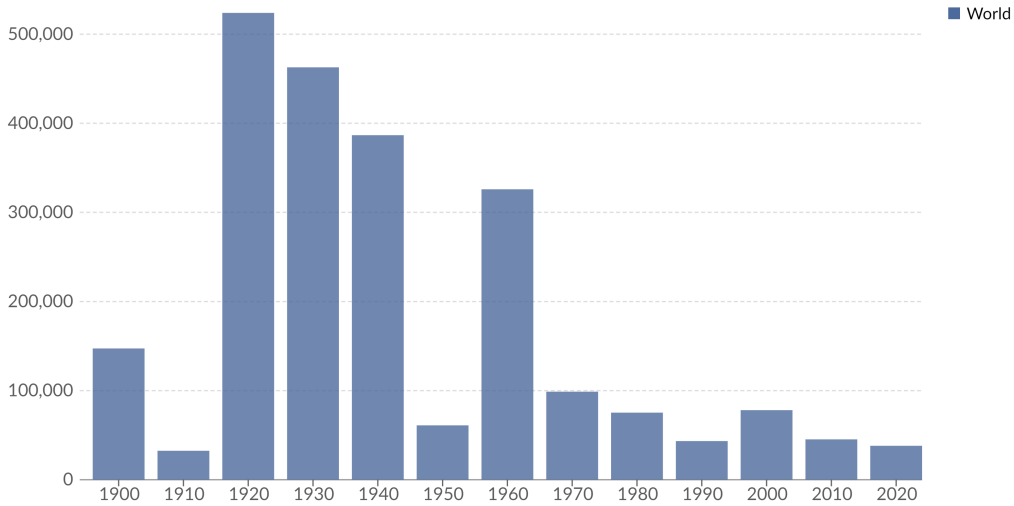
waves.

The importance of modelling dynamic loads is the primary motivation for the extension of the HybriDFEM method to modal analysis and dynamic simulation. By providing detailed simulations of the displacements and internal stresses over time, HybriDFEM aims to enhance the capability of modelling and designing structures subjected to dynamic loads. This will help prevent or at least mitigate damage to structures, ensuring greater resilience against earthquakes or any other dynamic force.

Decadal average: Annual number of deaths from disasters

Our World
in Data

Disasters include all geophysical, meteorological and climate events including earthquakes, volcanic activity, landslides, drought, wildfires, storms, and flooding. Decadal figures are measured as the annual average over the subsequent ten-year period.



Data source: Our World in Data based on EM-DAT, CRED / UCLouvain, Brussels, Belgium – www.emdat.be (D. Guha-Sapir)

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Note: Decadal figures are measured as the annual average over the subsequent ten-year period. This means figures for '1900' represent the average from 1900 to 1909; '1910' is the average from 1910 to 1919 etc. Data includes disasters recorded up to April 2024.

Figure 1.1: Annual number of deaths from disasters [16].

1.2 Objectives

Given the presence and importance of dynamic loads, static analysis is often not enough to correctly design a structure. The main goal of this master's thesis was the extension of HybriDFEM, a novel numerical model, to modal analysis and dynamic numerical simulation.

The first objective involved computing mode shapes and natural frequencies using the matrix eigenvalue problem. This required two elements: the stiffness matrix, which had already been developed and tested, and the mass matrix. A general formulation capable of computing this matrix for different block shapes was developed and implemented.

The second objective was to implement a series of numerical algorithms for dynamic analysis. Since these problems can be either damped or undamped, a formulation for the damping matrix was also developed. Although many algorithms are available in the literature, only a selection was implemented. However, this selection is broad enough to accommodate considerations such as numerical damping and to handle both linear and non-linear problems. The latter are particularly important because many structures, for reasons of cost and feasibility, are designed to exceed their elastic limit and behave plastically during events like earthquakes. A correct understanding of these instances is essential for accurately designing buildings and infrastructures.

Equally important was the testing and validation of these formulations. All implemented functions will be used for future developments, so it is crucial that they are fully validated and considered reliable. In cases where full validation is not possible, it is also important to be aware of potential issues or sources of error. Therefore, a significant portion of the thesis focuses on testing and validation. This involves finding suitable benchmarks in the literature, comparing them with HybriDFEM results and justifying eventual differences.

1.3 Research Novelty

Thanks to the dynamic analysis extension, the HybriDFEM method, which already offers a wide range of modelling capabilities, is further enhanced. This extension significantly increases the method's value, bringing it closer to other competitive, commercially used options. The ability to model non-linear dynamic systems is a significant advancement, enabling the design of masonry structures that are common in European countries, like Belgium, Portugal, and Italy and are particularly vulnerable to seismic activity.

The extension, however, offers much more than just increasing the value and capacity of the HybriDFEM method. In the literature, there is very little information available about modal analysis for discrete systems. Furthermore, conventional commercial Discrete Element Method codes have never been used for performing modal analysis [3]. In this context, the HybriDFEM extension proves exceptionally valuable, bridging a significant gap in the field.

1.4 Organization

Following this introductory chapter, Chapter 2 introduces the HybriDFEM method, outlining its purpose, the current state of its development, and the definition of the model.

Chapter 3 develops the formulation for computing the natural frequencies and mode shapes within the model. It addresses the absence of the mass concept in the method by developing a formulation for the mass matrix. By combining the mass matrix with the stiffness matrix, the mode shapes and natural frequencies are computed through solving the matrix eigenvalue problem.

In Chapter 4, the various formulations developed in the previous chapter are extensively tested and validated. The mass matrix is evaluated using different geometric configurations. Benchmarks for natural frequencies and mode shapes range from simple structures such as beams to more complex ones like arches and large walls. These tests are essential for confirming the correct implementation of the formulations detailed in the previous chapter and identifying any errors or limitations in the method.

Chapter 5 extends the method further by developing the formulation for dynamic analysis. It incorporates the concept of damping, or energy dissipation, using a damping matrix. Various numerical algorithms from the literature are implemented to provide users with a wide range of options for performing analyses. Both linear and non-linear, damped and undamped systems can be modelled.

The subsequent chapter tests and validates the formulations from Chapter 5. Linear formulations are compared with classical modal superposition, while non-linear ones are tested against results from relevant literature. These tests are crucial for confirming the correct implementation of the previous chapter and identifying any errors or limitations in the method.

Finally, Chapter 7 summarizes the main results and draws the general conclusions of this work. Additionally, it discusses possible future developments.

Important Note: For the writing of this thesis, the AI tool "ChatGPT" from OpenAI and the grammar correction tool "Antidote" were used to enhance the clarity, syntax, and overall flow of the text.

Chapter 2

HybriDFEM, midway between FEM and DEM

The goal of this chapter is to introduce the HybriDFEM method by providing a brief overview of its development, current state, and operational principles. This chapter summarizes information from various published papers on the method by I. Bouckaert et al. [7, 8].

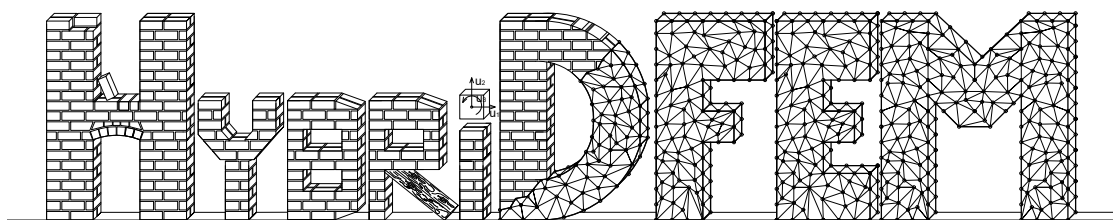


Figure 2.1: A suggested logo for the HybriDFEM method.

2.1 FEM, DEM, HybriDFEM

Among the numerous modelling approaches available, the Finite Element Method (FEM) [5] stands out as one of the most widely recognized and frequently employed methods in structural dynamics softwares. This method typically represents structures as continuous mediums but also offers techniques to model discontinuities when necessary [32, 40]. Homogenization techniques enable the averaging of discontinuous behaviour across elements, providing a computationally efficient model. However, this may lead to a loss of precision in results and the representation of

physical responses. Micro-modelling approaches offer a more explicit consideration of discontinuous behaviour, providing a higher level of detail but demanding significant computational resources.

Modeling discontinuities is a critical task in structural mechanics. Accurate and reliable modelling techniques are essential to simulate the response of various structures, including those inherently discontinuous (e.g., masonry), heterogeneous structural elements (e.g., mixed structures), cracked members (e.g., reinforced concrete), and interactions between different structural components (e.g., joints, structural details, etc.), as well as soil-structure interactions.

The Discrete Element Method (DEM) [15, 30] is a family of modelling techniques that can represent the discontinuous nature of material assemblies more accurately but requires more computational resources. It models a structure as an assembly of rigid elements connected through deformable contact interfaces. All DEM formulations share the characteristic of requiring a numerical time-stepping algorithm, resulting in drastically higher computational time and an inability to handle static or buckling problems effectively.

Other methods treat the problem as an assembly of rigid units interconnected by deformable interfaces modelled as distributed non-linear multidirectional springs, making them suitable for analysing large structures in static or dynamic problems. Various formulations, such as the Rigid Block model (RB) [38], the Rigid Body-Spring Model (RBSM) [11], the Applied Element Method (AEM) [31], and the Fiber Contact Element Method (FCEM) [19, 20], propose to model discontinuities with this approach.

The Hybrid Discrete-Finite Element Method (HybriDFEM) [7, 8] represents a novel approach to structural analysis, capable of handling discontinuities in structures. The method shares similarities with the Discrete Element Methods (DEM) in its ability to account for contact interfaces and/or block deformability, and with the Applied Element Method (AEM) in the representation of interfaces as a series of normal and shear springs. However, it is close to the FEM in the way it is formulated, offering the possibility to link both methods for potential hybrid applications. This hybrid capability enables explicit modelling of discontinuities while maintaining low computational demand.

Moreover, HybriDFEM, which is inherently discontinuous, can also serve as a versatile tool for modelling continuous structures through the use of scaled spring constitutive models. This versatility makes HybriDFEM highly flexible and suitable for a wide range of structural analysis tasks.

Currently, in its developmental stage, the HybriDFEM Method is limited to static problems. It enables the modelling of one-dimensional beam or arch elements where all blocks are aligned, as well as two-dimensional wall elements. The method accurately represents either continuous members or assemblies of distinct blocks, as done in the Discrete Element Method. Additionally, it accommodates for both geometric and material non-linearity. Numerous benchmarks were tested in the published papers [7, 8] to validate the method.

2.2 Block Discretisation

In the current state of the HybriDFEM method's development, a planar structure is modelled as a series of rigid blocks in contact through contact faces (CFs). The discretisation of the structure into n_b blocks defines the total number of degrees of freedom (DoFs) considered at the global level. The motion of each block is described by three kinematic variables: two translations and one rotation, all referenced to the block's origin, often its center of mass. These kinematic variables are stored in the vector of nodal displacements, $\mathbf{U}$. For a generic block i , the kinematic variables are the displacements $\mathbf{U}_{3i-2}$, $\mathbf{U}_{3i-1}$ and $\mathbf{U}_{3i}$, expressed in the global reference system (X , Y , Z).

In an HybriDFEM beam element, all blocks have the same dimensions except for the two extremity blocks, which are truncated at half-length to align their reference origin with the beam axis extremities. This block discretization is illustrated in Figure 2.2. This is just one specific case; the current iteration of the HybriDFEM method can also model wall elements or elements with blocks of varying dimensions, such as tapered beams.

Since the blocks are rigid, deformation occurs only at the CFs. Each CF is characterized by 6 DoFs, corresponding to the 3 DoFs of the two contiguous blocks. These are collected into the vector $\mathbf{q}_f^{glob} \in \mathbb{R}^6$, expressed in the global reference system (X , Y , Z).

2.3 Contact Pair Discretisation

Each CF is further subdivided into a series of n_c contact pairs (CPs) evenly distributed along the interface between two blocks (Figure 2.3b). Each CP consists of three points A , B and C . Points A and B are fixed to the edges of the first and second block, respectively. A non-linear spring is attached to point A and connected to another non-linear spring attached to point B through a hinged contact point C (Figure 2.3c). The kinematics of the contact pair is described by eight DoFs in total.

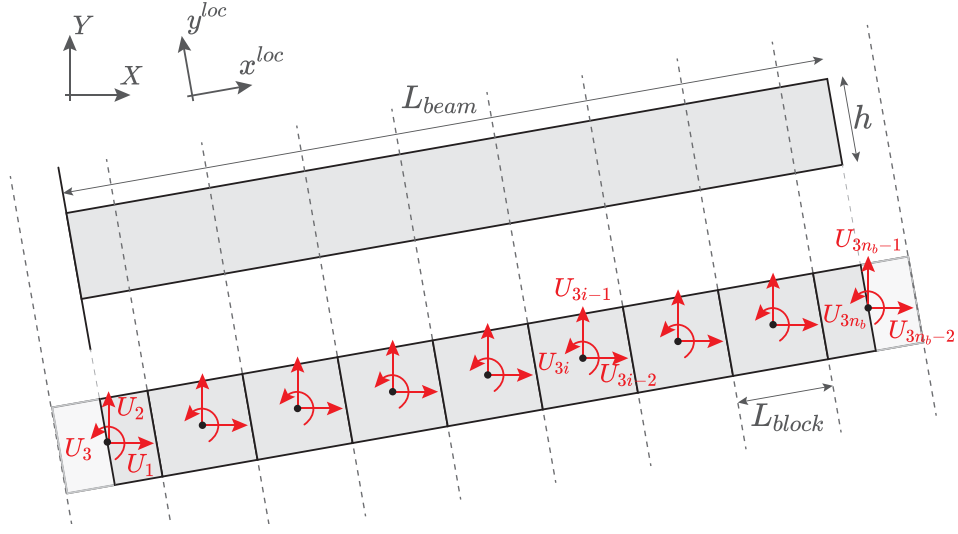


Figure 2.2: Block discretisation and kinematic variables of a beam at the structural level [7].

The first three refer to the displacement of point A and its spring's orientation, which follows the orientation of the block to which it is attached. Similarly, the next three DoFs of the CP refer to the displacement of point B and its spring's orientation. The last two DoFs refer to the displacement of point C and will be condensed for the computations at the structural level.

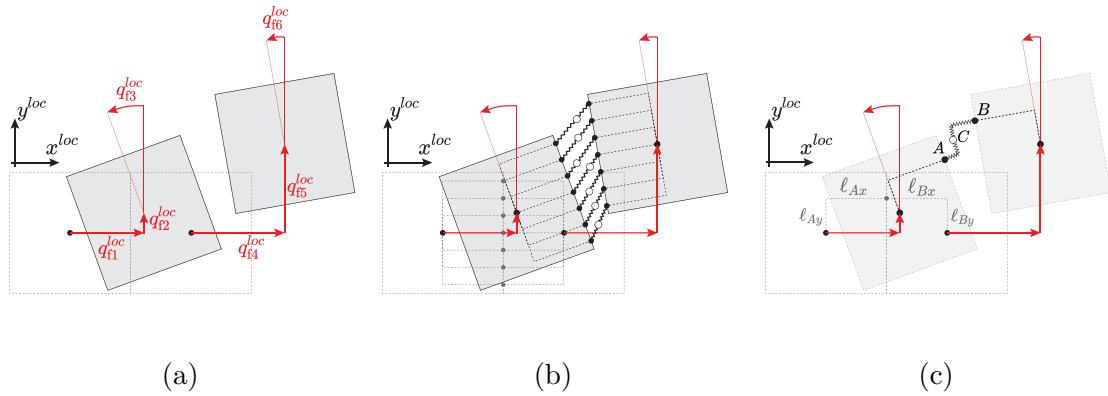


Figure 2.3: (a) Block kinematics in the local reference system, (b) contact pair discretisation of the contact face and (c) isolated contact pair [7].

The kinematics of the first six DoFs of the CP can be derived solely through compatibility relations, since the positions of points A and B with respect to their respective blocks do not change, and the blocks are rigid. These six DoFs of each CP can be directly related to the six DoFs of the adjacent blocks, $\mathbf{q}_1^{loc}$ to $\mathbf{q}_6^{loc}$, as represented in Figure 2.3a. To derive the position of point C (i.e., the last two DoFs of each CP), the relative elongations of the two pairs of springs need to be known. These elongations can be derived through a set of two linear equations related to compatibility and two possibly non-linear equations related to equilibrium.

The compatibility equations state that the total CP deformation must equal the sum of the elongations of the first and second spring, both in the normal and tangential directions (Figure 2.4a). By isolating the deformation of the CP without considering its rigid-body motion, non-linear geometry is accounted for. The equilibrium equations state that the forces through spring A must equal the forces through spring B (Figure 2.4b). These forces depend on the constitutive relations assigned to the springs, which relate the spring elongation to the spring forces. The constitutive relations can represent the behaviour at the contact between the two blocks (e.g., when modelling assemblies of rigid blocks) or can be scaled to represent the behaviour of a continuous material, where the deformation of the block is concentrated at each CF. By assigning a non-linear constitutive relation to the springs, non-linear material effects can be included.

The set of non-linear equations related to compatibility and equilibrium is solved using a Newton-Raphson solution procedure. At the end of this procedure, the forces through the springs attached to points A and B , along with their relative elongations and the displacement of point C , are known. This non-linear solution procedure is referred to as the internal solution procedure, in contrast to the external solution procedure at the global level, common to the non-linear FEM. From the spring forces, the resisting forces of each CP can be derived by equilibrium in the deformed configuration, accounting for non-linear geometric effects. Finally, the resisting forces of each CF can be computed by assembling the resisting forces of all its CPs, and the resisting forces $\mathbf{P}_r$ of the structure can be computed by assembling the resisting forces of all CFs.

2.4 Solution Procedure at the Structural Level

The steps described above can be used to derive the resisting forces $\mathbf{P}_r$ of the structure associated with a given set of nodal displacements $\mathbf{U}$. From the constitutive relations assigned to the springs, an incremental relationship between the spring forces and spring elongations can be derived. This relationship corresponds to the tangent stiffness matrix of the springs. The tangent stiffness matrices of the two

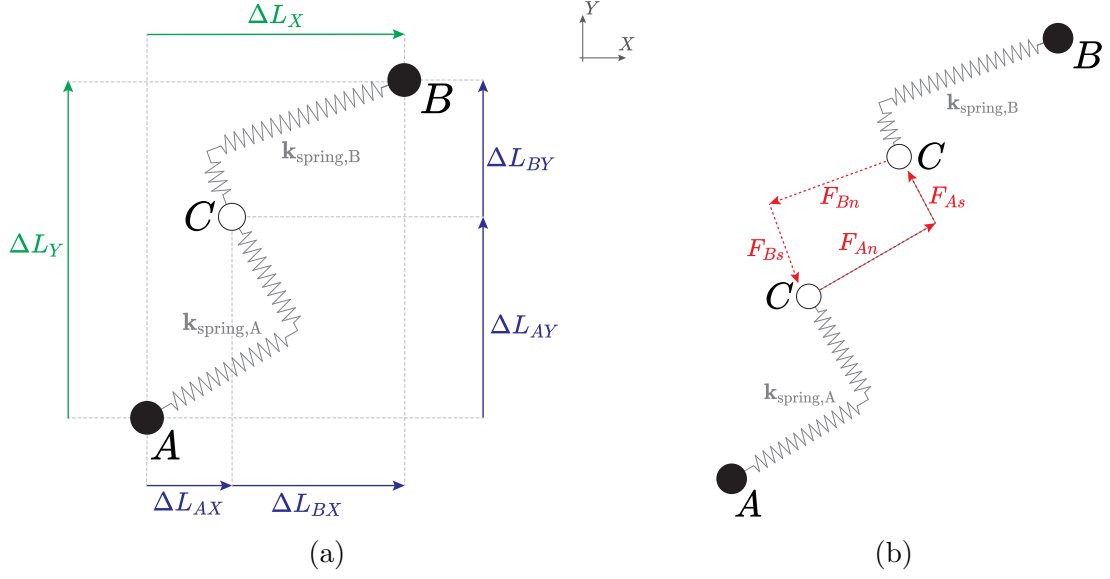


Figure 2.4: (a) Compatibility relation of the CP and (b) equilibrium of spring forces used to determine the position of point C , the relative spring elongations, and the spring forces [7].

springs composing one CP can be used to assemble the tangent stiffness matrix of the CP. Subsequently, this can be integrated into the tangent stiffness matrix of the CF. Finally, the tangent stiffness matrix of the entire structure can be assembled from the stiffness matrices of all the CFs, denoted as $\mathbf{K}$.

In the case of linear elastic structures, this stiffness matrix can be used to relate directly the nodal displacements stored in $\mathbf{U}$ to the global resisting forces $\mathbf{P}_r$. Equilibrium with a given set of applied forces $\mathbf{P}$ can be found by inverting the tangent stiffness matrix $\mathbf{K}$ (or, computationally, any solution method for linear equations).

When non-linear effects are considered, the nodal displacements that equilibrate the resisting forces with the applied forces cannot be computed by matrix inversion. Instead, an incremental solution procedure is required. This incremental-iterative solution procedure, which can be for instance a Newton-Raphson (load-control), displacement-control, or work-control procedure [5], computes the tangent stiffness matrix $\mathbf{K}$ of the structure at each iteration to achieve quadratic convergence. A detailed account of this solution procedure and all the equations involved in the operations described in the previous paragraph can be found in the relevant literature [7].

Chapter 3

Natural Frequencies and Mode Shapes: Formulation and Implementation

The primary objective of this chapter is to develop a formulation within the HybriDFEM method for computing the natural frequencies and mode shapes of a MDoF system. The process begins with development of a formulation for the mass matrix $\mathbf{M}$. Then, using both the mass matrix $\mathbf{M}$ and the already existing stiffness matrix $\mathbf{K}$ in HybriDFEM, the eigenvalue problem will be solved to determine the natural frequencies. Finally, the corresponding mode shapes will be computed based on these natural frequencies.

3.1 Mass Matrix

As previously mentioned, HybriDFEM's current development is limited to solving static problems. Static refers to an object or system in static equilibrium, where the sum of the forces applied equals zero. This balance is expressed using Newton's second law:

$$\sum \vec{F} = m \cdot \vec{a} = m \cdot 0 = 0 \quad (3.1.1)$$

Here, the acceleration of the system $\vec{a}$ is zero, as the problem has no time dependency. As a consequence, in static problems, mass generally has no influence, except when considering the weight of the structure as a static load. However, in dynamic conditions, such simplifications are not applicable due to the time dependency. The first step in addressing dynamic problems is to introduce the concept of mass and its formulation.

Many commercially used programs are based on finite elements. Among the common methodologies to represent the mass matrix, two stand out: the consistent mass matrix and the lumped mass matrix [43].

The **consistent mass matrix** is derived by considering the exact shape functions used for displacements in the element formulation. It is based on the integration of the element mass over its volume and takes into account the varying distribution of the mass within the element. This method is precise but computationally expensive.

The **lumped mass matrix**, on the other hand, assumes that the mass is concentrated at the nodes of the finite element. This implies that the mass distribution within the element is approximated as a set of point masses located at the element nodes. It is computationally more efficient, but may not capture the true distribution of mass within the element.

HybriDFEM is a discrete element method, where the analysed structure is divided into a series of blocks, each with their own mass. An approach similar to the lumped matrix was implemented. By assuming the mass is uniformly distributed across the block, HybriDFEM approximates the mass as a point mass located at the origin of the block.

Each block i has 3 degrees of freedom, and a 3×3 matrix represents the mass associated with the 3 possible displacements of the block. The matrix, referred to as the local matrix $\mathbf{M}_{i,\text{local}}$, is defined as:

$$\mathbf{M}_{i,\text{local}} = \begin{bmatrix} m_i & & \\ & m_i & \\ & & J_i \end{bmatrix} \quad (3.1.2)$$

Where m_i and J_i are the mass and the rotational inertia of the block i , respectively. The first two terms on the diagonal represent the mass tied to the translational degrees of freedom, while the last term relates to the rotational degree of freedom.

Local matrices, as the one from Equation 3.1.2, are associated with the local axis (x, y) of the block. However, since the kinematics of the structure at the global level are expressed in the global axis (X, Y) , an axis transformation is required before summing up the mass matrices of all the blocks.

Given that the mass matrix is diagonal, this transformation is trivial:

$$\begin{aligned}
\mathbf{M}_i &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \\ & & 1 \end{bmatrix} \begin{bmatrix} m_i & & \\ & m_i & \\ & & J_i \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \\ & & 1 \end{bmatrix} \\
&= \begin{bmatrix} m_i(\cos^2(\theta) + \sin^2(\theta)) & \cancel{m_i(\cos(\theta)\sin(\theta))} \cancel{-\sin(\theta)\cos(\theta)} \\ \cancel{m_i(\cos(\theta)\sin(\theta))} \cancel{-\sin(\theta)\cos(\theta)} & m_i(\cos^2(\theta) + \sin^2(\theta)) \\ & & J_i \end{bmatrix} \\
&= \begin{bmatrix} m_i & & \\ & m_i & \\ & & J_i \end{bmatrix}
\end{aligned} \tag{3.1.3}$$

where θ is the angle between the local and global axes. For a block i , the rotated mass matrix is always equal to the non-rotated one.

The mass matrix of the whole structure $\mathbf{M} \in \mathbb{R}^{3n \times 3n}$ is obtained by summing the mass matrices $\mathbf{M}_i \in \mathbb{R}^{3 \times 3}$ of all the blocks of all the elements.

$$\mathbf{M} = \sum_{i=1}^n \mathbf{A}_f^T \mathbf{M}_i \mathbf{A}_f = \begin{bmatrix} m_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & m_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & J_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & m_n & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & m_n & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & J_n \end{bmatrix} \tag{3.1.4}$$

with $3n$ the total number of DoFs and $\mathbf{A}_f \in \mathbb{R}^{3 \times 3n}$ the localisation matrix. This matrix relates the DoFs of the blocks to those of the structure. All the entries of this matrix are equal to 0 except at indexes where a DoF of the structure corresponds to a DoF of the block, where the entry is equal to 1.

3.1.1 Mass and Rotational Inertia of the Block i

As the volumetric mass density of the block ρ_i is uniform, the value of the mass m_i can be easily computed by multiplying it with the volume V_i of the block i . Since HybriDFEM works in two dimensions, with a constant depth e_i for each block, the volume V_i can be defined as the block surface A_i times the depth e_i .

$$m_i = V_i \rho_i = A_i e_i \rho_i \tag{3.1.1.1}$$

The value of the rotational inertia J_i is expressed relative to a specific reference point $(X_R; Y_R)$, which in the case of HybriDFEM is the origin of the block (which is not necessarily its center of mass). The rotational inertia J_i can be computed by solving the following integral on the block surface A_i :

$$J_i = I_{x,i} + I_{y,i} = \int_{A_i} y^2 dm + \int_{A_i} x^2 dm \quad (3.1.1.2)$$

where $I_{x,i}$ and $I_{y,i}$ are the rotational inertia relative to the x- or y-axis, respectively. For reference, Figure 3.1 reports an example of a rectangular block. The reference point or origin of the block is in this case its center of mass.

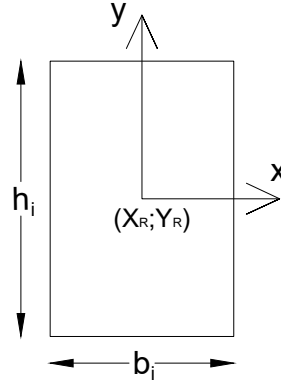


Figure 3.1: Example of a rectangular block.

While the rotational inertia J_i typically has a minor impact on the natural frequencies and could be neglected to simplify mass matrix formulations, its effect might become noticeable for a coarse discretization or short beams. Additionally, its presence is crucial for dynamic simulations that consider rotational degrees of freedom. Neglecting J_i could result in a singular (or badly conditioned) mass matrix, making it impossible (or very difficult) to invert. Therefore, it was deemed beneficial for HybriDFEM to incorporate a more complete formulation, and an algorithm for determining this term was developed. By default, rotational inertias are considered, but users have the option to neglect these terms in the mass function arguments.

3.1.2 Implementation

As discussed in the previous section, both the mass m_i and the rotational inertia J_i depends on the block surface A_i . In the current iteration of HybriDFEM, blocks are quadrilateral. Rectangular blocks are used to model beams or walls, while

trapezoidal or parallelogram blocks are used to model more complex shapes like arches or tapered beams. Some possible shapes are reported in Figure 3.2.

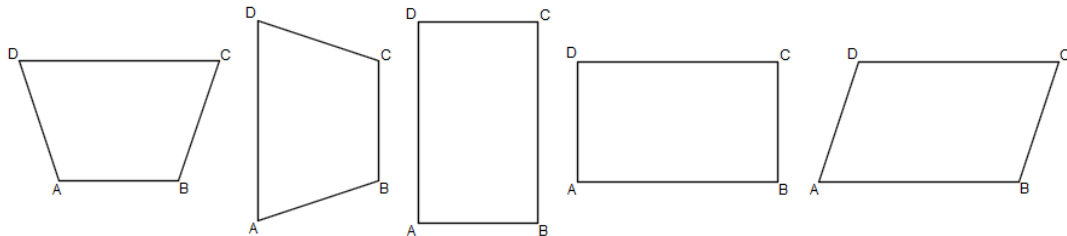


Figure 3.2: Possible block types in the current HybriDFEM iteration.

One of the potential approaches explored was to develop a formulation capable of identifying the shape of the block and use mathematical expressions derived from relevant literature to compute the mass and rotational inertia of the block. However, it seemed more intriguing and compelling to create a general formulation applicable to any quadrilateral.

Drawing inspiration from the Roman saying "Divide et Impera", any quadrilateral can be subdivided into triangles. This strategy simplifies computations by dealing with multiple, albeit simpler, shapes. Moreover, this formulation allows for adaptability in scenarios where HybriDFEM might be extended to encompass more complex and curved shapes, which can also be approximated by a series of small triangles.

In the initial stages of the thesis, each block was divided into four triangles parallel to the global axis (X, Y), as depicted in Figure 3.3a. This approach aimed to leverage the powerful parallel axis theorem [49] and avoid the necessity for rotational matrices to compute the rotational inertia. Although effective, forgoing the use of rotational matrices was not the most advisable approach, as they are straightforward to implement and enable a simpler two-triangle division, as illustrated in Figure 3.3b, leading to faster computations. Consequently, an optimization of the implementation was undertaken during the course of the thesis.

The parallel axis theorem [49], also referred to as Huygens–Steiner theorem, or simply Steiner’s theorem, provides a method to determine the rotational inertia of a rigid body about any axis p' , given the body’s rotational inertia about a parallel axis through the object’s center of gravity p and the perpendicular distance between the axes d . Mathematically, these two inertias are related by:

$$J_{p'} = J_p + md^2 \quad (3.1.2.1)$$

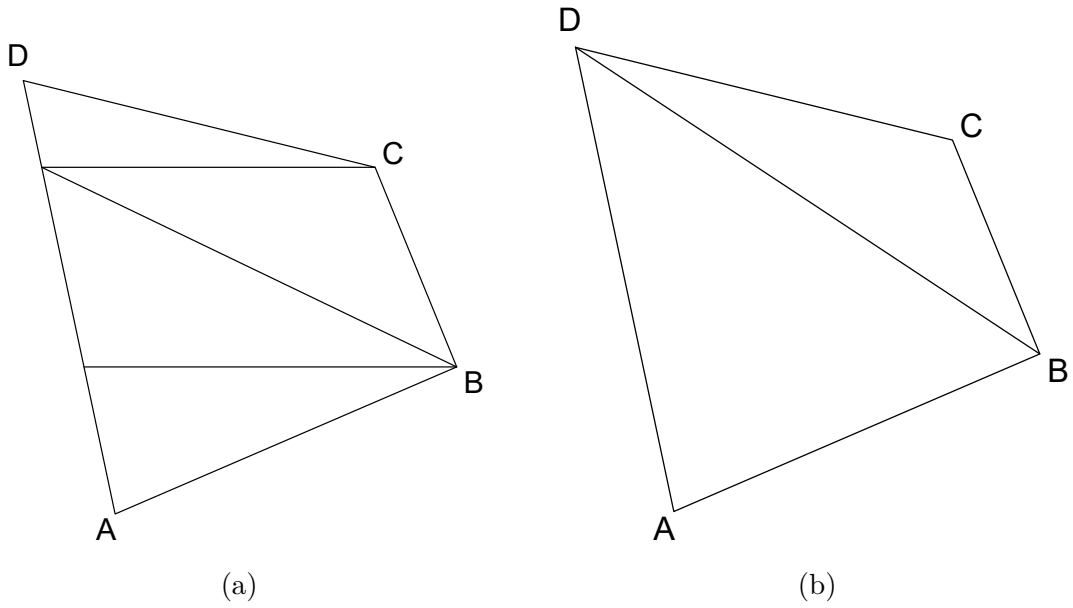


Figure 3.3: Division of a random quadrilateral in four (a) and two (b) triangles.

Figure 3.4 illustrates an example of a rectangular block where the parallel axis theorem is applicable. The selected axes are y and y' . The origin of the block is positioned on the edge of the block, at mid-height.

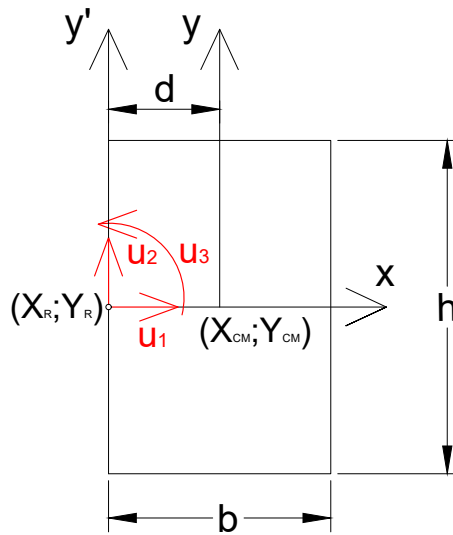


Figure 3.4: Visual representation of a rectangle block where the reference point is different from the center of mass.

The following formulations are generalized for any n triangle division, both the cases of two and four triangles. For each triangle j , the volume, mass, and rotational inertia are computed.

Mass implementation

The mass of a block i is the sum of the masses of n triangles j .

$$m_i = \sum_{j=1}^n m_{ij} = \rho_i \sum_{j=1}^n V_{ij} = \rho_i e_i \sum_{j=1}^n A_{ij} \quad (3.1.2.2)$$

with i the index of the block, j the index of the triangle, ρ_i and e_i respectively the volumetric mass density and the depth of the block i and A_{ij} the surface of a triangle j computed as :

$$A_{ij} = \sqrt{s_{ij} (s_{ij} - l_{1,ij}) (s_{ij} - l_{2,ij}) (s_{ij} - l_{3,ij})} \quad (3.1.2.3)$$

with s_{ij} the semi-perimeter of the triangle j and $l_{1,ij}$, $l_{2,ij}$ and $l_{3,ij}$ the lengths of the three sides of the triangle j [48].

Rotational inertia implementation

Using the parallel axis theorem in combination with Equation 3.1.1.2, the rotational inertia of the block i can be expressed as:

$$J_i = \sum_{j=1}^n J_{ij} = \sum_{j=1}^n (I_{x,ij} + I_{y,ij} + m_{ij} d_{ij}^2) \quad (3.1.2.4)$$

with i the index of the block, j the index of the triangle, $I_{x,ij}$ (or $I_{y,ij}$) the rotational inertia of the triangle j with respect to the axis x (or y) passing through the center of mass of the triangle j and d_{ij} the distance between the center of mass of the triangle j and the origin of the block i .

Rotational inertia through the axis x

For any triangle j , the rotational inertia $I_{x,ij}$ with respect to the axis x , passing through the center of mass of the triangle, is computed using the parallel axis theorem on the rotational inertia $I_{x',ij}$ with respect to the axis x' , passing through the triangle base:

$$I_{x,ij} = I_{x',ij} - m_{ij} d^2 \quad (3.1.2.5)$$

where d is the distance between the two axes x and x' .

$$I_{x',ij} = \int_{A_{ij}} y^2 dm_{ij} = \int_0^{h_{ij}} L_{ij} y^2 dm_{ij} \quad (3.1.2.6)$$

where dm_{ij} is the mass of the infinitesimal rectangle $L_{ij} dy$ and h_{ij} is the height of the triangle (see Figure 3.5).

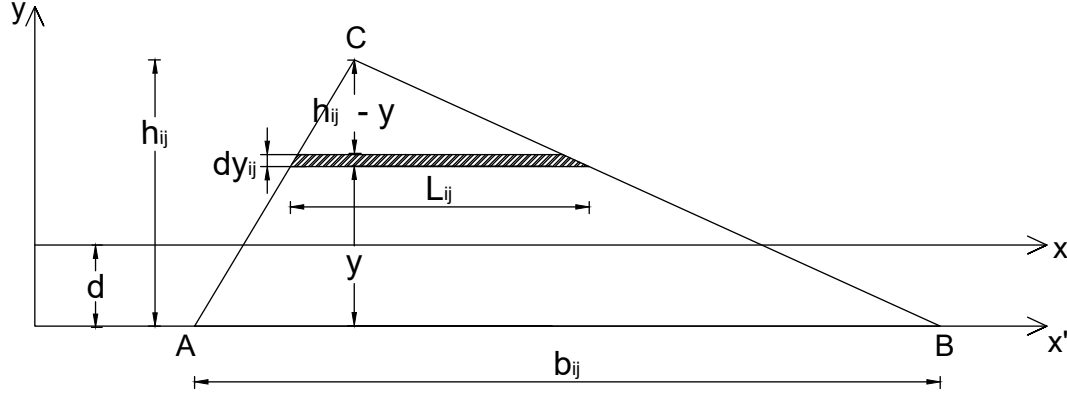


Figure 3.5: Triangle j with base b_{ij} and height h_{ij} . Rotational inertia of axis x .

The length L_{ij} can be computed as:

$$L_{ij} = \frac{(h_{ij} - y)}{h_{ij}} b_{ij} \quad (3.1.2.7)$$

where b_{ij} is the base of the triangle. As the mass distribution is constant, the mass dm_{ij} can also be written as:

$$\frac{dm_{ij}}{dA_{ij}} = \frac{m_{ij}}{A_{ij}} \Rightarrow dm_{ij} = \frac{m_{ij}}{\frac{1}{2}b_{ij}h_{ij}} L_{ij} dy_{ij} \quad (3.1.2.8)$$

By combining Equation 3.1.2.7 and 3.1.2.8:

$$dm_{ij} = \frac{2m_{ij}}{b_{ij}h_{ij}} L_{ij} dy = \frac{2m_{ij}}{b_{ij}h_{ij}} \frac{(h_{ij} - y)b_{ij}}{h_{ij}} dy_{ij} = \frac{2m_{ij}(h_{ij} - y)}{h_{ij}^2} dy_{ij} \quad (3.1.2.9)$$

By replacing dm_{ij} in Equation 3.1.2.6 with Equation 3.1.2.9:

$$\begin{aligned} I_{x',ij} &= \frac{2m_{ij}}{h_{ij}^2} \int_0^{h_{ij}} y^2 (h_{ij} - y) dy_{ij} = \frac{2m_{ij}}{h_{ij}^2} \left[\frac{h_{ij} y^3}{3} - \frac{y^4}{4} \right]_0^{h_{ij}} \\ &= \frac{2m_{ij}}{h_{ij}^2} \left[\frac{h_{ij}^4}{3} - \frac{h_{ij}^4}{4} \right]_0^{h_{ij}} = \frac{m_{ij}}{6} h_{ij}^2 = \frac{b_{ij}}{12} h_{ij}^3 \rho_{ij} e_{ij} \end{aligned} \quad (3.1.2.10)$$

Finally, combining Equation 3.1.2.5 and 3.1.2.10:

$$I_{x,ij} = I_{x',ij} - m_{ij}d^2 = \frac{m_{ij}}{6}h_{ij}^2 - \frac{m_{ij}}{9}h_{ij}^2 = \frac{m_{ij}}{18}h_{ij}^2 = \frac{b_{ij}}{36}h_{ij}^3 \rho_{ij} e_{ij} \quad (3.1.2.11)$$

Rotational inertia through the axis y

For any triangle j , the rotational inertia $I_{y,ij}$ with respect to the axis y is computed by dividing the triangle in two sub-triangles, as done in Figure 3.6.

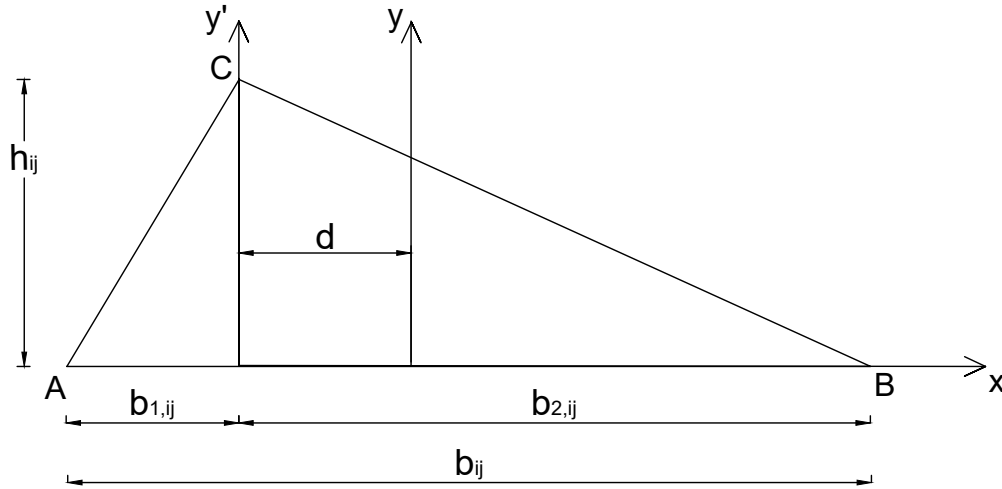


Figure 3.6: Triangle j with base b_{ij} and height h_{ij} . Rotational inertia of axis y .

The inertias of these two smaller triangles are computed with respect to the axis y' and then adapted using the parallel axis theorem to obtain the inertia with respect to the axis y . The rotational inertia of a triangle in respect to an axis passing through its base was already computed in Equation 3.1.2.10.

$$\begin{aligned} I_{y',ij} &= \frac{h_{ij} b_{1,ij}^3}{12} \rho_{ij} e_{ij} + \frac{h_{ij} b_{2,ij}^3}{12} \rho_{ij} e_{ij} \\ &= \frac{h_{ij} b_{1,ij}^3}{12} \rho_{ij} e_{ij} + \frac{h_{ij} (b_{ij} - b_{1,ij})^3}{12} \rho_{ij} e_{ij} \\ &= \frac{h_{ij} \rho_{ij} e_{ij}}{12} (b_{1,ij}^3 + b_{ij}^3 - b_{1,ij}^3 - 3b_{1,ij} b_{ij} (b_{ij} - b_{1,ij})) \\ &= \frac{b_{ij} h_{ij} \rho_{ij} e_{ij}}{12} (b_{ij}^2 - 3b_{1,ij} (b_{ij} - b_{1,ij})) \\ &= \frac{m_{ij}}{6} (b_{ij}^2 - 3b_1 (b_{ij} - b_{1,ij})) \end{aligned} \quad (3.1.2.12)$$

The rotational inertia is then computed from y' to y .

$$\begin{aligned}
I_{y,ij} &= I_{y',ij} - md^2 \\
&= I_{y',ij} - m_{ij} \left(\frac{b_{ij} - 2b_{1,ij}}{3} \right)^2 \\
&= I_{y',ij} - m_{ij} \frac{b_{ij}^2 - 4b_{ij}b_{1,ij} + 4b_{1,ij}^2}{9} \\
&= \frac{m_{ij}}{18} (3b_{ij}^2 - 9b_{1,ij}b_{ij} + 9b_{1,ij}^2 - 2b_{ij}^2 + 8b_{1,ij}b_{ij} - 8b_{1,ij}^2) \\
&= \frac{m_{ij}}{18} (b_{ij}^2 - b_{1,ij}b_{ij} + b_{1,ij}^2) \\
&= \frac{b_{ij}h_{ij}}{36} (b_{ij}^2 - b_{1,ij}b_{ij} + b_{1,ij}^2) \rho_{ij} e_{ij}
\end{aligned} \tag{3.1.2.13}$$

3.2 Natural Frequencies and Mode Shapes

The following section comes from the textbook in structural dynamics: Dynamics of Structures: Theory and Applications to Earthquake Engineering by Anil K. Chopra [14].

Once the mass matrix is defined and implemented, systems in free vibration can be studied and the concept of natural frequencies and mode shapes can be introduced. Free vibration refers to the motion of a structure without any dynamic excitation, external forces or support motion. Free vibration is initiated by disturbing the structure from its equilibrium position by some initial displacements and/or by imparting some initial velocities.

Free vibration of a linear multi degree of freedom (MDoF) system is governed by:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(\mathbf{t}) \text{ with } \mathbf{p}(\mathbf{t}) = \mathbf{0} \tag{3.2.1}$$

which, for undamped systems, simplifies to:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{3.2.2}$$

For any n -DoF system, there are n characteristic deflected shapes. If the system is displaced in one of these shapes and then released, it will vibrate in simple harmonic motion, maintaining the initial deflected shape. Each of these characteristic deflected shapes is referred to as a natural mode of vibration for the specific n -DoF system.

The displacement of the structure, tied to one of its modes n , can be described mathematically by:

$$\mathbf{u}_n(\mathbf{t}) = q_n(t)\boldsymbol{\phi}_n \quad (3.2.3)$$

where the deflected shape $\boldsymbol{\phi}_n$ does not vary with time. The time variation of the displacements is described by the simple harmonic function:

$$q_n(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \quad (3.2.4)$$

where A_n and B_n are constants that can be determined from the initial conditions that initiate the motion. Combining Equations 3.2.3 and 3.2.4 gives:

$$\mathbf{u}_n(\mathbf{t}) = \boldsymbol{\phi}_n (A_n \cos(\omega_n t) + B_n \sin(\omega_n t)) \quad (3.2.5)$$

where the deflected shape $\boldsymbol{\phi}_n$ and the natural frequency ω_n tied to a specific mode n are unknown. Substituting this form of $\mathbf{u}_n(\mathbf{t})$ in Equation 3.2.2 gives:

$$[-\omega_n^2 \mathbf{M} \boldsymbol{\phi}_n + \mathbf{K} \boldsymbol{\phi}_n] q_n(t) = \mathbf{0} \quad (3.2.6)$$

This equation can be satisfied in one of two ways. Either $q_n(t) = 0$, which implies that $\mathbf{u}_n(\mathbf{t}) = \mathbf{0}$ and there is no motion of the system (this is the so-called trivial solution), or the natural frequencies ω_n and modes $\boldsymbol{\phi}_n$ must satisfy the following algebraic equation:

$$\mathbf{K} \boldsymbol{\phi}_n = \omega_n^2 \mathbf{M} \boldsymbol{\phi}_n \quad (3.2.7)$$

which provides a useful condition. This algebraic equation is called the matrix eigenvalue problem.

The stiffness and mass matrices $\mathbf{K}$ and $\mathbf{M}$ are known; the problem is to determine the scalar ω_n^2 and the vector $\boldsymbol{\phi}_n$. To indicate the formal solution to Equation 3.2.7, it is rewritten as:

$$[\mathbf{K} - \omega_n^2 \mathbf{M}] \boldsymbol{\phi}_n = \mathbf{0} \quad (3.2.8)$$

which can be interpreted as a set of N homogeneous algebraic equations for the N elements ϕ_{jn} with $(j = 1, 2, \dots, N)$. This set always has the trivial solution $\boldsymbol{\phi}_n = \mathbf{0}$ which is not useful because it implies no motion. It has non-trivial solutions if:

$$\det[\mathbf{K} - \omega_n^2 \mathbf{M}] = 0 \quad (3.2.9)$$

When the determinant is expanded, a polynomial of order N in ω_n^2 is obtained. This equation is called characteristic equation or frequency equation and has N real and positive roots for ω_n^2 because $\mathbf{M}$ and $\mathbf{K}$, the structural mass and stiffness matrices, are symmetric and positive definite. The positive definite property of matrix $\mathbf{K}$ is guaranteed for all structures that are supported in a manner preventing rigid-body motion, a condition commonly satisfied by civil engineering structures.

The N roots, ω_n^2 determine the N natural frequencies ω_n of vibration, conventionally arranged in sequence from smallest to largest. Natural frequencies are certain discrete frequencies at which a system is prone to vibrate. They hold significant importance as they govern the dynamic response of a structure to external loads or excitations. When the frequency of an external force matches one of the natural frequencies of the system, resonance can occur. Resonance amplifies the vibrations of the structure, potentially causing excessive displacements, stresses, or even structural failure if not properly controlled. These roots of the characteristic equation are also known as eigenvalues, characteristic values, or normal values.

When a natural frequency ω_n is known, Equation 3.2.8 can be solved for the corresponding vector ϕ_n to within a multiplicative constant. The eigenvalue problem does not fix the absolute amplitude of the vectors ϕ_n , only the shape of the vector given by the relative values of the N displacements ϕ_{jn} with $(j = 1, 2, \dots, N)$.

Corresponding to the N natural vibration frequencies ω_n of an N -DOF system, there are N independent vectors ϕ_n , which are known as natural modes of vibration, or natural mode shapes of vibration. They represent the shapes that a structure would show when vibrating at specific natural frequencies. These vectors are also known as eigenvectors, characteristic vectors, or normal modes.

In both the natural frequencies and mode shapes, the term natural is used to emphasize the fact that these are natural properties of the structure in free vibration, and they depend only on its mass and stiffness. The subscript n denotes the mode number, and the first mode ($n = 1$) is also known as the **fundamental mode**.

During free vibration in each natural mode, an undamped system oscillates at its natural frequency with all DoFs of the system vibrating in the same phase, passing through their zero, maximum, or minimum displacement positions at the same instant of time. In damped systems, this property is generally violated, and classical natural modes may not exist.

Modal and spectral matrices

The N eigenvectors can be assembled compactly in a single square matrix Φ , called modal matrix, wherein each column is a natural mode.

Similarly, the N eigenvalues ω_i^2 can be assembled into a diagonal matrix Ω^2 , called spectral matrix. Therefore, is possible to assemble all the N relations from Equation 3.2.7 into a single equation:

$$\mathbf{K} \Phi = \mathbf{M} \Phi \Omega^2 \quad (3.2.10)$$

Orthogonality of modes

It can be proven that the following mode orthogonality conditions hold when $\omega_i \neq \omega_j$:

$$\phi_i^T \mathbf{K} \phi_j = 0 \quad \phi_i^T \mathbf{M} \phi_j = 0 \quad (3.2.11)$$

If Equation 3.2.9 has a k -fold multiple root (i.e., if the system has one frequency repeated k times), then it is always possible to find k modes associated with this frequency that satisfy the orthogonality conditions of Equation 3.2.11.

The orthogonality of natural modes implies that the following square matrices are diagonal:

$$\mathbf{K}_g = \Phi^T \mathbf{K} \Phi \quad \mathbf{M}_g = \Phi^T \mathbf{M} \Phi \quad (3.2.12)$$

where the diagonal elements are:

$$K_i = \phi_i^T \mathbf{K} \phi_i \quad M_i = \phi_i^T \mathbf{M} \phi_i \quad (3.2.13)$$

Since $\mathbf{M}$ and $\mathbf{K}$ are positive definite, the diagonal elements of $\mathbf{K}_g$ and $\mathbf{M}_g$ are positive and they are related by:

$$K_i = \omega_i^2 M_i \quad (3.2.14)$$

Orthogonality implies that the work done by the i -th mode inertia forces in going through the j -th mode displacements is zero. It also implies that the work done by the resisting forces associated with displacements in the i -th mode in going through the j -th mode displacements is zero.

3.2.1 Type of Natural Frequencies and Mode Shapes

In the literature, various forms of vibration are distinguished, including longitudinal, transverse, and torsional vibrations. These forms can be recognized not only on simple structural elements such as beams but also for more complex ones like frames or even bridges [37, 53]. For instance, considering a basic structural element such as a beam, these forms can be defined as follows:

- **Longitudinal (or axial) vibrations:** they occur when the particles of a beam move parallel to its longitudinal axis. These vibrations cause the beam to alternately elongate and shorten, resulting in the induction of tensile and compressive stresses within the beam.
- **Transverse vibrations:** they occur when the particles of the beam move approximately perpendicular to the axis of the beam. In this scenario, the beam alternately straightens and bends, leading to the induction of bending and shear stresses within the beam.

- **Torsional vibrations:** they occur when the particles of the beam undergo alternate twisting and untwisting due to vibratory motion. In this case, torsional shear stresses are induced in the beam.

This distinction helps to categorize and differentiate the various types of vibrational motion that can occur, specifying the direction and nature of the movement of particles within the structure.

It's important to acknowledge that HybriDFEM, for the time being, is conceived as a program that works in two dimensions and not three, so each rigid block has 3 DoF instead of 6. This means that it can't simulate torsional vibrations, and transverse vibrations are restricted to a single axis.

3.2.2 Implementation

Given that the mass matrix $\mathbf{M}$ and the initial stiffness matrix $\mathbf{K}$ are known, the implementation in HybriDFEM becomes relatively straightforward. Equation 3.2.9 can be solved by utilizing either the 'linalg.eig' [41] or 'sparse.linalg.eigs' [42] function from the SciPy Python package, which grants access to the eigenvalues ω_n^2 and eigenvectors ϕ_n . Subsequently, the frequencies are sorted from the smallest to the largest to align with conventions.

The implemented function for computing such modes, named '`get_frequencies()`', provides the flexibility to select the number of modes that HybriDFEM needs to compute. In systems with a high degree of freedom, computational time becomes critical due to the considerable size of matrices involved. However, typically, only the first few modes are essential to compute. By calculating only a subset of the modes, significant reductions in program computation time can be achieved.

Chapter 4

Natural Frequencies and Mode Shapes: Tests and Validations

This chapter will focus on testing the mode shapes and frequencies of HybriDFEM across various benchmarks sourced from the literature. These benchmarks will range from simple structures, such as beams, to more intricate ones, such as arches and walls. This approach will facilitate a deeper understanding of the strengths and limitations of HybriDFEM. Additionally, some tests will be conducted to verify the mass matrix formulation.

4.1 Mass Matrix

To validate the correctness of the proposed formulation, it was tested with various cases including rectangles, trapezoids, and parallelograms, each with different dimensions and reference points [14, 39, 44]. This last condition is particularly important for the first and last blocks of any HybriDFEM beam element, which are half-blocks where the origin of the block does not coincide with the center of mass of the half-block.

All the results are reported in Tables 4.1, 4.2, 4.3 and 4.4. For each case, the formulas for computing the rotational inertias with respect to the center of mass are reported on top of the table. The value relative to any other point can be computed using the parallel axis theorem.

For all the cases tested, the results of HybriDFEM, for both 2-triangle and 4-triangle divisions, matched the analytical expressions, confirming the formulation's reliability. The main difference between the 2-triangle and 4-triangle versions is the computation speed, with the former being five times faster, which is highly

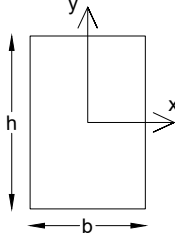
Rectangle					$J = \frac{h^2 + b^2}{12} m$			
			Analytical		4 triangles		2 triangles	
Reference point	h [m]	b [m]	m [kg]	J [kg m ²]	m [kg]	J [kg m ²]	m [kg]	J [kg m ²]
(0,0)	2	2	400	266.6667	400	266.6667	400	266.6667
	5	2	1000	2416.667	1000	2416.667	1000	2416.667
	2	5	1000	2416.667	1000	2416.667	1000	2416.667
	0.5	3	150	115.625	150	115.625	150	115.625
	3	0.5	150	115.625	150	115.625	150	115.625
(-b/2,0)	2	2	400	666.6667	400	666.6667	400	666.6667
	5	2	1000	3416.667	1000	3416.667	1000	3416.667
	2	5	1000	8666.667	1000	8666.667	1000	8666.667
	0.5	3	150	453.125	150	453.125	150	453.125
	3	0.5	150	125	150	125	150	125
(0,h/2)	2	2	400	666.6667	400	666.6667	400	666.6667
	5	2	1000	8666.667	1000	8666.667	1000	8666.667
	2	5	1000	3416.667	1000	3416.667	1000	3416.667
	0.5	3	150	125	150	125	150	125
	3	0.5	150	453.125	150	453.125	150	453.125

Table 4.1: Tests for mass and rotational inertia of a rectangle - different cases.

relevant for finer discretizations. Additionally, the updated formulation includes fewer conditional statements, is simpler to understand, and has an overall 67 % fewer lines of code, outperforming its predecessor. For all tests $\rho = 100 \text{ kg/m}^3$ and $e = 1 \text{ m}$ were used.

As the mass matrix is positive and diagonal, future optimizations could involve storing it in a vector instead of a matrix. This change would significantly reduce the required space but would also necessitate adaptations of all the mass-dependent functions.

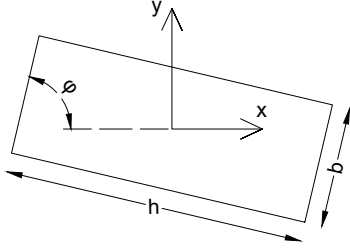
Rotated Rectangle						$J = \frac{b^2 \cos^2(\phi) + h^2 \sin^2(\phi)}{12} m + \frac{h^2 \cos^2(\phi) + b^2 \sin^2(\phi)}{12} m$			
				Analytical		4 triangles		2 triangles	
Reference Point	h [m]	b [m]	ϕ [-]	m [kg]	J [kg m ²]	m [kg]	J [kg m ²]	m [kg]	J [kg m ²]
(0,0)	2	2	1.57	400	266.66	400	266.66	400	266.66
	5	5	1.57	2500	10416.67	2500	10416.67	2500	10416.67
	5.657	1.4425	0.785	816	2317.498	816	2317.49	816	2317.49

Table 4.2: Tests for mass and rotational inertia of a rotated rectangle - different cases.

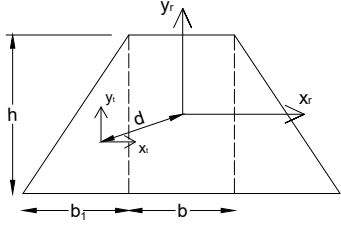
Symmetric Trapezoid						$J = \frac{b^2 + h^2}{12} m_r + \frac{b_1^2 + h^2}{9} m_t + 2d^2 m_t$			
				Analytical		4 triangles		2 triangles	
Reference point	b [m]	b ₁ [m]	h [m]	m [kg]	J [kg m ²]	m [kg]	J [kg m ²]	m [kg]	J [kg m ²]
(0,0)	2	0	2	400	266.66	400	266.66	400	266.66
	2	2	2	800	1600	800	1600	800	1600
	5	2	2	1400	6650	1400	6650	1400	6650
	2	2	5	2000	7500	2000	7500	2000	7500
	20	10	2	6000	502000	6000	502000	6000	502000
	3	3	30	1800	1417500	1800	1417500	1800	1417500
(0,3)	2	0	2	400	3866.66	400	3866.66	400	3866.66
	2	2	2	800	8800	800	8800	800	8800
	5	2	2	1400	19250	1400	19250	1400	19250
	2	2	5	2000	25500	2000	25500	2000	25500
	20	10	2	6000	556000	6000	556000	6000	556000
	3	3	30	1800	1579500	1800	1579500	1800	1579500

Table 4.3: Tests for mass and rotational inertia of a trapezoid - different cases.

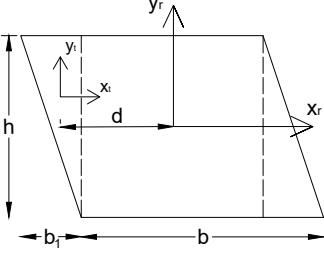
Symmetric Parallelogram						$J = \frac{h^2}{12}m + \frac{b_1^2}{9}m_t + \frac{(b - b_1)^2}{12}m_r + 2d^2 m_t$			
				Analytical		4 triangles		2 triangles	
Reference point	b [m]	b ₁ [m]	h [m]	m [kg]	J [kg m ²]	m [kg]	J [kg m ²]	m [kg]	J [kg m ²]
(0,0)	2	0	2	400	266.66	400	266.66	400	266.66
	3	1	2	400	300	400	300	400	300
	6	1	2	1000	2500	1000	2500	1000	2500
	4	1	1.5	450	459.375	450	459.375	450	459.375
	3.5	1	6	1500	5406.25	1500	5406.25	1500	5406.25
(2,0)	2	0	2	400	1866.66	400	1866.66	400	1866.66
	3	1	2	400	1900	400	1900	400	1900
	6	1	2	1000	6500	1000	6500	1000	6500
	4	1	1.5	450	2259.375	450	2259.375	450	2259.375
	3.5	1	6	1500	11406.25	1500	11406.25	1500	11406.25
(2,2)	2	0	2	400	3466.66	400	3466.66	400	3466.66
	3	1	2	400	3500	400	3500	400	3500
	6	1	2	1000	10500	1000	10500	1000	10500
	4	1	1.5	450	4059.375	450	4059.375	450	4059.375
	3.5	1	6	1500	17406.25	1500	17406.25	1500	17406.25

Table 4.4: Tests for mass and rotational inertia of a parallelogram - different cases.

4.2 Longitudinal Vibration of a Beam

From the literature [1, 28], analytical expressions for the longitudinal natural modes of a beam can be found for different boundary conditions. Table 4.5 presents three support conditions along with their respective frequency equations, natural frequencies, and mode shape equations. For all three cases, L is the length of the beam and n is the mode number. Additionally, the beams are composed of a linear elastic material where E and ρ are the Young modulus and volumetric mass density, respectively.

Beam Boundary Conditions	Frequency equation	Natural frequencies	Mode shape equation
C-F	$\cos\left(\omega L \sqrt{\frac{\rho}{E}}\right) = 0$	$\omega_n = \frac{(2n-1)\pi}{2L} \sqrt{\frac{E}{\rho}}$	$\phi_n(x) = \sin\left(\frac{2n-1}{2L}\pi x\right)$
F-F	$\sin\left(\omega L \sqrt{\frac{\rho}{E}}\right) = 0$	$\omega_n = \frac{n\pi}{L} \sqrt{\frac{E}{\rho}}$	$\phi_n(x) = \cos\left(\frac{n\pi}{L}x\right)$
C-C	$\sin\left(\omega L \sqrt{\frac{\rho}{E}}\right) = 0$	$\omega_n = \frac{n\pi}{L} \sqrt{\frac{E}{\rho}}$	$\phi_n(x) = \sin\left(\frac{n\pi}{L}\pi x\right)$

Table 4.5: Natural frequencies, frequency equations and mode shape equations for three different type of beams. A cantilever beam (C-F), which has one side clamped and one free. A clamped beam (C-C), which has both sides clamped. A free beam (F-F), which has both sides free.

Testing for a cantilever beam

The numerical example consists of a cantilever beam (Figure 4.1) of length $L = 5$ m and rectangular cross-section of dimension $h \times b = 0.4 \text{ m} \times 0.2 \text{ m}$. It is composed of a homogeneous linear elastic material with $E = 210$ GPa, $\rho = 7850 \text{ kg/m}^3$ and $\nu = 0.3$. Shear deformations and rotational inertias are considered in the analysis, but they do not affect the observed frequencies, as these are solely dependent on the axial stiffness.

The benchmark involves computing the natural frequencies and mode shapes of the beam using HybriDFEM with different levels of block discretization. Ten contact pairs were used for all discretizations. The amount of contact pairs chosen does not influence the results, as these frequencies solely depend on the axial stiffness. This can be demonstrated easily, as in HybriDFEM, the axial stiffness of one contact

face k_n is computed as:

$$k_n = \sum_{i=1}^N \frac{E A}{L} \frac{1}{N} = \frac{E A}{L} \quad (4.2.1)$$

with A the surface of the cross-section and N the number of contact pairs. As N cancels out, there is no dependency on the contact pairs.

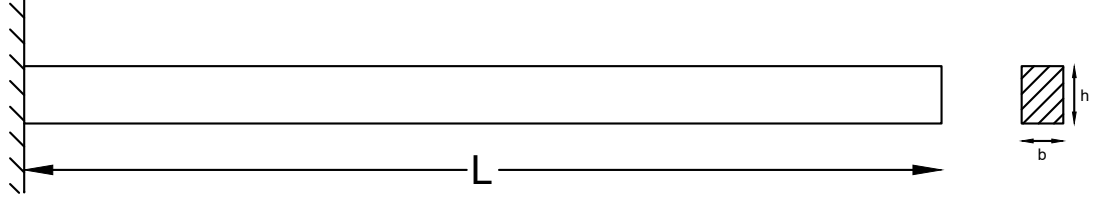


Figure 4.1: Example of cantilever beam.

The first ten natural frequencies obtained with HybriDFEM are compared with the analytical solution of a cantilever beam from Table 4.5. This comparison is visible in Figure 4.2. The abscissa reports the mode number while the ordinate the percentage error, corresponding to the difference between the numerical and analytical frequency, divided by the latter.

The figure illustrates a clear convergence towards the analytical solution. The error of the finest discretization, composed of 200 blocks, reaches -0.00026% for the first frequency, while -0.094% for the tenth. The error for the coarser discretization, composed of 10 blocks, reaches -0.127% for the first mode, which is relatively close to the analytical solution, while -32.8% for the ninth, which is much more distant.

This underscores that achieving accurate results for the first mode, which is usually the most important, is feasible even with a coarse discretization. Achieving accurate results for the second, third, and fourth modes necessitates a finer discretization, ideally around 50 blocks. However, for higher modes, a very fine discretization becomes imperative, despite the increased simulation time. Fortunately, typically only the first few modes are of interest in structural computations, so this computational overhead should be manageable.

The first five numerical mode shapes are visible in Figure 4.4 for a coarse (10 blocks) and very fine (200 blocks) discretization. As it could be expected, no rotation or vertical displacement are visible, there is only movement in the longitudinal direction. In both cases, zones where the rigid blocks accumulate create compression stresses, and zones where the rigid blocks move apart result in tensile stresses.

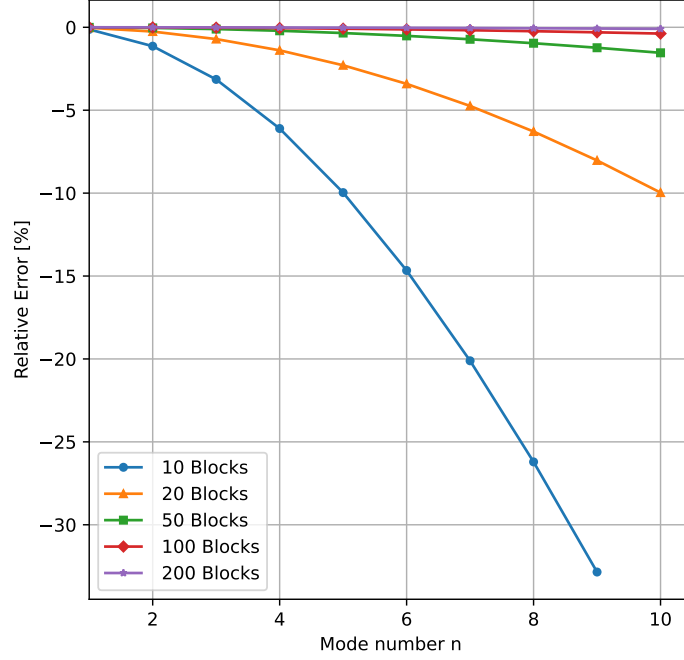


Figure 4.2: The first ten axial frequencies of a cantilever beam, for different block discretizations and using ten contact pairs (CPs).

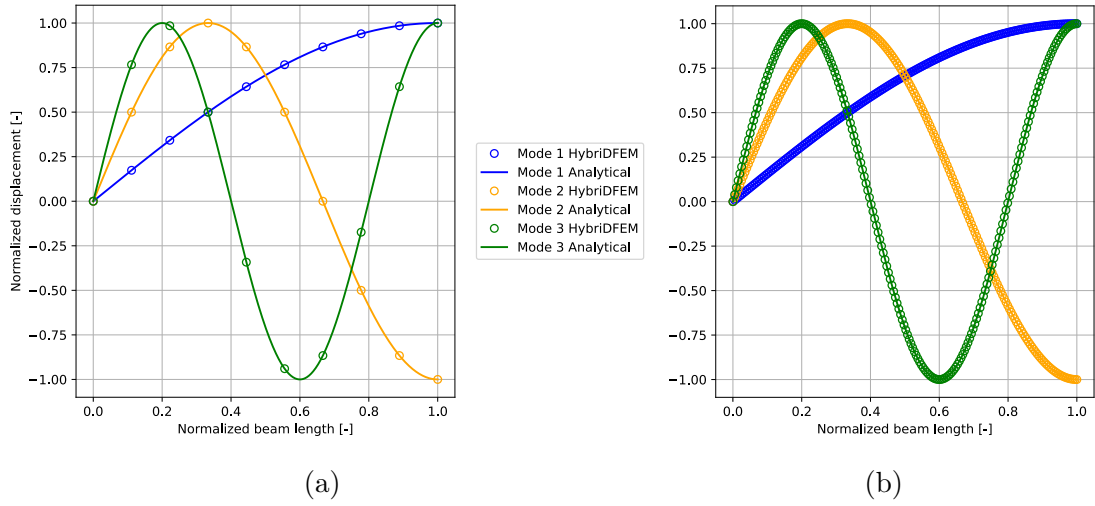


Figure 4.3: First three mode shapes for coarse (a) and very fine (b) block discretizations. Only the vertical displacement is shown.

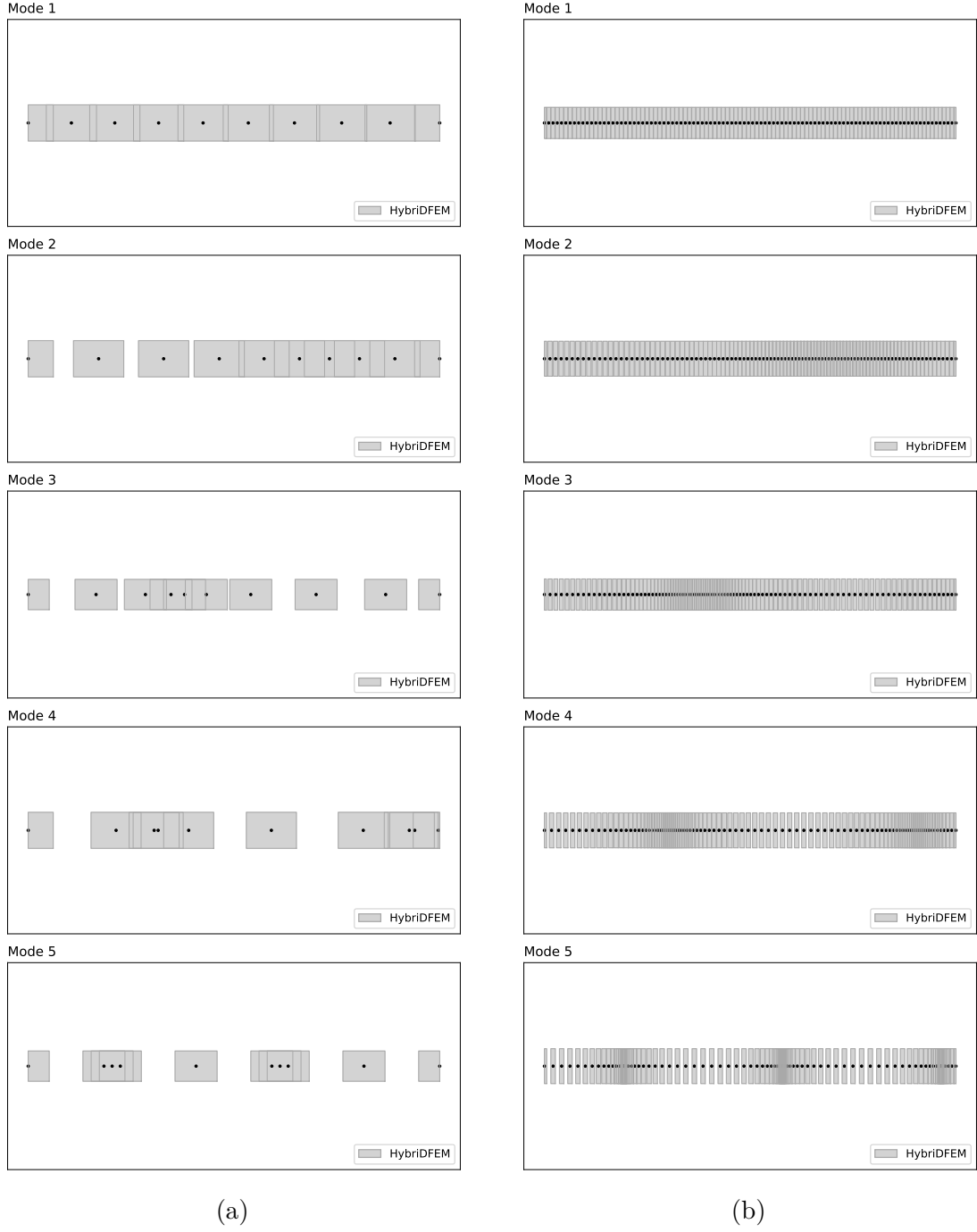


Figure 4.4: First five mode shapes for a coarse (a) and very fine (b) block discretization. Horizontal displacement, vertical displacement and rotation are shown. The scale of (b) was doubled to better see the compression/tension effects.

The vertical displacements from HybriDFEM are compared with the analytical solution in Figure 4.3. Both coarse and very fine discretizations are presented. The abscissa shows the normalized beam length, while the ordinate shows the normalized displacement in the longitudinal direction. Even with a very coarse discretization (ten blocks), HybriDFEM results fit well with the analytical curve. Using a finer block discretization provides more points on the curve for a better approximation of the shape.

Testing for a clamped beam

The numerical example consists of a clamped beam (Figure 4.5) of length $L = 5$ m and rectangular cross-section of dimension $h \times b = 0.4$ m $\times$ 0.2 m. It is composed of a homogeneous linear elastic material with $E = 210$ GPa, $\rho = 7850$ kg/m³ and $\nu = 0.3$. Shear deformations and rotational inertias are considered in the analysis, but they do not affect the observed frequencies, as these are solely dependent on the axial stiffness.

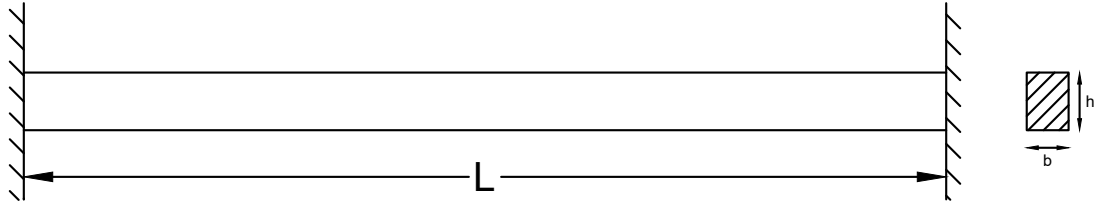


Figure 4.5: Example of clamped beam.

Similarly to the previous example, the benchmark involves computing the natural frequencies and mode shapes of the beam using HybriDFEM with different levels of block discretization. Ten contact pairs were used for all discretizations.

The first ten natural frequencies obtained with HybriDFEM are compared with the analytical solution of a clamped beam from Table 4.5. This comparison is visible in Figure 4.6. The error of the finest discretization, composed of 200 blocks, reaches -0.001% for the first frequency, while -0.1% for the tenth. The error for the coarser discretization, composed of 10 blocks, reaches -0.51% for the first mode, while -29.5% for the eighth.

It is noticeable that the numerical results are less precise than in the previous example. However, convergence to the analytical solution is still present. Apart from the different boundary conditions, the discrepancy could be caused by the fact that in the clamped example, there are three fewer degrees of freedom (DoFs) compared to the cantilever example, as two blocks are fixed instead of one.

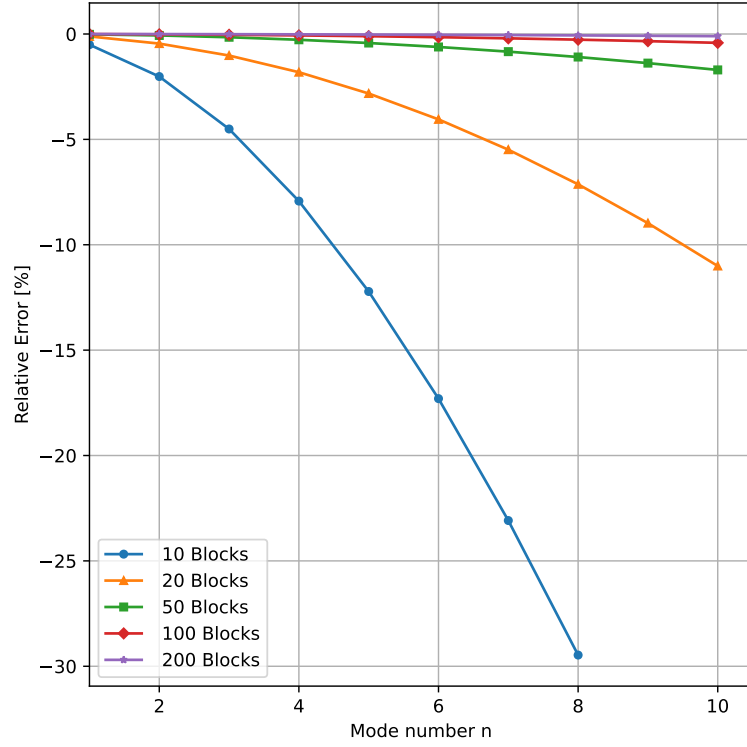


Figure 4.6: The first ten axial frequencies of a clamped beam, for different block discretizations and using ten contact pairs (CPs).

As also observed in the other example, the first mode, which is the most important, can be obtained with acceptable precision even with a coarse discretization. In this case, 20 blocks, instead of 10, are recommended to achieve the same degree of precision as the cantilever case. For higher modes, a more precise discretization is necessary to obtain reliable approximations.

The first five numerical mode shapes are visible in Figure 4.7 for a coarse (10 blocks) and very fine (200 blocks) discretization. No rotation or vertical displacement is visible, there is only movement in the longitudinal direction. In both cases, there are zones where the rigid blocks accumulate that create compression stresses, and zones where the rigid blocks move away from each other, resulting in tensile stresses. Compared to the cantilever case, some of the mode shapes for the clamped configuration (like mode 2 or 4) exhibit symmetry, as identical supports are present on both sides.

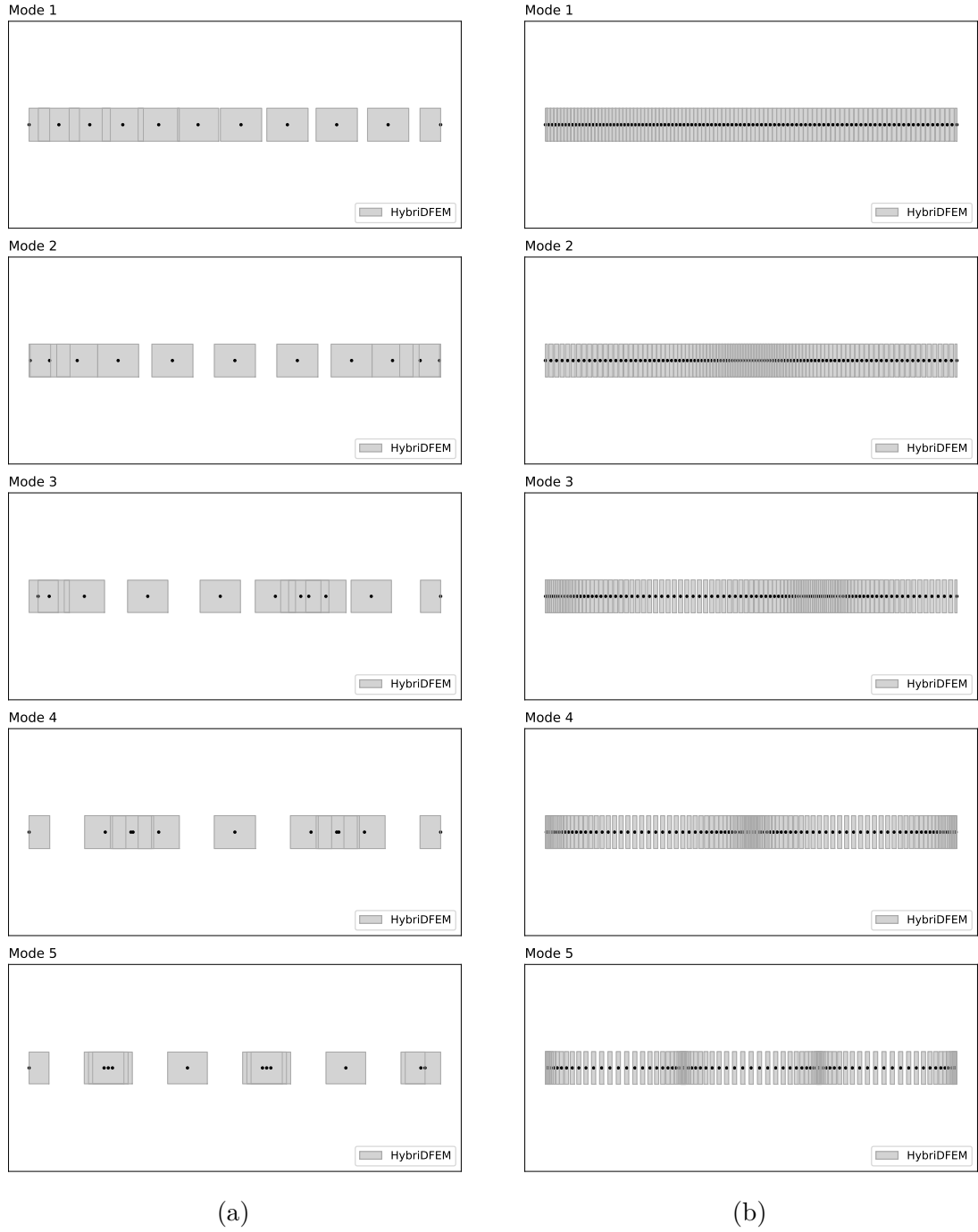


Figure 4.7: First five mode shapes for a coarse (a) and very fine (b) block discretization. Horizontal displacement, vertical displacement and rotation are shown. The scale of (b) was doubled to better see the compression/tension effects.

The vertical displacements from HybriDFEM are compared with the analytical solution in Figure 4.8. A coarse and very fine discretization are presented. Even with a very coarse discretization of 10 blocks, HybriDFEM results align well with the analytical curve for the first two modes. However, for the third mode, there are some discrepancies near the peaks of the sinusoids. This is caused by the normalization of the values computed with HybriDFEM. With a ten-block discretization, the peak of the mode shape, situated in 0.25 (or 0.75), couldn't be computed, causing HybriDFEM to normalize with respect to a smaller value and generating errors. Using a finer block discretization would provide more points on the curve, allowing for a better approximation of the shape and resolving these differences.

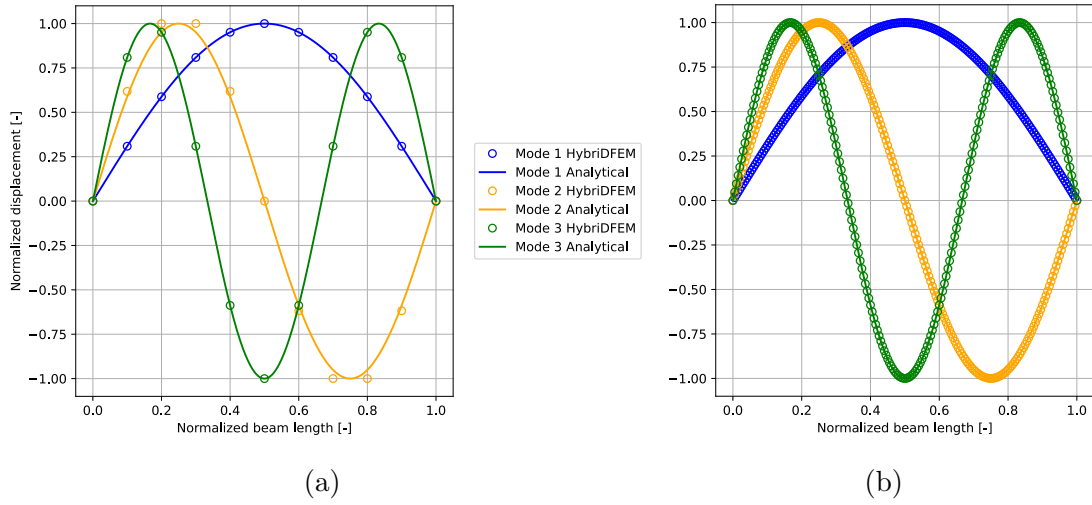


Figure 4.8: First three mode shapes for coarse (a) and very fine (b) block discretizations. Only the vertical displacement is shown.

4.2.1 Section Summary and Reflections

In summary, HybriDFEM demonstrates excellent performance when analysing longitudinal modes. Two cases were tested: a cantilever and a clamped beam. In both cases, with a discretely fine discretization, a very good approximation of the frequencies can be obtained, with negligible errors. Similarly, the mode shapes are very well approximated and seem to correctly represent what is expected. Overall, HybriDFEM emerges as a powerful tool for effectively modelling longitudinal modes. Undoubtedly, exploring more complex cases, such as tapered beams, would provide valuable insights. However, the results obtained from these examples are highly promising, laying a solid foundation for further investigation.

4.3 Transverse Vibration of a Beam

In the literature, there is a greater emphasis on computing the mode shapes and frequencies for transverse vibration rather than for longitudinal vibration. This emphasis is attributed to the significant impact bending modes have on the structural behaviour of beams in practical applications. Bending modes govern crucial aspects such as deflection, stress distribution, and overall structural integrity. Therefore, more benchmarks will be tested for this scenario.

4.3.1 Euler-Bernoulli Beam Theory

The Euler–Bernoulli beam theory, also referred to as classical beam theory [50], is a simplification of the linear theory of elasticity, which provides a means of calculating the load-carrying and deflection characteristics of beams. It covers the case corresponding to small deflections of a beam that is subjected to transverse loads only. It is based on the major assumption that 'plane sections remain plane'. By disregarding the influences of shear deformation and rotational inertia, it can be considered a specific case of the Timoshenko–Ehrenfest beam theory. The Euler–Bernoulli theory is particularly effective when applied to slender beams, where the impacts of shear deformation and rotational inertia are generally minimal and often deemed negligible. It is in fact also referred to as "Euler–Bernoulli theory of slender beams".

The subject is strongly explored in the literature [14, 22, 50], here, only the final formulation for the natural frequencies and mode shapes will be reported.

In the case of free vibration for a beam of constant flexural rigidity EI , the natural frequencies for the n -th mode can be computed as:

$$\omega_n = (\beta_n L)^2 \sqrt{\frac{EI}{\rho A L^4}} \quad (4.3.1.1)$$

with A the surface of the beam cross-section, ρ the volumetric mass density and L the length of the beam. The term β_n depends on the boundary conditions of the problem and the mode number n and it's computed by solving specific frequency equations.

Table 4.6 presents a series of frequency equations for various boundary conditions. As there are infinite solutions for each equation, there are an infinite number of values possible for β , so also an infinite number of frequencies. The fundamental frequency is computed with the smallest possible positive value for β , denoted β_1 . The table also includes corresponding mode shape equations.

Beam boundary condition	Frequency equation	Mode shape equation
C-F	$\cosh \beta L \cos \beta L + 1 = 0$	$y = (\sin(\beta_n x) - \sinh(\beta_n x)) - \frac{\sin(\beta_n L) + \sinh(\beta_n L)}{\cos(\beta_n L) + \cosh(\beta_n L)} (\cos(\beta_n x) - \cosh(\beta_n x))$
C-C	$\cosh \beta L \cos \beta L - 1 = 0$	$y = (\cosh(\beta_n x) - \cos(\beta_n x)) - \frac{\cosh(\beta_n L) + \cos(\beta_n L)}{\sinh(\beta_n L) + \sin(\beta_n L)} (\sinh(\beta_n x) - \sin(\beta_n x))$
C-P	$\sin \beta L = 0$	$y = (\cosh(\beta_n x) - \cos(\beta_n x)) - \frac{\cosh(\beta_n L) + \cos(\beta_n L)}{\sinh(\beta_n L) + \sin(\beta_n L)} (\sinh(\beta_n x) - \sin(\beta_n x))$
P-R	$\sin \beta L = 0$	$y = \sin \beta_n x$

Table 4.6: Frequency and mode shape equation for four different types of beams. A cantilever beam (C-C), which has one side clamped and one free. A clamped beam (C-C), which has both sides clamped. A clamped-pinned beam (C-P), which has one side clamped and one pinned. A simply supported beam (P-R), which has a one side pinned and with a roller.

The first mode shapes for these boundary conditions, computed using the equations from Table 4.6, are illustrated in Figure 4.9.

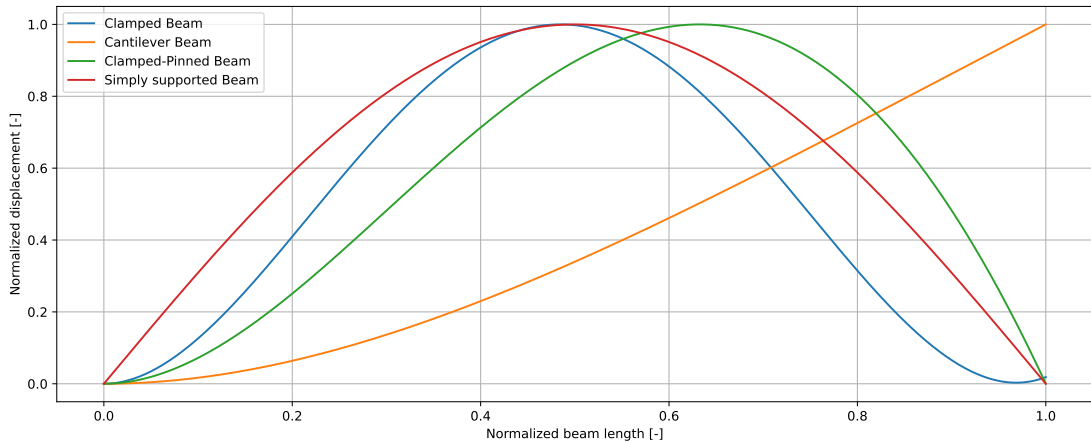


Figure 4.9: First normalized mode shapes for four different boundary conditions.

Testing for a simply supported Euler-Bernoulli beam

The numerical example consists of a simply supported beam (Figure 4.10) of length $L = 5$ m and rectangular cross-section of dimension $h \times b = 0.4$ m $\times$ 0.2 m. It is composed of a homogeneous linear elastic material with $E = 210$ GPa, $\rho = 7850$ kg/m³ and $\nu = 0.3$. Shear deformations and rotational inertias are neglected according to the Euler-Bernoulli beam hypothesis.

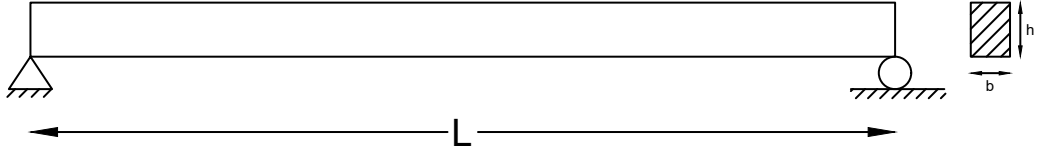


Figure 4.10: Example of simply supported beam.

The benchmark consists of computing the natural frequencies and mode shapes for such beam using HybriDFEM with different levels of discretization, both at the block and CP level.

A more refined block discretization will result in a more evenly distributed mass. Since the natural frequencies are influenced by the mass (see Equation 4.3.1.1), this refinement will lead to a more accurate approximation of the natural frequencies.

A more refined CP discretization will result in a better approximation of the flexural stiffness EI . Since the natural frequencies are influenced by the flexural stiffness (see Equation 4.3.1.1), this refinement will also lead to a more accurate approximation of the natural frequencies.

Comparisons between the first ten natural frequencies obtained with HybriDFEM and the analytical solution (Equation 4.3.1.1) are shown in Figure 4.11, with the mode number on the abscissa and the percentage error on the ordinate, computed as the difference between the numerical and analytical frequencies divided by the latter.

For Figure 4.11a, computed with 100 CPs, the error of the finest discretization, composed of 400 blocks, reaches -0.002% for the first frequency, while 0.01% for the tenth. The error for the coarser discretization, composed of 25 blocks, reaches 0.033% for the first mode, which is relatively close to the analytical solution, while 4.42% for the tenth, which is significantly more distant.

For Figure 4.11b, computed with 200 blocks, the error of the finest discretization, composed of 200 CPs, reaches 0.000% for the first frequency, while 0.051% for the tenth. The error for the coarser discretization, composed of 25 blocks, reaches -1.015% for the first mode, and -0.963% for the tenth, seemingly giving better results for higher modes. This is because the error from the block discretization slightly compensates the error from the CPs.

It is interesting to notice that while a finer block discretization significantly improves higher modes and only slightly the first, using a finer contact pair discretization improves the precision of all the modes, shifting the whole curve closer to the analytical solution.

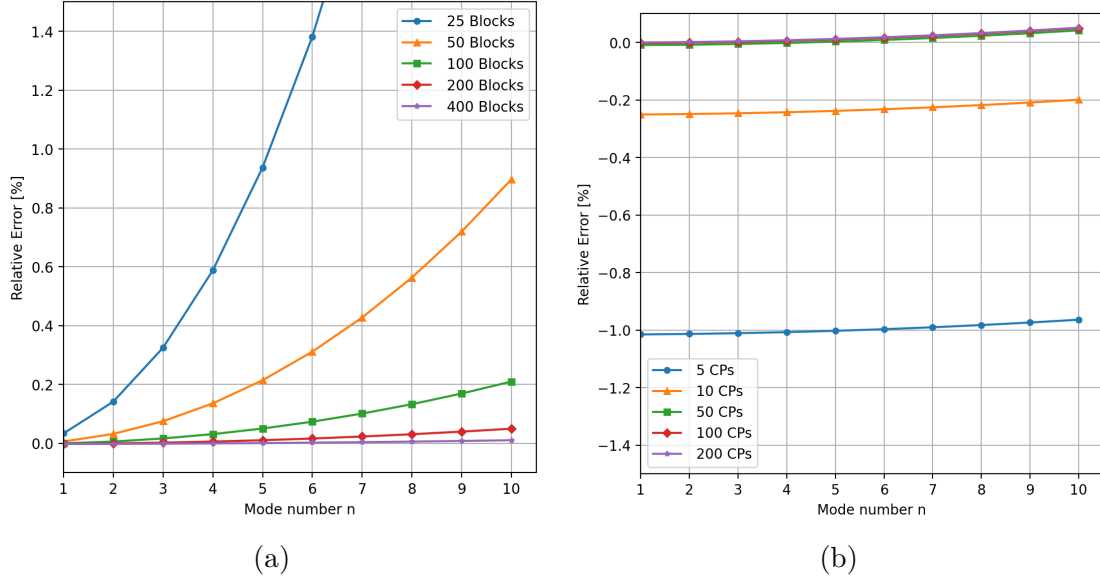


Figure 4.11: Influence of the block (a) and CP (b) discretization on the first 10 frequencies. On (a) the CP number is fixed at 100, while on (b) the number of blocks is fixed at 200.

The fact that the HybriDFEM predicts higher frequencies than the analytical value for coarse block discretization, but smaller frequencies for coarse CPs discretization, suggests that a few blocks tend to overestimate the stiffness or underestimate the mass of the beam, whereas a few CPs tends to underestimate it stiffness. Overall, in both cases, a clear convergence to the analytical solution is evident, thereby validating the correctness of the formulation for this specific case.

The first five mode shapes obtained with HybriDFEM are compared with the analytical solution from Table 4.6 and are visible on Figure 4.12. The number of CP was kept at 100 as they don't influence the mode shapes. The reason for this is that, unlike frequencies, mode shapes primarily rely on the overall geometry and boundary conditions of the beam rather than on its flexural stiffness.

It's visible on Figure 4.12a that, even with a coarse block discretization of 25 blocks, HybriDFEM fits really well with the analytical solution. Using a finer block discretization, as the one from Figure 4.12b, where 200 blocks were used, will cause an increase in the number of points on the curve which will allow for a better approximation of the shape.

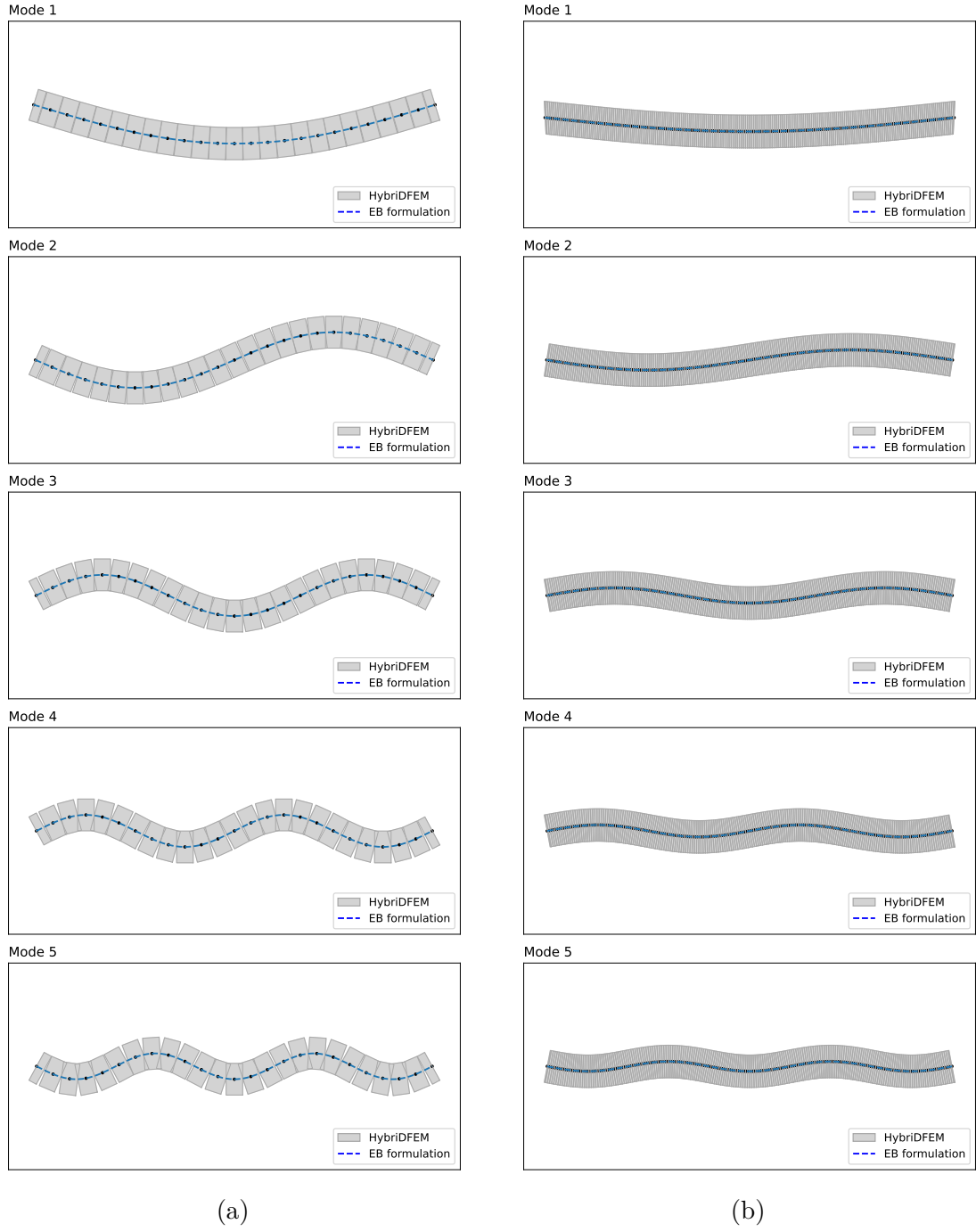


Figure 4.12: Comparison between HybriDFEM and Euler Bernoulli first 5 mode shapes. A coarse (a) and fine (b) block discretization were used, with 100 CP for both cases.

For further insights, another beam of the same dimensions, clamped at both edges, underwent analysis, detailed in Appendix A. Notably, the errors between simply supported and clamped beams are comparable for contact pair discretization, whereas for block discretization, convergence to the analytical solution is slower for the clamped beam.

4.3.2 Timoshenko–Ehrenfest Beam Theory

The Timoshenko–Ehrenfest beam theory takes into account shear deformation and rotational inertia effects, making it suitable for describing the behaviour of thick beams, sandwich composite beams, or beams subject to high-frequency excitation when the wavelength approaches the thickness of the beam. The resulting equation is of 4th order but, unlike Euler–Bernoulli beam theory, there is also a second-order partial derivative present. Physically, considering the additional deformation mechanism effectively reduces the stiffness of the beam. Consequently, this leads to greater deflection under a static load and lower predicted eigenfrequencies for a given set of boundary conditions. If the shear modulus of the beam material approaches infinity, making the beam rigid in shear, and rotational inertia effects are neglected, Timoshenko beam theory converges toward ordinary beam theory [47].

Similar to the Euler–Bernoulli beam, the Timoshenko–Ehrenfest beam theory has been extensively studied in the literature [12, 13, 18]. Here, only the final formulation for the mode shapes and frequencies will be reported. In contrast to the Euler–Bernoulli formulation, the Timoshenko–Ehrenfest formulation provides not only the transverse displacement of the mode shapes but also the section rotation, which depends on the shear deformation.

The Timoshenko beam frequency spectrum consists of two parts, separated by a transition frequency $\tilde{\omega}$. The second part is also referred to as the shear propagation spectrum.

$$\tilde{\omega} = \sqrt{\frac{G\kappa A}{\rho I}} \quad (4.3.2.1)$$

with A the surface of the beam cross-section, EI the flexural rigidity of the beam and ρ the volumetric mass density of the beam material.

From the physical point of view, it represents the ratio between shear stiffness and rotational inertia, it marks the transition between the bending-dominated and shear-dominated deformation modes. It also appears to be coinciding with the cut-off frequency for wave propagation in an infinite Timoshenko beam in K. F. Graff [23]. The transition frequency value itself may or may not be part of the

frequency spectrum, depending on the applied boundary conditions. In the case of a simply supported beam, it happens to be included in the spectrum.

To be somewhat representative of most of the classical boundary conditions, two cases were tested: a simply supported and clamped beam.

For a **simply supported beam** the natural frequencies can have three different values depending on their position relative to $\tilde{\omega}$ [12].

Case 1: $\omega_n^2 < \tilde{\omega}^2$

$$\omega_n = +\sqrt{\omega_2^2(k_2)} \text{ with } k_2 = 1, \dots, k_2^* \quad (4.3.2.2)$$

where $k_2^* = \max\{k_2 \in \mathbb{N} \mid \omega_n < \tilde{\omega}\}$.

Case 2: $\omega^2 = \tilde{\omega}^2$

$$\omega_n = \tilde{\omega} = \sqrt{\frac{G\kappa A}{\rho I}} \quad (4.3.2.3)$$

Case 3: $\omega^2 > \tilde{\omega}^2$

$$\omega_n = +\sqrt{\omega_2^2(k_2)} \text{ or } \omega_n = +\sqrt{\omega_1^2(k_1)} \quad (4.3.2.4)$$

with $k_2 = k_2^*, \dots, \infty$ and with $k_1 = 1, \dots, \infty$.

The frequencies $\omega_1^2(k_1)$ and $\omega_2^2(k_2)$ can be computed using:

$$\omega_1^2(k_1) = \frac{1}{2} \left(-b^*(k_1) - \sqrt{b^*(k_1)^2 - 4c^*(k_1)} \right) \quad (4.3.2.5)$$

$$\omega_2^2(k_2) = \frac{1}{2} \left(-b^*(k_2) + \sqrt{b^*(k_2)^2 - 4c^*(k_2)} \right) \quad (4.3.2.6)$$

where

$$b^*(k_{1,2}) = \frac{G\kappa}{\rho} \left[\frac{A}{I} + \left(\frac{k_{1,2}\pi}{L} \right)^2 \left(1 + \frac{E}{G\kappa} \right) \right] \quad (4.3.2.7)$$

$$c^*(k_{1,2}) = \frac{EG\kappa}{\rho^2} \left(\frac{k_{1,2}\pi}{L} \right)^4 \quad (4.3.2.8)$$

where $G = E/(2(1 + \nu))$ is the shear modulus and κ is the shear reduction factor.

Finally, the mode shapes can be computed for the transverse displacement and section rotation as:

$$y_n(x) = \sin(\lambda_{k_{1,2}}x), \quad \phi_n(x) = \frac{\alpha_{1,2}}{\lambda_{k_{1,2}}} \cos(\lambda_{k_{1,2}}x) \quad (4.3.2.9)$$

with :

$$\lambda_{k_{1,2}} = \frac{\pi k_{1,2}}{2}; \quad \alpha_{1,2} = \omega_n^2 \frac{\rho}{G\kappa} - \lambda_{1,2}^2 \quad (4.3.2.10)$$

The only exception is the transition frequency $\tilde{\omega}$ where:

$$y_{\tilde{\omega}}(x) = 0, \quad \phi_{\tilde{\omega}}(x) = 1 \quad (4.3.2.11)$$

Differently from what happens in the case of a **simply supported beam**, in the **clamped case** it is not possible to identify natural frequencies by a closed-form procedure. It is indeed necessary to numerically find the roots of two transcendental equations, for the first and second part of the spectrum. Some solutions computed in A. Cazzani et al, 2016 [12] obtained with a computer algebra system (CAS), namely Mathematica™ (version 6.0), are compared in Appendix A.

It can also be noted that the transition frequency is not part of the spectrum in general (except for particular special values of the beam length), differently from what happens in the simply supported beam, where it is always part of the spectrum.

Testing for a simply supported Timoshenko-Ehrenfest beam

The numerical example consists of a simply supported beam (Figure 4.13) of length $L = 1.6$ m and rectangular cross-section of dimension $h \times b = 0.4$ m $\times$ 0.2 m. It is composed of a homogeneous linear elastic material with $E = 210$ GPa, $\rho = 7850$ kg/m³ and $\nu = 0.3$. A ratio of $L/h = 4$ was chosen to better notice the effects of the shear deformation. Shear deformations and rotational inertias are not neglected according to the Timoshenko-Ehrenfest beam hypothesis.

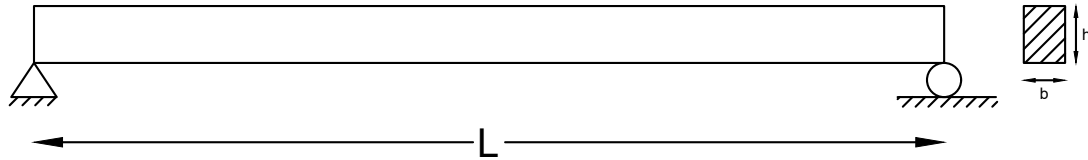


Figure 4.13: Example of simply supported beam.

The benchmark consists of computing natural frequencies and mode shapes for such beam using HybriDFEM with different levels of discretization, both at the block and CP level. The beam is computed with the shear correction factor for a rectangular section from T. Kaneko (1975) [29]:

$$\kappa = \frac{5(1 + \nu)}{6 + 5\nu} = \frac{13}{15} \quad (4.3.2.12)$$

The first ten natural frequencies obtained with HybriDFEM are compared with the analytical solution presented previously. This comparison is visible in Figure 4.14. The abscissa reports the mode number while the ordinate the percentage error, corresponding to the difference between the numerical and analytical frequency, divided by the latter.

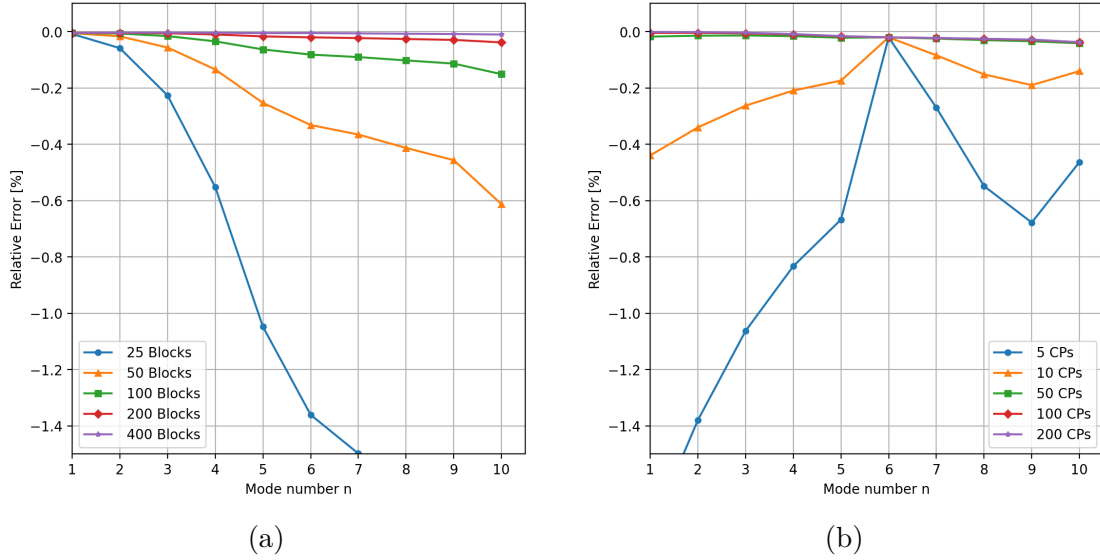


Figure 4.14: Influence of the block (a) and CP (b) discretization on the first 10 frequencies. On (a) the CP number is fixed at 100, while on (b) the number of blocks is fixed at 200.

For Figure 4.14a, computed with 100 CPs, the error of the finest discretization, composed of 400 blocks, reaches -0.004% for the first frequency, while -0.01% for the tenth. The error for the coarser discretization, composed of 25 blocks, reaches -0.008% for the first mode, which is relatively close to the analytical solution, while -2.54% for the tenth, which is significantly more distant, but closer to the analytical solution than the Euler-Bernoulli beam.

For Figure 4.14b, computed with 200 blocks, the error of the finest discretization, composed of 200 CPs, reaches -0.001% for the first frequency, while -0.037% for

the tenth. The error for the coarser discretization, composed of 25 blocks, reaches -1.77% for the first mode, and -0.463% for the tenth.

It's interesting to notice that the error for the sixth mode is always -0.02% for any CP discretization. This mode represents the transition mode $\tilde{\omega}$, a pure-shear vibration mode, so flexural effects do not enter into the play, which explains the non-dependency on the contact pairs. This is evident by observing the mode shapes in Figure 4.15. In contrast to the Euler-Bernoulli beam, HybriDFEM, when considering rotational inertia and shear effects, predicts smaller frequencies for both a block and CP coarse discretization, which suggests that a few blocks tend to underestimate the stiffness or overestimate the mass of the beam and a few CPs tend to underestimate its stiffness.

The transversal displacements of the first ten mode shapes obtained with HybriDFEM are compared with the analytical solution from Equation 4.3.2.9 and are visible on Figure 4.15. The number of CP and blocks were kept at 100. As it was seen for the Euler-Bernoulli beam, HybriDFEM approximation fits really well with the analytical curve. In this benchmark, the first five mode shapes, are part of the bending-dominated spectrum, which explains the similarity with the Euler-Bernoulli mode shapes from Figure 4.12. The transition mode $\tilde{\omega}$, which in this case is the sixth mode, can also be easily distinguished, as the transverse displacement is equal to zero and the rotation is constant. One notable observation is the recurrence of certain mode shapes between the two parts of the spectrum, even if with different amplitudes. For instance, the first mode shares similarities with the seventh, and the second mode with the ninth. Additionally, the seventh and ninth modes exhibit more pronounced shear deformations. This phenomenon explains the nomenclature of the second part of the spectrum as the shear-propagation spectrum."

The appearance of 'similar' modes, as also noted by A. Cazzani et al. [12], occurs only in case of a simply supported beam: other combinations of boundary conditions never produce modes having the same shape in the first and in the second part of the spectrum. This 'similar' modes can be told apart by observing the rotation, as visible in Figure 4.16. This figure reports the section rotations of the first 10 mode shapes computed with HybriDFEM (100 blocks $\times$ 100 CPs) compared with the analytical solution from Equation 4.3.2.9. Shear spectrum modes appear to have a much higher section rotation in comparison with bending modes.

For further insights, an additional analysis was conducted on a beam with different dimensions, clamped at both edges. Somewhat similar results were found. Two things were verified, the absence of the transition frequency in the natural frequencies, and the absence of repeating mode shapes. Details of this investigation can be found in Appendix A.

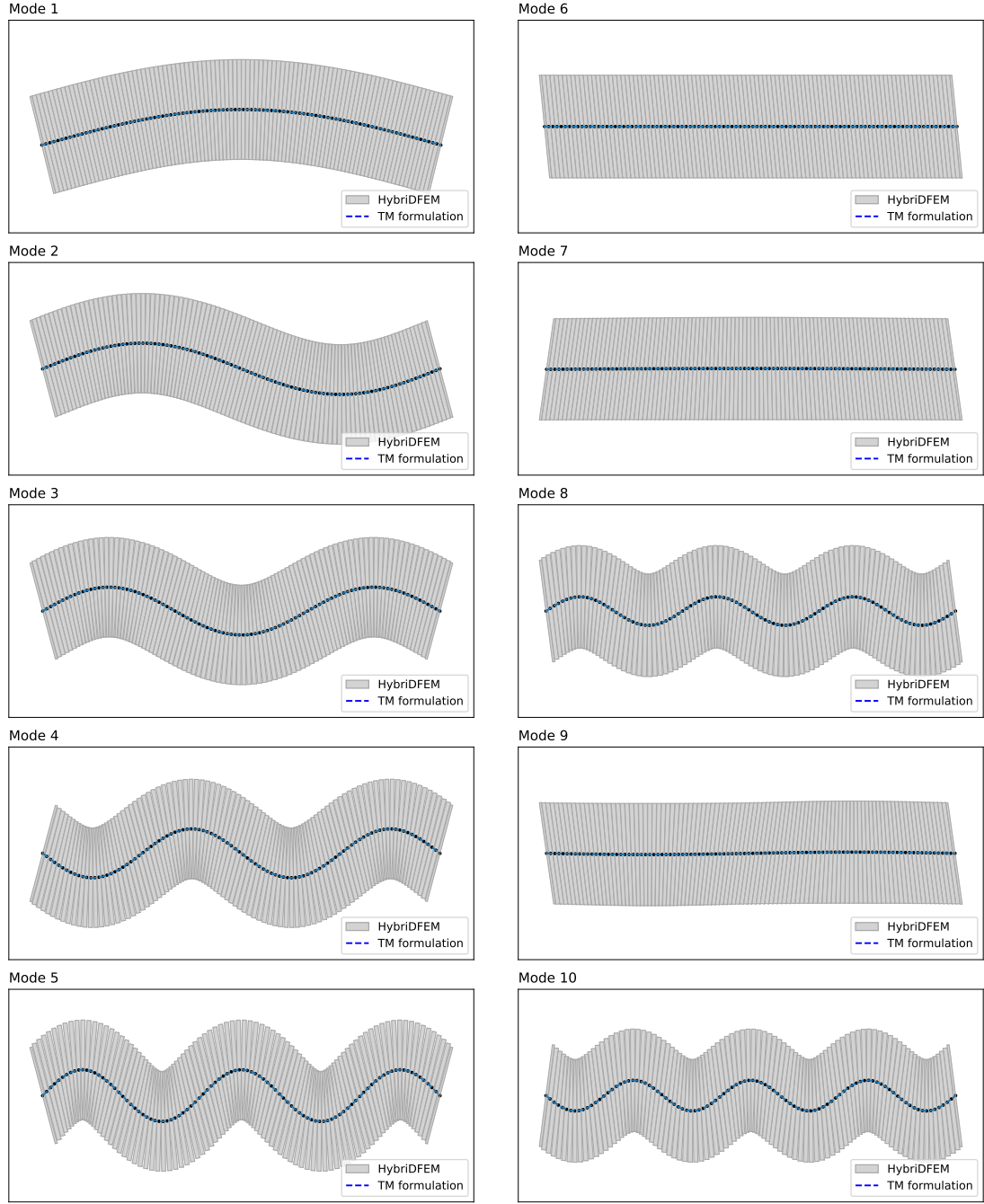


Figure 4.15: Comparison between HybriDFEM and Timoshenko-Ehrenfest first 10 mode shapes. A fine block and CP discretization was used ($100 \text{ blocks} \times 100 \text{ CPs}$).

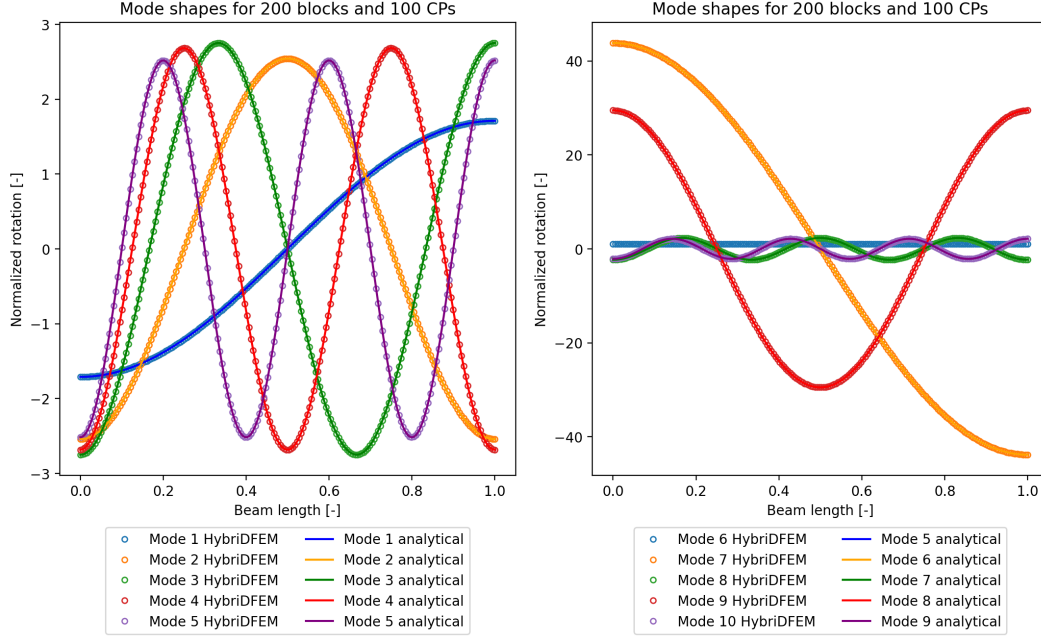


Figure 4.16: First 10 mode shapes for and very fine block discretization. Only the rotation is shown.

4.3.3 Tapered Beams

Tapered beams are structural elements commonly used in civil engineering and construction. Unlike traditional beams with a constant cross-section along their length, tapered beams gradually change in size. Such beams hold significant importance as they allow for material optimization, which is favourable economically and environmentally, and they allow weight reduction without sacrificing the load-bearing capacity of the beam. Given the somewhat frequent presence of such beams, HybridFEM performance was also tested for tapered beams.

All the following benchmarks are based on the paper by J.R. Banerjee and A. Ananthapuvirajah [2], where the natural frequencies were obtained by solving the free vibration problem of tapered beams using Bessel functions. Shear deformations and rotational inertias are neglected, similarly to what was done in the Euler-Bernoulli formulation. The natural frequencies and mode shapes are given for different boundary conditions and taper ratios c .

$$c = \frac{h_2}{h_1} - 1 \quad (4.3.3.1)$$

where h_1 and h_2 are the heights of the cross-section on the left and right extremity of the beam as depicted in Figure 4.17.

It is noteworthy that all the tests performed are limited to beams where the height varies linearly along the length, as depicted in Figure 4.17. HybriDFEM formulation assumes a constant beam depth, making variations in depth unfeasible.

This section is limited to simple cases to establish a robust foundation for the formulation. More complex scenarios, where analytical solutions could be harder to obtain, such as beams with multiple height changes across the length or non-linear variations, are left for future exploration.

Testing for a simply supported tapered beam

The numerical example consists of a simply supported tapered beam (Figure 4.17) of length $L = 5$ m and rectangular cross-section varying linearly along the length. On the side of the roller support, it is of dimension $h_1 \times b = 0.4 \text{ m} \times 0.2 \text{ m}$. On the side of the pinned support, it is of dimension $h_2 \times b = 0.8 \text{ m} \times 0.2 \text{ m}$. It is composed of a homogeneous linear elastic material with $E = 210 \text{ GPa}$, $\rho = 7850 \text{ kg/m}^3$ and $\nu = 0.3$. The shear deformation and rotational inertia are neglected.

The benchmark consists of computing the natural frequencies and mode shapes for such beams using HybriDFEM with different levels of discretization, both at the block and CP level.

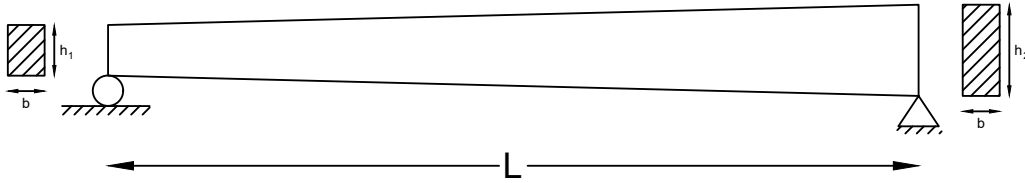


Figure 4.17: Example of simply supported tapered beam.

The first five natural frequencies obtained with HybriDFEM are compared with the analytical solution of a simply supported beam from the analysed paper. This comparison is visible in Figure 4.18.

For Figure 4.18a, computed with 100 CPs, the error of the finest discretization, composed of 400 blocks, reaches 0.000% for the first frequency, while 0.001% for the fifth. The error for the coarser discretization, composed of 25 blocks, reaches 0.065% for the first mode, and 1.93% for the fifth.

For Figure 4.18b, computed with 200 blocks, the error of the finest discretization, composed of 200 CPs, reaches 0.000% for the first frequency, while 0.025% for the fifth. The error for the coarser discretization, composed of 25 blocks, reaches -2.02% for the first mode, and -1.99% for the fifth.

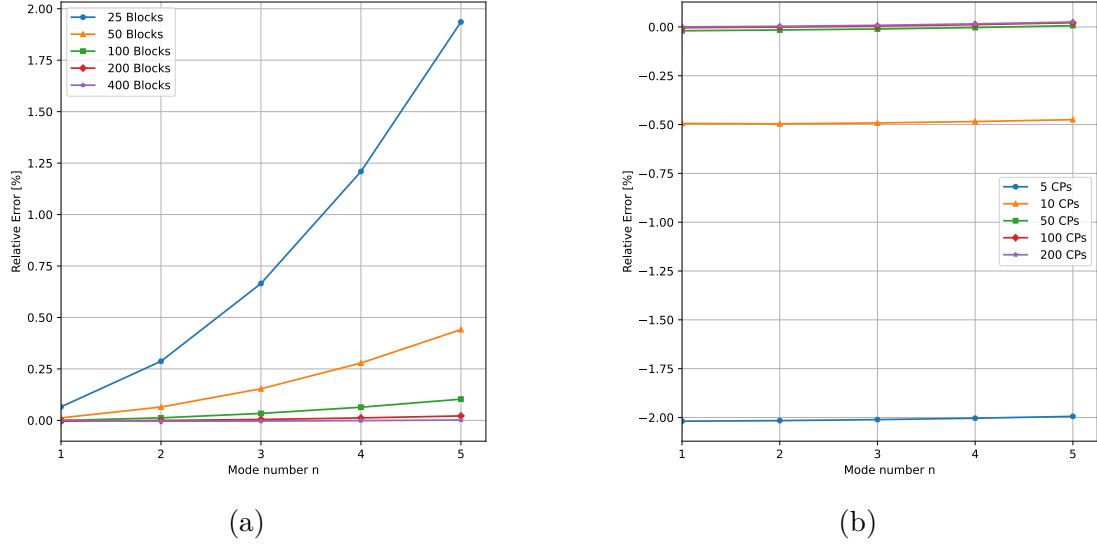


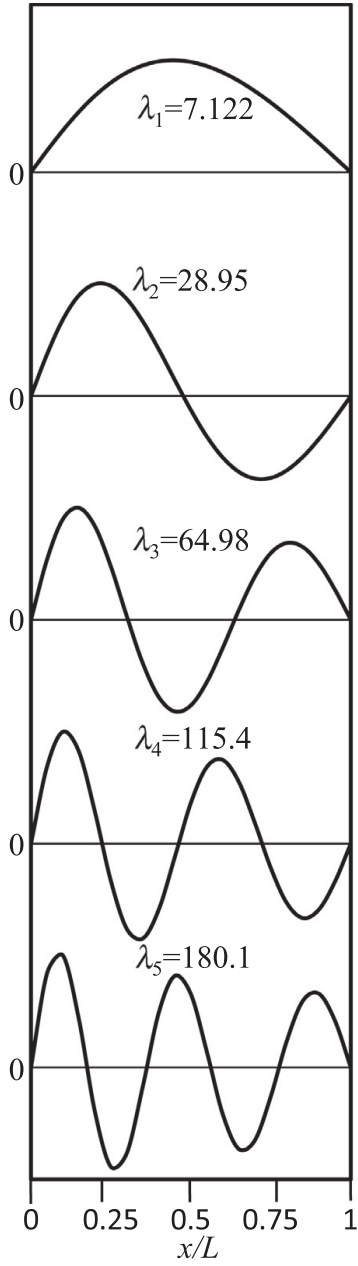
Figure 4.18: Influence of the block (a) and CP (b) discretization on the first 5 frequencies. On (a) the CP number is fixed at 100, while on (b) the number of blocks is fixed at 200.

For both discretizations, the convergence to the analytical solution closely resembles that of a simply supported Euler-Bernoulli beam, as shown in a previous section in Figure 4.11. Similarly, while a more precise block discretization improves significantly higher modes and only slightly the first, using a more precise contact pair discretization improves the precision of all the modes altogether, shifting the whole curve closer to the analytical solution.

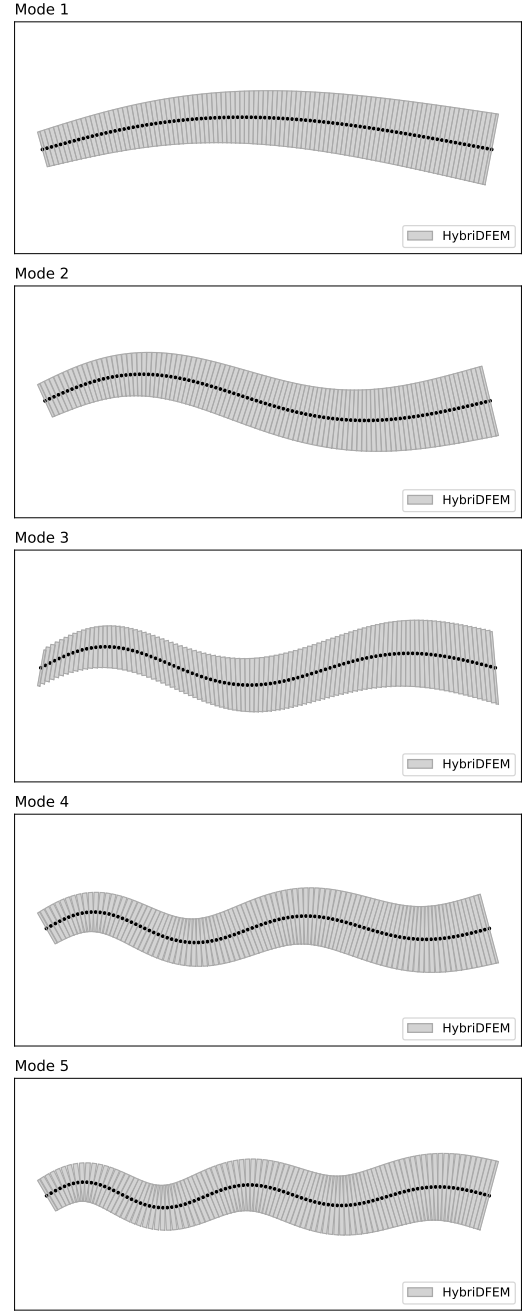
It is interesting to notice that HybriDFEM tapered beams yield less accurate results compared to HybriDFEM Euler-Bernoulli beams for coarser discretization, but achieve greater precision for finer discretizations. This discrepancy is likely due to the precision of the frequency from the article, where the decimal numbers are restricted to four.

The first five mode shapes obtained with HybriDFEM are compared with the analytical solution from the article and are visible on Figure 4.19. The number of CP and blocks were kept at 100. As already mentioned in other examples, contact pairs do not influence the mode shapes, and the number of blocks just provides a greater number of points to approximate the mode shape. Even with a coarser discretization, the fit remains quite good.

J.R. Banerjee's mode shapes (Figure 4.19a) and those obtained through HybriDFEM (Figure 4.19b) exhibit a striking similarity to each other. Although the paper lacks equations or specific values for direct comparison, the noticeable close resemblance



(a)



(b)

Figure 4.19: First 5 mode shapes obtained with (a) J.R. Banerjee et al. [2] and (b) HybriDFEM with 100 blocks and 100 CPs.

of the shapes, coupled with HybriDFEM's accurate computation of frequencies, suggests that HybriDFEM approximates this benchmark quite effectively.

Testing for Different Boundary Conditions and Taper Ratios

The numerical example consists of three different types of beams: a simply supported, a cantilever and clamped beam. All the beams are of length $L = 5$ m and rectangular cross-section varying linearly along the length. On the left extremity of the beam, it is of dimension $h_2 \times b = 0.4 \text{ m} \times 0.2 \text{ m}$. On the right extremity, it is of dimension $h_2 \times b = 0.4(1+c) \text{ m} \times 0.2 \text{ m}$. Three different taper ratios were tested $c = 0.25, 1$ or 4 . All beams are composed of a homogeneous linear elastic material with $E = 210 \text{ GPa}$, $\rho = 7850 \text{ kg/m}^3$ and $\nu = 0.3$. Shear deformations and rotational inertias are neglected.

Given the promising results of the prior convergence analysis concerning a simply supported beam, this benchmark will exclusively focus on comparing the frequencies' values achieved with fine discretization settings: 100 blocks and 100 contact pairs.

In order to make the results universal, the natural frequencies are non-dimensional with respect to the length L , the bending rigidity EI_h and the mass per unit length ρA_h at the thick end of the tapered beam so that:

$$\lambda_n = \omega_n \sqrt{\frac{\rho A_h L^4}{EI_h}} \quad (4.3.3.2)$$

where ω_n is the nth computed natural frequency in rad/s and λ_n is the corresponding nth non-dimensional natural frequency of the tapered beam [2].

In Tables 4.7, 4.8 and 4.9, the non-dimensional natural frequencies of all conducted tests are provided.

Taking into consideration the data from the three tables, the average relative error for all modes and boundary conditions, with a taper ratio $c = 0.25$, is 0.047%. For $c = 1$ and 4 , the relative errors are 0.048% and 0.068% respectively. This suggests that HybriDFEM provides more accurate approximations for smaller taper ratios, which correspond to a beam closer to a straight configuration. This aligns intuitively with the notion that modelling a straight beam is inherently simpler.

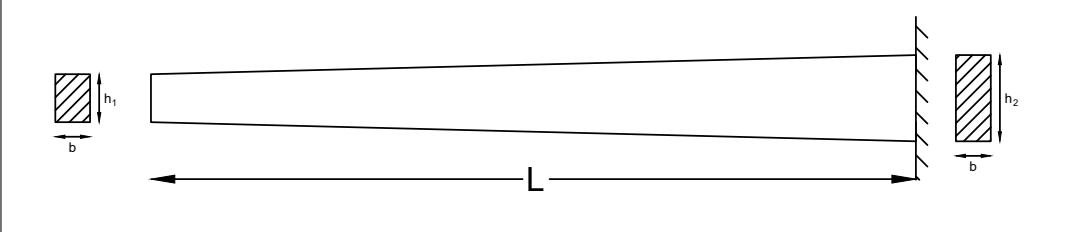
Taking into consideration the data from the three tables, the average relative error for all modes and taper ratios in a cantilever beam is 0.017%. For a simply supported and clamped beam, the average relative error is 0.043% and 0.10% respectively. This indicates that the convergence rate of HybriDFEM to the analytical solution is also influenced by the boundary conditions. Specifically, a simply supported beam exhibits faster convergence compared to a clamped beam.

This observation aligns well with previous findings, as discussed in Section 4.3.1, where a classical Euler-Bernoulli straight beam exhibited slower convergence to the analytical solution for a clamped configuration compared to a simply supported one.

Additionally, it's worth noting that smaller frequency values generally indicate smaller relative errors. This is potentially influenced by the higher precision of reported decimal numbers in the paper, which allow for a more accurate comparison. While this nuance shouldn't significantly impact the results, it's still worth it to be kept in mind.

Finally, considering the values from the three tables, the average relative error for all boundary conditions and taper ratios, for the first mode is 0.013%. The relative errors for the second, third, fourth and fifth mode are 0.028%, 0.048%, 0.073% and 0.109% respectively. Similarly to the previous benchmark (see Figure 4.18) higher modes give bigger relative errors.

Overall, HybriDFEM approximations for a fine block and CP discretization closely align with the analytical solutions, so they can be deemed reliable.



Mode	Non-dimensional natural frequencies λ_n					
	$c = 0.25$		$c = 1$		$c = 4$	
	Analytical	HBFEM	Analytical	HBFEM	Analytical	HBFEM
1	3.60827	3.60785	3.82379	3.82338	4.29249	4.29186
2	20.6210	20.6189	18.3173	18.3144	15.7427	15.7388
3	56.1923	56.1904	47.2648	47.2590	36.8846	36.8725
4	109.318	109.3317	90.4505	90.4496	68.1164	68.0948
5	180.163	180.228	148.002	148.030	109.594	109.570

Table 4.7: Non-dimensional natural frequencies of a tapered cantilever beam (C-F) for various taper ratios c . Comparison between analytical and HybriDFEM solution (100 blocks and CPs).

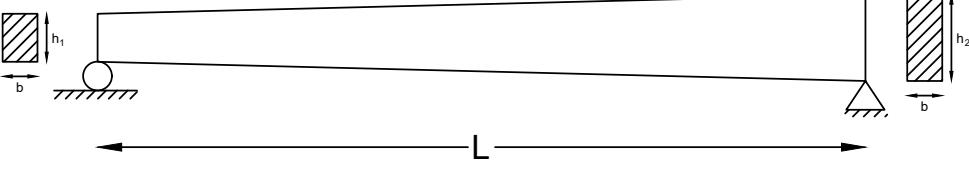
						
Mode	Non-dimensional natural frequencies λ_n					
	$c = 0.25$		$c = 1$		$c = 4$	
	Analytical	HBFEM	Analytical	HBFEM	Analytical	HBFEM
1	8.84619	8.84610	7.12153	7.12149	4.91976	4.91970
2	35.4452	35.4494	28.9518	28.9553	21.3445	21.3472
3	79.7303	79.7566	64.9788	65.0007	47.4820	47.4997
4	141.721	141.809	115.351	115.424	83.8216	83.8816
5	221.420	221.6428	180.089	180.275	130.436	130.587

Table 4.8: Non-dimensional natural frequencies of a tapered simply supported beam (S-R) for various taper ratios c . Comparison between analytical and HybridFEM solution (100 blocks and CPs).

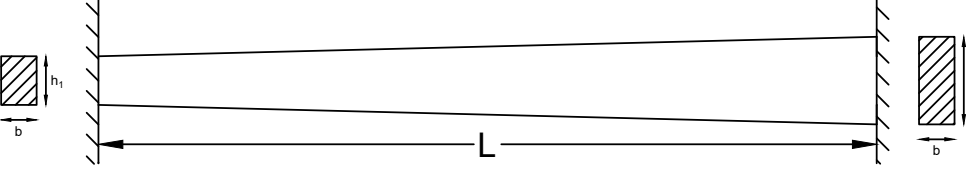
						
Mode	Non-dimensional natural frequencies λ_n					
	$c = 0.25$		$c = 1$		$c = 4$	
	Analytical	HBFEM	Analytical	HBFEM	Analytical	HBFEM
1	20.0782	20.0815	16.3356	16.3389	11.8417	11.8473
2	55.3399	55.3631	44.9806	45.0018	32.4755	32.5016
3	108.483	108.567	88.1382	88.2103	63.5118	63.5896
4	179.324	179.534	145.665	145.847	104.867	105.050
5	267.875	268.325	217.572	217.958	156.554	156.924

Table 4.9: Non-dimensional natural frequencies of a tapered clamped beam (C-C) for various taper ratios c . Comparison between analytical and HybridFEM solution (100 blocks and CPs).

4.3.4 Similar DEM Model

Given HybriDFEM's demonstrated effectiveness against classical beam theories, it becomes pertinent and interesting to evaluate its performance against alternative formulations. A study by A.C. Neves in 2015 [33, 34, 35] employed a similar discrete model to analyse the natural frequencies and mode shapes of simply supported and cantilever beams. Additionally, the study delved into the realm of cracked beams, a topic not covered in this paper but worthy of future investigation.

The model represents a beam element as a discrete system composed of rigid blocks. Each block possesses two degrees of freedom: one for transverse translation and one for rotation. Displacements and moment of inertia for each block are computed relative to their center of mass. The interaction between adjacent blocks is simulated by rotational and transverse springs, allowing taking into account bending and shear distortion effects. The beam's ends consist of half blocks, with the supports positioned at the edges. For a visual depiction of this model, refer to Figure 4.20. It's important to note that for a simply supported beam, the length is defined as the distance between two supports, while, for a cantilever beam, the length is measured from the support to the center of mass of the furthest block.

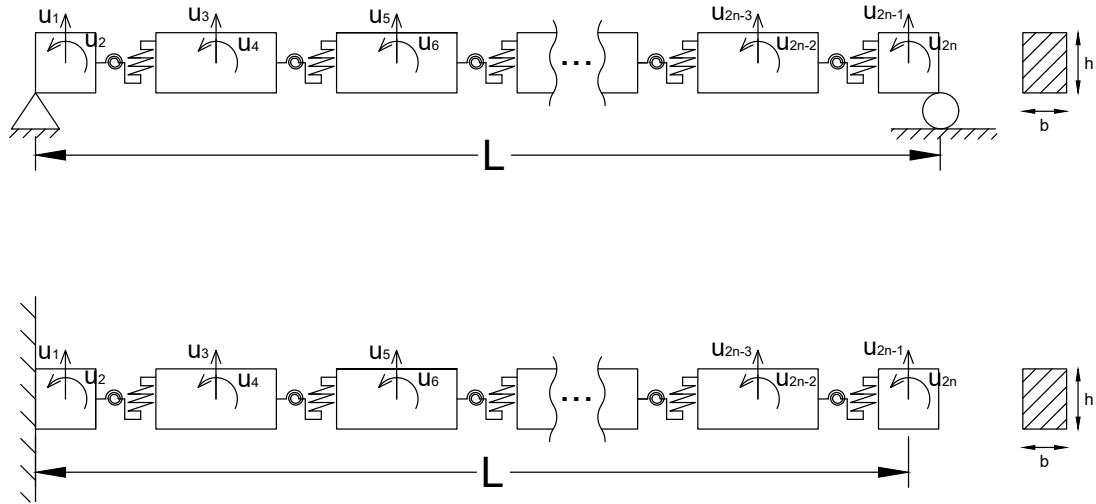


Figure 4.20: Simply supported and cantilever beam from [34].

In a typical HybriDFEM beam element, there are three degrees of freedom: two for translation and one for rotation. The interaction between adjacent blocks is modelled by axial and shear springs, which are discretized along the block's face. This setup enables the consideration of axial, bending, and shear distortion effects. Additionally, the displacements and rotational inertia of the half blocks at the

extremities are computed with respect to the block's edge, rather than the center of mass.

For a simpler formulation, the tests from Neves neglect shear distortion. Consequently, the interaction between blocks is achieved solely through rotational springs, with the stiffness of these springs determined as:

$$k_n = \frac{EI}{l} \quad (4.3.4.1)$$

with EI being the flexural stiffness of a block and l its length. The rotational stiffness of HybriDFEM's contact faces is computed using axial springs, and for a very precise CP discretization, is equal to:

$$k_n = \lim_{n \rightarrow \infty} \frac{2Eb}{l} \sum_{i=1}^{n/2} \left(\frac{h}{2n} (2i-1)^2 \right) = \frac{EI}{l} \quad (4.3.4.2)$$

with n the number of springs, E being Young's modulus of the material, I , b , h and L the inertia, the depth, the height and length of a block.

Testing for a homogeneous cantilever beam

The numerical example consists of a cantilever beam of length $L = 8$ m and rectangular cross-section of dimension $h \times b = 0.22$ m $\times$ 0.06 m. It is composed of a homogeneous linear elastic material with $E = 210$ GPa and $\rho = 7800$ kg/m³. Shear deformations are neglected, while the rotational inertias are not.

Neves' model is compared with two HybriDFEM beams, a standard HybriDFEM beam and an adapted version, where the degrees of freedom of the half blocks were shifted to the center of mass to accurately reflect Neves' formulation.

The benchmark consists of comparing such beam with different levels of block discretization. For HybriDFEM, the contact pair discretization is fixed at 100 CPs, to obtain a close approximation of k_n .

Following a similar approach to Neves' paper, the results are compared to the Euler-Bernoulli beam theory. As discussed in Section 4.3.1, it's important to remember that this theory neglects rotational inertia effects, which means that differences will arise. However, given that the ratio of the beam's height to its length (h/L) is relatively small, especially for the first modes, the results should be comparable. For higher modes, the error could be more significant.

The results are visible in Figure 4.21. The abscissa reports the block discretization while the ordinate the percentage error, corresponding to the difference between the numerical and analytical frequency, divided by the latter.

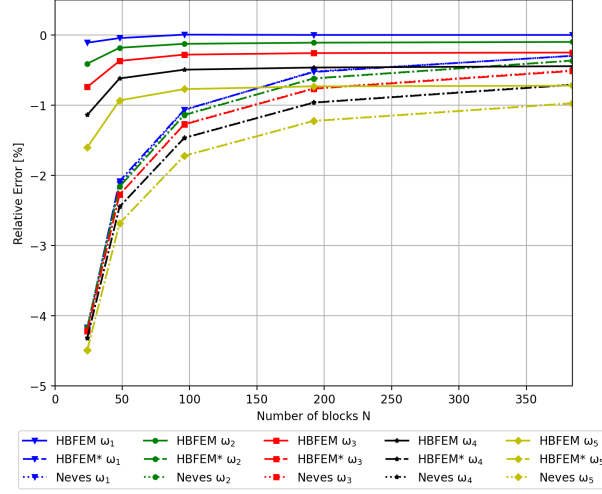


Figure 4.21: Comparison between Neves, HybriDFEM Beam, adapted HybriDFEM Beam* models for different block discretizations.

The adapted HybriDFEM beam and Neves' model give exactly the same results, as they are based on the same hypothesis. The error of the finest discretization, 384 blocks, reaches -0.294% for the first frequency and -0.972% for the fifth. The error for the coarser discretization, composed of 24 blocks, reaches -4.19% for the first mode and -4.49% for the fifth. Despite employing a very fine discretization, the results for the first mode did not fully converge to the analytical solution, but the error can be considered acceptable. As it could be expected, as the rotational inertia is taken into account, higher modes don't converge exactly to the Euler-Bernoulli analytical solution.

What is interesting is that HybriDFEM beam element converges to the analytical solution much faster. The error of the finest discretization, 384 blocks, reaches 0.003% for the first frequency while -0.720% for the fifth. The first mode is extremely close to Euler-Bernoulli solution, while the fifth, still gives closer result than Neves' one.

Overall, it appears that all the models tend to converge to the same results, despite requiring different block discretizations. The difference in convergence rate is caused by how Neves' model was defined. Neves' slightly overestimates the mass because the beam length is measured from the clamped support to the center of mass of the furthest block. This additional mass results in a slight decrease in frequency. As the discretization becomes more precise, this additional mass diminishes, eventually converging to the correct result. This is further confirmed by the fact that HybriDFEM and Neves' formulation give the same results for a

simply supported beam (see Appendix A).

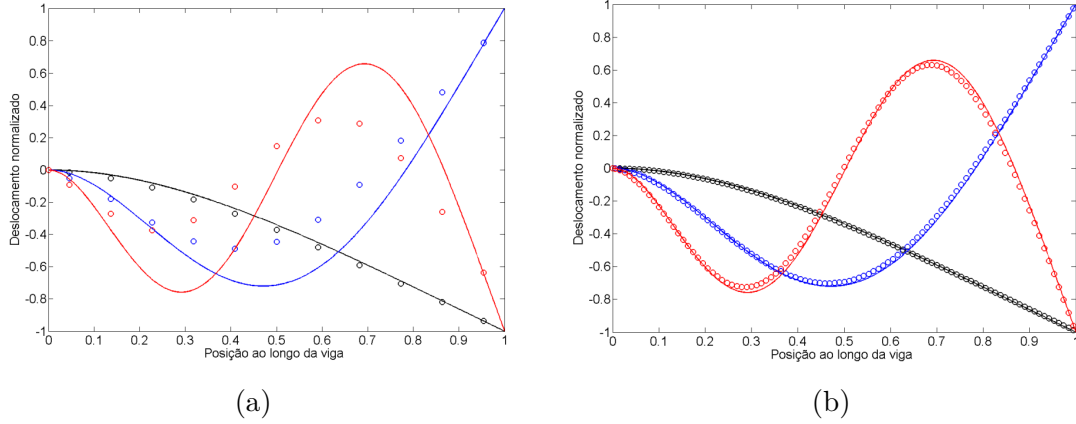


Figure 4.22: Comparison between NEVES formulation and Euler Bernoulli first 3 mode shapes. A coarse (a) and fine (b) block discretization were used.

- Numerical first mode — Euler-Bernoulli first mode
- Numerical second mode — Euler-Bernoulli second mode
- Numerical third mode — Euler-Bernoulli third mode

In Section 4.3.1, HybriDFEM was compared with the Euler-Bernoulli analytical mode shape and gave very good results, even for coarse discretizations. On the other hand, reported in Figure 4.22, Neves' formulation yields notably different results. Even with a fine discretization (96 blocks), a small discrepancy can be observed with Neves' model, while HybriDFEM accurately replicates the expected shape. This discrepancy is likely caused by the same previously mentioned reason, wherein Neves considers a beam that is slightly longer than the analytical solution.

Overall, the two formulations give similar results, even if needing different block discretizations. Neves' errors could be easily rectified by adjusting the length of the beam to align more closely with how the simply supported beam is defined.

It's important to note that both models consider rotational inertia, whereas Euler-Bernoulli theory neglects it. Consequently, differences, especially for higher modes, are expected. However, these differences are somewhat mitigated by selecting a very long and thin beam. In hindsight, choosing a different theory for comparison might have provided a more comprehensive assessment of both formulations' correctness.

4.3.5 Section Summary and Reflections

In summary, HybriDFEM proves to be a highly proficient tool for computing transverse natural frequencies and mode shapes for beam elements.

Through rigorous testing and comparison with prominent beam theories such as Euler-Bernoulli and Timoshenko-Ehrenfest, HybriDFEM consistently demonstrated its capability to effectively approximate analytical frequencies and mode shapes. This proficiency was particularly evident with a fine block and CP discretization. Notably, refining both discretizations was crucial for accurate natural frequencies, while mode shapes were significantly impacted only by block discretization. HybriDFEM's adaptability shines through its ability to incorporate or neglect elements like rotational inertia and shear deformations depending on the model requirements.

Moreover, HybriDFEM's precision was also validated through testing with tapered beams, commonly encountered in civil engineering applications. Across various taper ratios and boundary conditions, while neglecting rotational inertia and shear deformation, HybriDFEM exhibited remarkable accuracy in approximating analytical solutions.

Comparative analysis with another DEM formulation revealed HybriDFEM's efficiency, achieving comparable results with fewer blocks, thus offering a faster modelling solution.

In conclusion, HybriDFEM emerges as an exceptionally powerful tool for modelling transverse modes in beams. Its extensive validation across simple and complex scenarios establishes a solid foundation for its reliability. While the model's capabilities have been comprehensively demonstrated, future investigations could delve into more intricate scenarios such as non-linear tapered beams or composite beams.

4.4 Arch-Frame

While beams and columns are commonly encountered in structural engineering, they represent just a fraction of the multitude of structures in the field. This section will focus on assessing HybriDFEM's performance when modelling continuous arch structures.

The following benchmark is based on the paper by B. Bozyigit et al. [9], which explores the natural frequencies of an arch using the Dynamic Stiffness Method. The paper investigates the free vibration characteristics of arch-frames, which consist of two columns and a semicircular arch. The columns are modelled as Timoshenko beams, while the arch is modelled as an exact semicircular curved beam element. This element accounts for axial and shear deformations, as well as rotational inertia.

Testing for a homogeneous arch-Frame structure

The numerical example consists of an arch-frame structure (visible in Figure 4.23). The columns are of height $H_c = 4$ m with a rectangular cross-section of dimension $b_a \times h_c = 0.5$ m $\times$ 1 m. The semicircular arch is of height $H_a = 4$ m, has a radius of curvature $R_a = 4$ m with a rectangular cross-section of dimension $b_a \times h_a = 0.5$ m $\times$ 1 m and an opening angle $\alpha = 180^\circ$. The whole structure is composed of a homogeneous linear elastic material with $E = 30$ GPa, $\rho = 2500$ kg/m³ and $\nu = 0.3$. Shear deformations and rotational inertias are considered in the analysis.

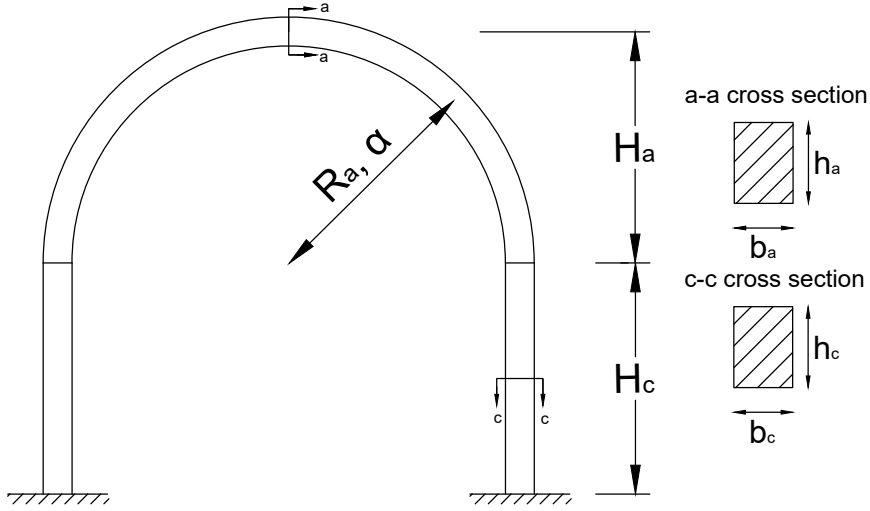


Figure 4.23: Benchmark arch-frame structure.

The benchmark consists of computing frequencies and mode shapes for such structure using HybriDFEM with different levels of discretization, both at the block and CP level.

The first five natural frequencies obtained with HybriDFEM are compared with the analytical solution from the paper. This comparison is visible in Figure 4.24.

For Figure 4.24a, computed with 100 CPs, the error of the finest discretization, composed of 400 blocks, reaches 1.14% for the first frequency, while 1.36% for the fifth. The error for the coarser discretization, composed of 25 blocks, reaches 1.43% for the first mode, while 2.24% for the fifth. Overall, the frequencies converge to an error of 1.14%.

For Figure 4.24b, computed with 100 blocks, the error of the finest discretization, composed of 200 CPs, reaches 1.15% for the first frequency, while 1.39% for the fifth. The error for the coarser discretization, composed of 25 blocks, reaches -0.86% for the first mode, and -0.42% for the fifth.

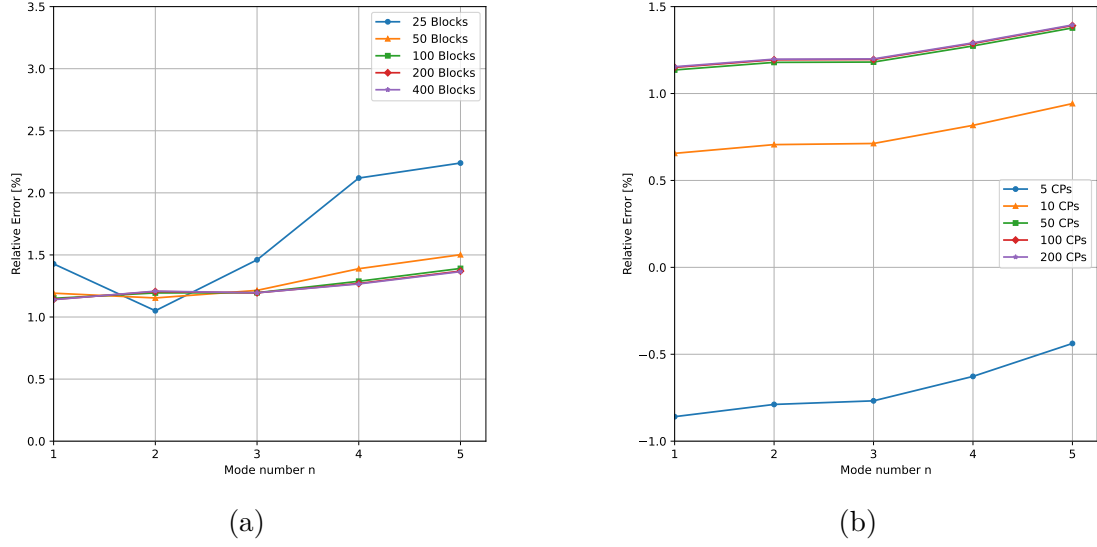


Figure 4.24: Influence of the block (a) and CP (b) discretization on the first 5 frequencies. On (a) the CP number is fixed at 100, while on (b) the number of blocks is fixed at 100.

HybriDFEM appears to overestimate the natural frequency of the arch structure. This discrepancy could be partly attributed to the slight difference in the shear correction factor k , which is $5/6$ in the analytical solution compared to $13/15$ in HybriDFEM. Another cause could be the use of trapezoidal blocks in HybriDFEM, which only approximate the roundness of an arch (visible in Figure 4.26a). However, the most likely cause is that in HybriDFEM, the columns are not perfectly comparable to true Timoshenko beams. While Timoshenko beams consider only transverse modes and neglect longitudinal ones, HybriDFEM accounts for both, potentially resulting in higher natural frequencies. In the case of simple beams, as discussed in Section 4.3.2, the two types of modes were independent of each other, allowing for easy identification and exclusion of the axial modes. However, in the case of more complex structures such as an arch, this distinction becomes less straightforward. Essentially, HybriDFEM also considers the axial stiffness of the columns, leading to an increase in the overall stiffness of the structure and consequently higher natural frequencies.

The first five mode shapes obtained with HybriDFEM, for both a coarse (25 blocks) and fine (200 blocks) block discretization, are shown in Figure 4.26. The amount of contact CPs was kept at 100 because, as noted in other sections, CPs do not influence the mode shapes. It is worth noting that mode 4 is similar for both discretizations, differing only by a scale factor of -1.

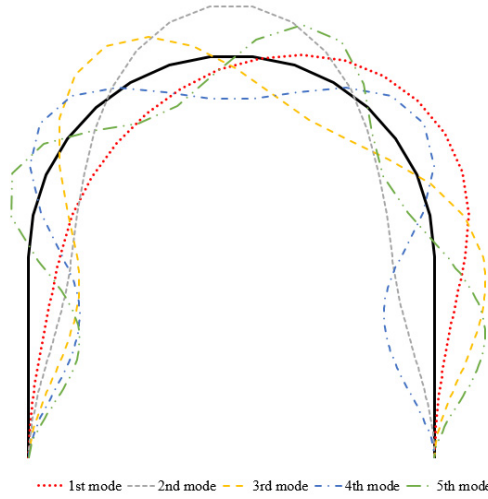


Figure 4.25: Mode shapes computed in B. Bozyigit et al. [9].

The mode shapes depicted in the paper (Figure 4.25) and those from HybriDFEM (Figure 4.26) appear remarkably similar. Unfortunately, the paper lacks sufficient information about the mode shapes for a thorough comparison. Nevertheless, given the close resemblance of the shapes and the relatively accurate frequency computation by HybriDFEM (with an error of approximately 1%), HybriDFEM can be considered quite accurate in computing the frequencies of arch-like structures.

4.4.1 Section Summary and Reflections

Only one benchmark was tested for arch-like structures. The natural frequencies computed with HybriDFEM are quite close to the analytical solution from Bozyigit's paper, with an error of approximately 1%. The overestimation of the frequencies is likely attributed to the different column formulation of the two models; HybriDFEM considers axial deformations, whereas Timoshenko beams do not.

It is noteworthy that HybriDFEM is expected to be extended in the future to combine HybriDFEM and FEM elements. This will enable modelling the columns as true Timoshenko beams, allowing for an accurate assessment of whether the upper-mentioned justification for the differences is valid.

Overall, a 1% error is a reasonable approximation, demonstrating that HybriDFEM performs well in computing the natural frequencies and mode shapes of arch-frames. However, only one benchmark was tested, which is an insufficient sample size to thoroughly assess the method's performance. More benchmarks should be tested in the future to ensure a comprehensive evaluation.

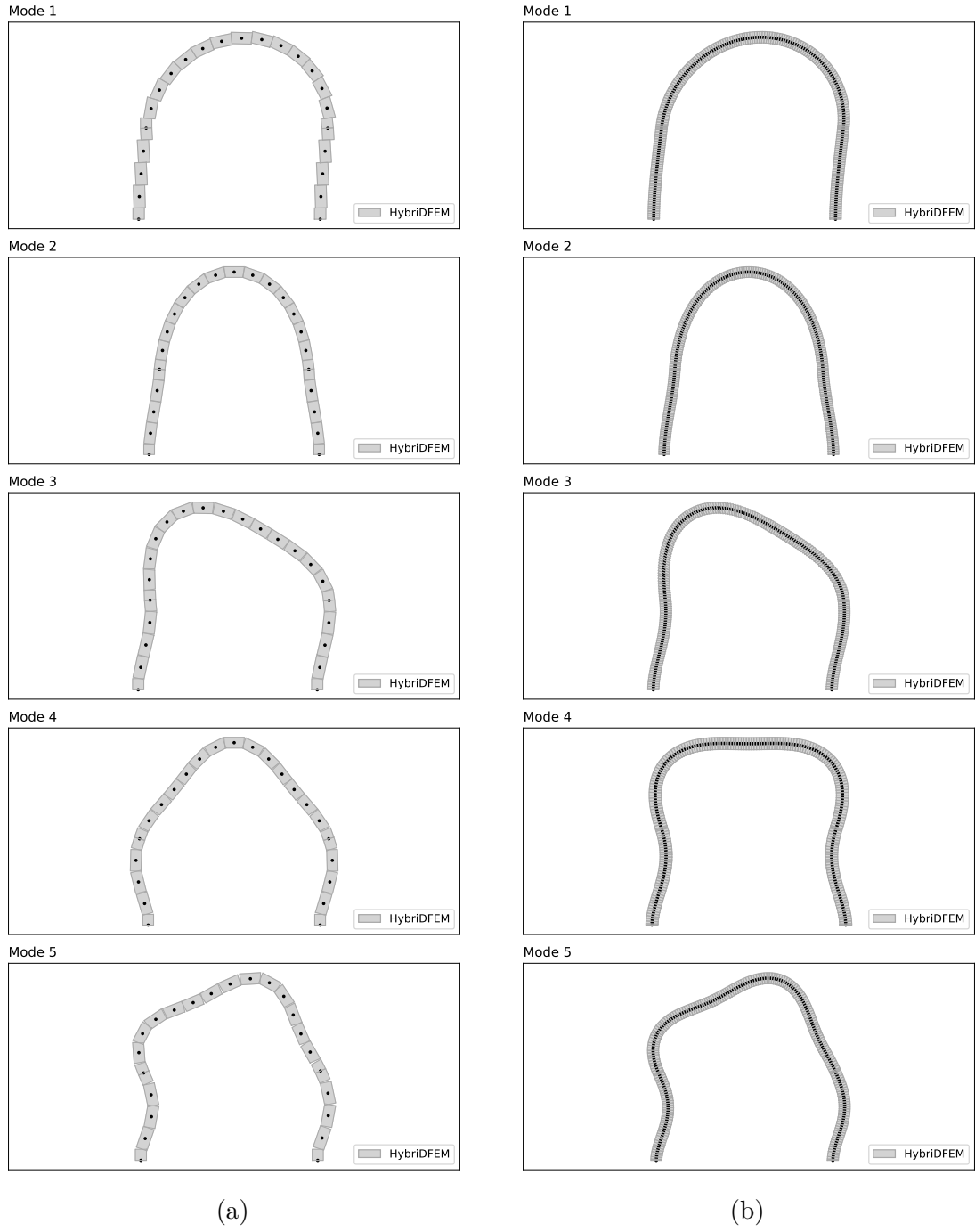


Figure 4.26: First 5 mode shapes of the structures computed with HybriDFEM. A coarse (a) and fine (b) block discretization were used, with 100 CP for both cases.

4.5 Masonry Wall

So far, all tests have been conducted on homogeneous linear materials. However, HybriDFEM is based on a discrete formulation, conceived with the goal of modelling discrete structures such as masonry. Therefore, it is essential to evaluate the performance of HybriDFEM specifically in these scenarios.

Three benchmarks were tested. The first two benchmarks are based on the work of D. Baraldi et al. [3], analysing a simple wall composed of four bricks. The first benchmark compares HybriDFEM's results with a simplified analytical model, while the second compares it with the discrete element method used in the paper. The third benchmark, based on the work of S. Brasile et al. [10], is more complex, modelling a large wall composed of 1960 bricks.

4.5.1 Simple Wall: Analytical Model

Frequently, when testing models, it is beneficial to use simple analytical solutions to confirm that the formulation behaves as intended. To this end, a simplified version of the wall from D. Baraldi et al. [3] was modelled.

The numerical example consists of a small masonry wall (Figure 4.28) of dimensions $B = 0.25$ m and $H = 0.185$ m. The wall is made by UNI bricks ($b = 250$ mm, $s = 120$ mm, $a = 55$ mm [45]) connected by a standard mortar joint of thickness $e = 10$ mm. The bricks are rigid blocks with density $\rho = 1800$ kg/m³. The mortar joint is composed of a homogeneous linear elastic material with $E = 1$ GPa, $\rho = 1800$ kg/m³ and $\nu = 0.2$.

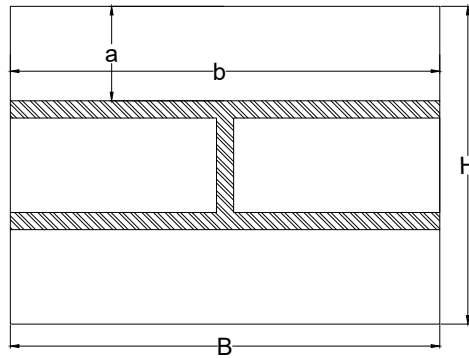


Figure 4.27: Benchmarked wall.

Shear deformations and rotational inertias are considered in the analysis. The wall is modelled without displacement restraints, but the blocks cannot rotate. This choice was made because their influence can be challenging to model accurately

analytically. It is important to note that the purpose of this simplified example is not to develop a new analytical formulation, but rather to partially validate the accuracy of HybriDFEM.

Analytical model

The wall can be represented as a system of four masses connected by shear and normal springs (see Figure 4.28).

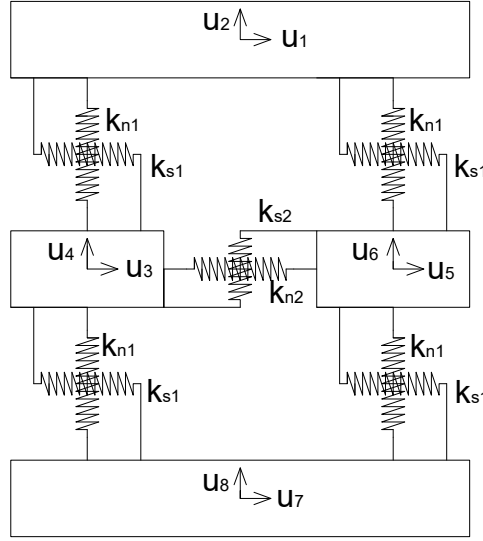


Figure 4.28: Mass-spring model of the small masonry wall.

The stiffness matrix is defined as:

$$K = \begin{bmatrix} 2k_{s1} & 0 & -k_{s1} & 0 & -k_{s1} & 0 & 0 & 0 \\ 0 & 2k_{n1} & 0 & -k_{n1} & 0 & -k_{n1} & 0 & 0 \\ -k_{s1} & 0 & 2k_{s1} + k_{n2} & 0 & -k_{n2} & 0 & -k_{s1} & 0 \\ 0 & -k_{n1} & 0 & 2k_{n1} + k_{s2} & 0 & -k_{s2} & 0 & -k_{n1} \\ -k_{s1} & 0 & -k_{n2} & 0 & 2k_{s1} + k_{n2} & 0 & -k_{s1} & 0 \\ 0 & -k_{n1} & 0 & -k_{s2} & 0 & 2k_{n1} + k_{s2} & 0 & -k_{n1} \\ 0 & 0 & -k_{s1} & 0 & -k_{s1} & 0 & 2k_{s1} & 0 \\ 0 & 0 & 0 & -k_{n1} & 0 & -k_{n1} & 0 & 2k_{n1} \end{bmatrix} \quad (4.5.1.1)$$

Where the different spring stiffnesses are defined as:

$$k_{s1} = \frac{GA_b}{t_m}, \quad k_{s2} = \frac{GA_h}{t_m}, \quad k_{n1} = \frac{KA_b}{t_m}, \quad k_{n2} = \frac{KA_h}{t_m}, \quad (4.5.1.2)$$

With t_m the mortar thickness, A the contact surface between the masses, G the shear modulus and K the bulk modulus. The mass matrix is defined as:

$$M = \begin{bmatrix} 2m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2m \end{bmatrix} \quad (4.5.1.3)$$

By solving the eigenvalue problem from Equation 3.2.9 the natural frequencies are obtained.

$$\begin{aligned} \omega_1 &= 0 \text{ rad/s}, & \omega_2 &= 0 \text{ rad/s}, \\ \omega_3 &= \sqrt{\frac{k_{s1}}{m}} = 8664.6 \text{ rad/s}, & \omega_4 &= \sqrt{\frac{k_{n1}}{m}} = 13423.1 \text{ rad/s}, \\ \omega_5 &= \sqrt{\frac{3k_{s1}}{m}} = 15007.5 \text{ rad/s}, & \omega_6 &= \sqrt{\frac{2(k_{s1} + k_{n2})}{m}} = 15460.9 \text{ rad/s}, \\ \omega_7 &= \sqrt{\frac{2(k_{n1} + k_{s2})}{m}} = 19934.8 \text{ rad/s}, & \omega_8 &= \sqrt{\frac{3k_{n1}}{m}} = 23249.5 \text{ rad/s}. \end{aligned} \quad (4.5.1.4)$$

Comparison with HybriDFEM

The benchmark involves computing the frequencies and mode shapes for this structure using HybriDFEM and comparing the results with those of the presented analytical model. The wall consists of four blocks, with the number of CPs fixed at 10. The CP discretization does not influence the results, as the rotational degrees of freedom are fixed.

Table 4.10 presents the computed values of all eight natural frequencies using both HybriDFEM and the analytical solution. The values are practically identical, with a difference of 10^{-12} , which is considered negligible.

Figure 4.29 shows the corresponding mode shapes computed with HybriDFEM. The first two modes are characterized by rigid translation, and their corresponding frequencies are considered negligible relative to the other modes. Modes 3 and 5 represent pure shear modes, with the two middle blocks remaining attached. Modes 4 and 8 correspond to pure axial modes, also with the two middle blocks remaining attached. Finally, modes 6 and 7 are influenced by both shear and axial stresses.

	Modes							
	1	2	3	4	5	6	7	8
Analytical	0	0	8664.6	13423.1	15007.5	15460.9	19934.8	23249.5
HybriDFEM	0	0	8664.6	13423.1	15007.5	15460.9	19934.8	23249.5

Table 4.10: Natural frequencies in rad/s computed with the analytical model and HybriDFEM.

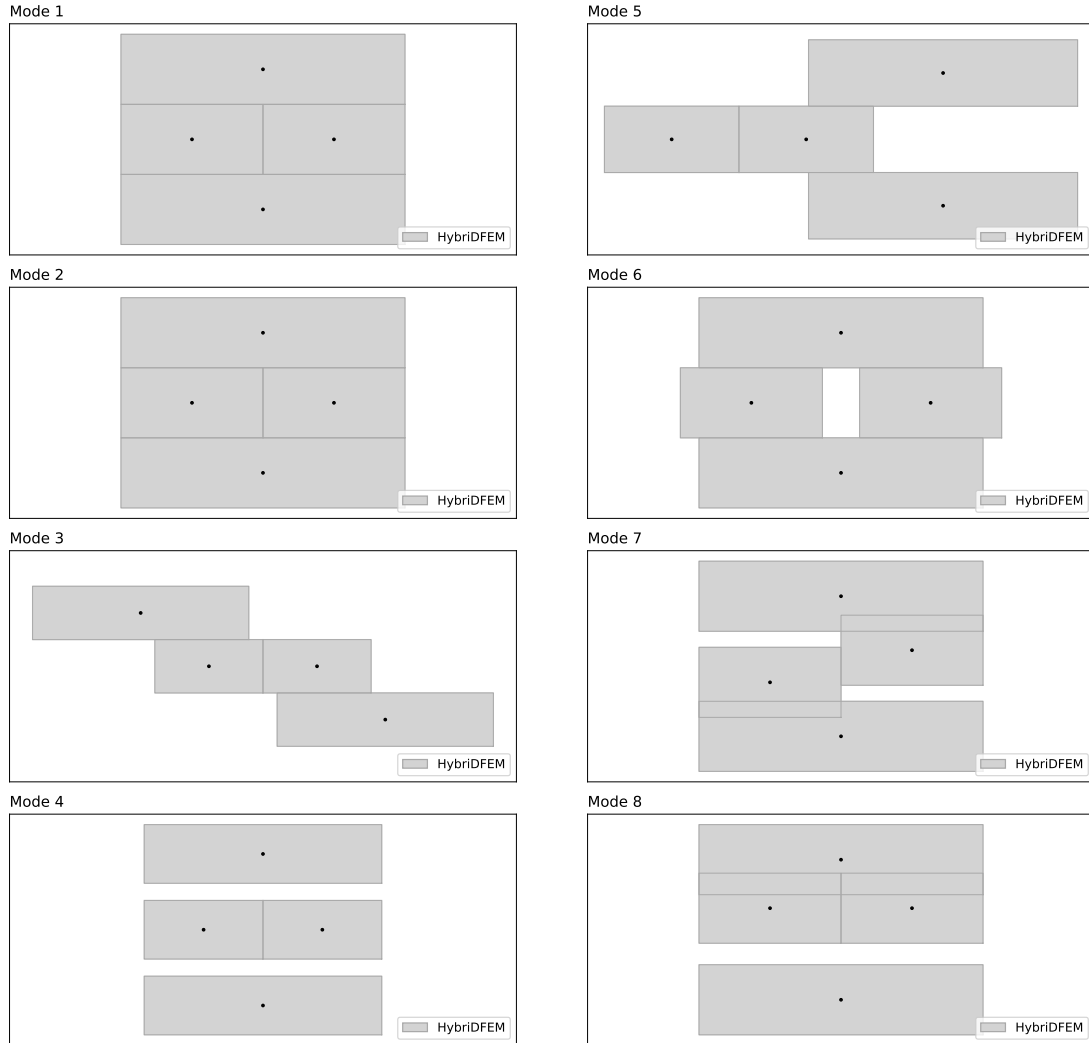


Figure 4.29: All mode shapes of a simple masonry wall where the rotations are blocked, computed with HybriDFEM.

4.5.2 Simple Wall: DEM Formulation

The benchmark is based on the paper from D. Baraldi et al. [3], where a small masonry wall, composed of four bricks, was analysed. The natural frequencies and mode shapes were computed using a specific discrete model precisely described in the same paper.

There are two main distinctions between HybriDFEM and the compared model. Firstly, at the contact face level, the model calculates axial, shear, and rotational stiffnesses pointwise at the center of the contact face, whereas HybriDFEM discretizes these different stiffnesses along the face. Secondly, in computing the block stiffness matrix, the model employs partial differentiation of the elastic energy, while HybriDFEM relies on constitutive relations [8].

Comparison with HybriDFEM

The numerical example is nearly equivalent to the one presented in the previous chapter (see Figure 4.28). The only difference is that in this case, the blocks are also allowed to rotate. The benchmark consists of computing frequencies and mode shapes for this structure using HybriDFEM. The wall is composed of four blocks, with the number of CPs fixed at 10.

Table 4.11 reports all twelve natural frequencies presented in D. Baraldi et al. [3] and computed with HybriDFEM. The frequencies from the paper were reordered using the mode shapes (from Figures 4.30 and 4.31) to correctly compare them with HybriDFEM.

The first three modes are characterized by rigid roto-translation, and their corresponding frequencies are negligible compared to the other eigenvalues. Modes 6 and 12 are pure axial modes, similar to those described in Section 4.5.1. Mode 8 is also very similar, being affected by both shear and axial deformations. Modes 5, 8, and 11 are similar to the shear modes from the previous example but now include some rotations. Modes 4, 9, and 10 are flexural modes.

When comparing the frequencies computed with HybriDFEM to those from the paper, significant differences arise. While the order of magnitude is somewhat close for certain modes, the values generally differ. This disparity is likely due to differences in the model formulations. Specifically, the discretization along the face in HybriDFEM contrasts with the pointwise approach adopted by Baraldi.

It is reassuring to note that despite the differences in formulation, the overall shapes of the modes align, even if they appear in different orders in the two formulations. However, an exception arises with the 11th mode: while HybriDFEM identifies it as a pure axial mode, Baraldi observes some rotations in the two half blocks. This

Mode	Baraldi et al.	HybriDFEM
1	0	0
2	0	0
3	0	0
4	4325	3471
5	4184	4795
6	5163	5630
7	4732	5887
8	5732	6080
9	2946	8042
10	9172	8834
11	3430	9586
12	8755	9751

Table 4.11: Natural frequencies in Hz from D. Baraldi et al. and computed with HybriDFEM.

observation highlights further differences between the two models: in HybriDFEM, the axial stiffness is independent of the rotational stiffness. As a result, axial modes always remain independent of rotations.

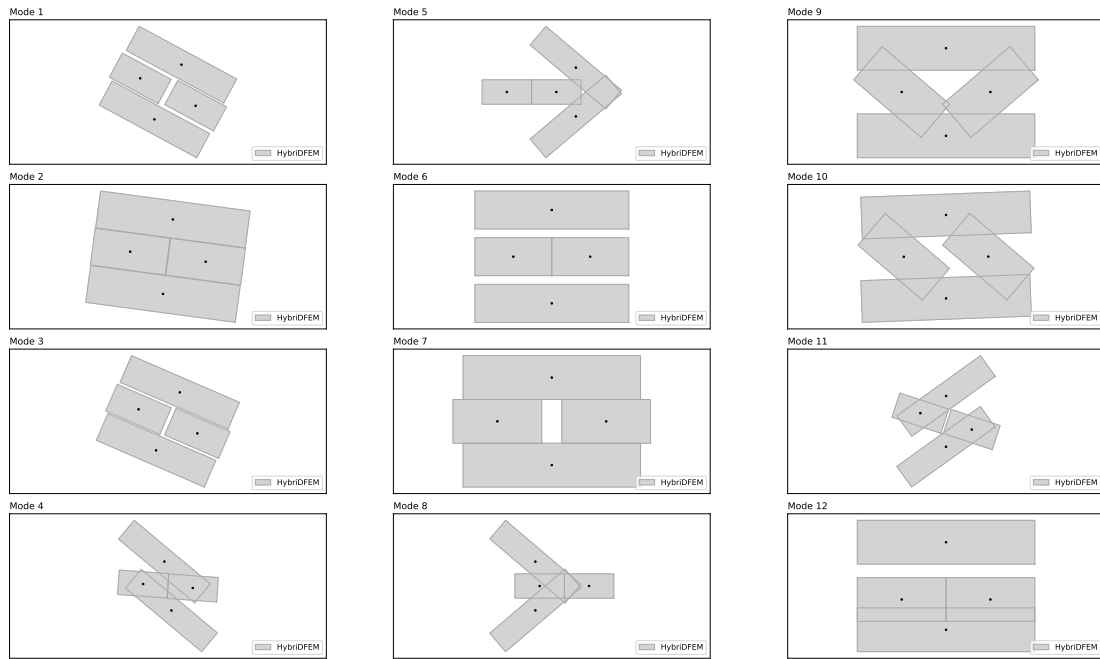


Figure 4.30: All mode shapes of a simple masonry wall, computed with HybriDFEM.

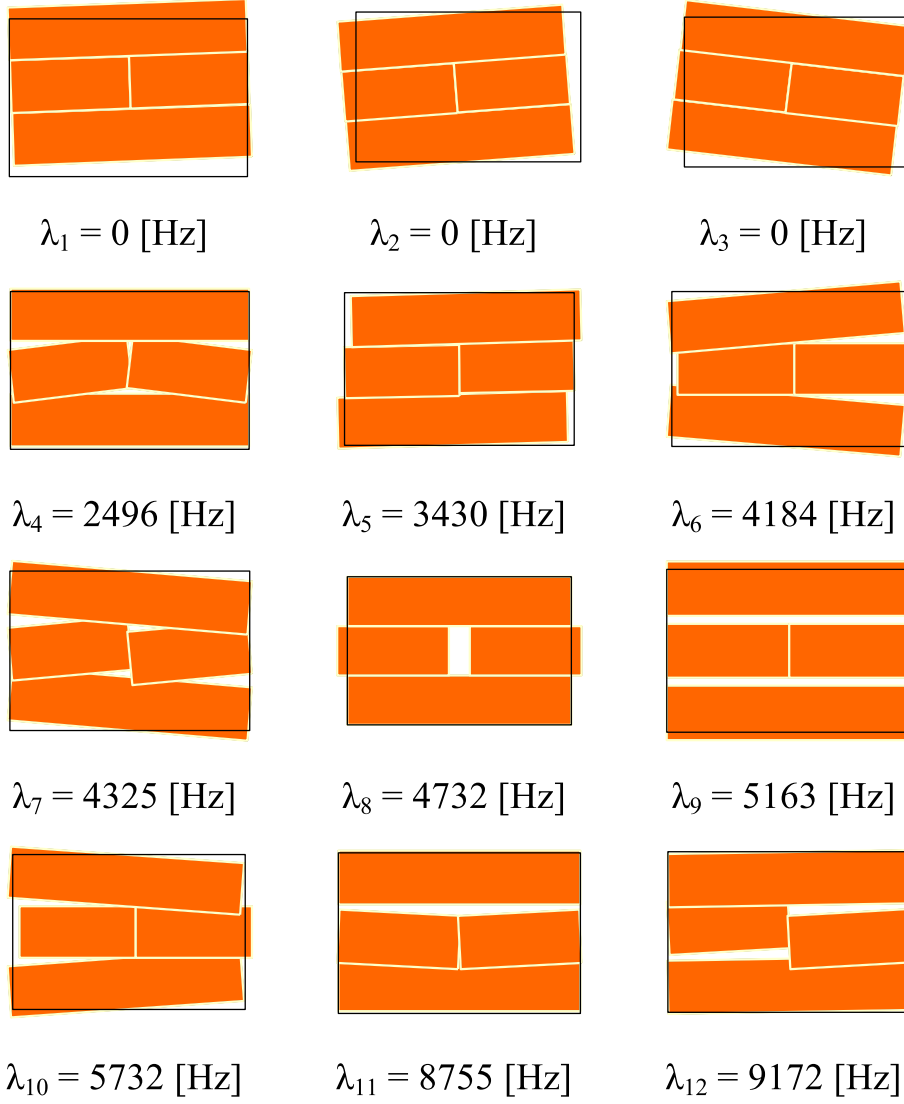


Figure 4.31: All mode shapes of a simple wall from D. Baraldi et al. [3].

4.5.3 Large Wall: DEM Formulation

The benchmark is based on the paper from S. Brasile et al. [10], where a large masonry wall composed of 1960 bricks was analyzed. The natural frequencies and mode shapes were computed using a specific discrete model described in detail in the same paper.

There are two main distinctions between HybriDFEM and the compared model. Firstly, at the contact face level, the model represents the mortar as a set of two axial and two shear springs in parallel, spaced at $\sqrt{3}l/3$, where l is the height of the block. HybriDFEM acts similarly if the number of CPs is fixed at two, but the distance between them is $l/2$. Secondly, in computing the block stiffness matrix, the model employs energy equivalence, while HybriDFEM relies on constitutive relations [8].

Testing for a large masonry Wall

The numerical example consists of a large masonry wall (Figure 4.32) with dimensions $B = 6.24$ m and $H = 5.2$ m. The wall is made of UNI bricks ($b = 250$ mm, $s = 120$ mm, $a = 55$ mm [45]) connected by a standard mortar joint of thickness $e = 10$ mm. The bricks are rigid blocks with density $\rho = 1800$ kg/m³. The mortar joint is composed of a homogeneous linear elastic material with $E = 200$ MPa, $G = 10$ MPa and $\rho = 1800$ kg/m³. Shear deformations and rotational inertias are considered in the analysis. The wall is modelled without restraints.

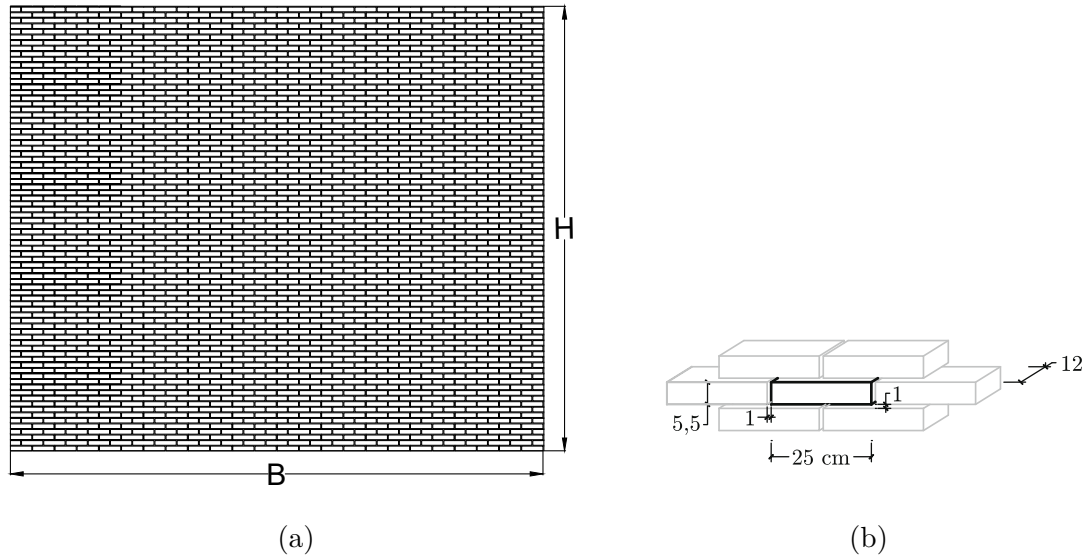


Figure 4.32: The benchmarked wall (a) and the brick size (b) [10].

The benchmark consists of computing frequencies and mode shapes for such structure using HybriDFEM. The wall is composed by 1960 blocks, with a number of CPs fixed at 2 to truly reflect what is done in the paper.

Table 4.12 reports the first 8 non-zero natural frequencies presented in S. Brasile et al. and computed with HybriDFEM. The respective mode shapes are reported in Figure 4.33. The computed frequencies highly resemble each other, with minor differences caused by the differences in the models.

Mode	S. Brasile et al.	HybriDFEM
1	0	0
2	0	0
3	0	0
4	147.84	148.44
5	208.05	204.25
6	235.32	237.53
7	327.25	321.42
8	371.79	373.93
9	479.63	459.32
10	483.32	487.09
11	556.03	562.4

Table 4.12: Natural frequencies in rad/s from S. Brasile et al. [10] and computed with HybriDFEM.

The first three modes are characterized by a rigid roto-translation, and their corresponding frequencies are negligible compared to the other eigenvalues. The respective mode shapes for these modes were not present in S. Brasile et al., so they are not reported here.

In Figure 4.33, three distinct types of modes are identified. Firstly, there are pure axial modes, like mode 11, where there is no lateral relative movement between different rows of bricks, but a compression of the bricks in the vertical direction can be observed. Secondly, shear modes, such as modes 5, 6, 9, and 10, are observed. In these modes, rows (or columns) of bricks slide with each other, creating a sinusoidal shape on two edges of the wall, and two straight sides on the other two. Lastly, flexural modes, like modes 4, 7, and 8 are characterized by trigonometric functions on all four edges of the wall.

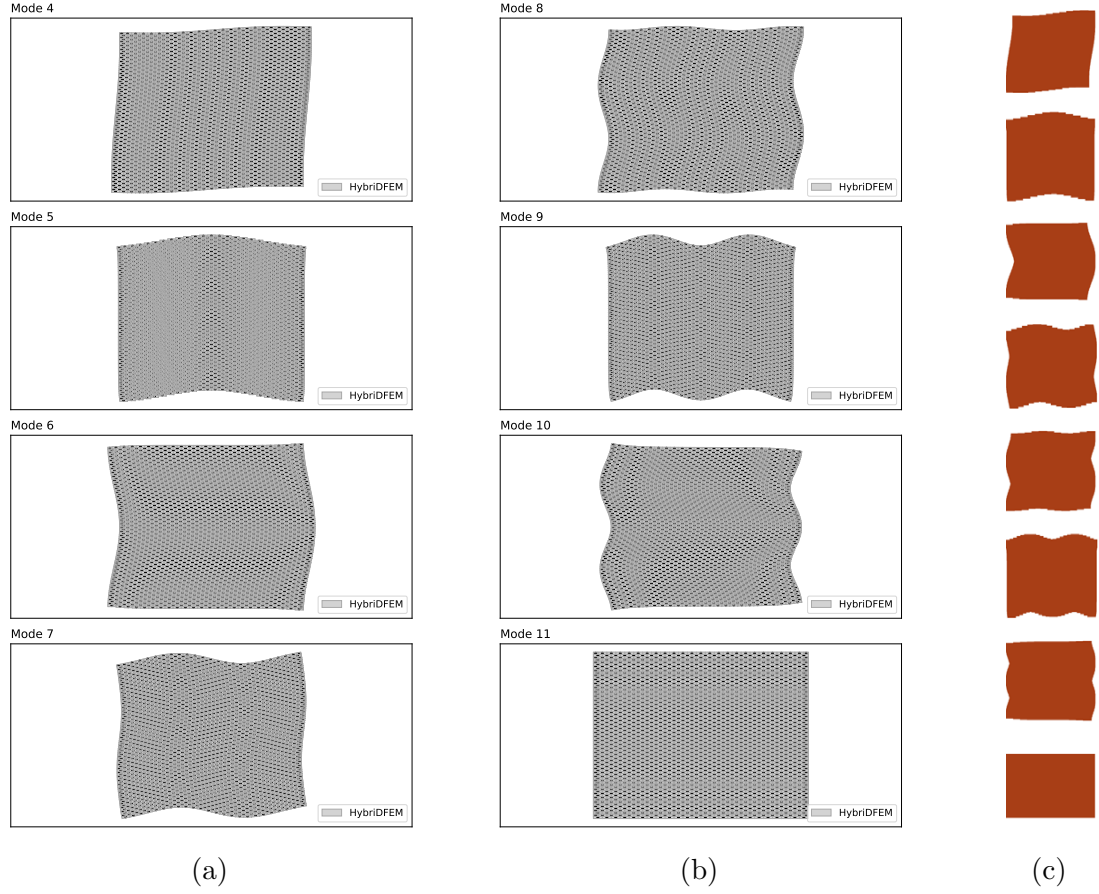


Figure 4.33: First 8 mode shapes computed with (a, b) HybriDFEM and (c) from S. Brasile et al. [10].

4.5.4 Section Summary and Reflections

Two of the three tested benchmarks show very good agreement between HybriDFEM and the results from the literature. The benchmark from D. Baraldi et al. presents the same mode shapes as HybriDFEM, but the order of the modes and the computed frequencies differ. This discrepancy is likely due to the differences between the formulations. S. Brasile et al. benchmark confirms this, as the model is more similar to HybriDFEM, and a good agreement is found, with only minor differences in the frequency values.

Overall, HybriDFEM effectively models masonry walls, even when dealing with numerous degrees of freedom. This capability makes it a powerful tool, particularly considering the limited number of papers in the literature that delve into the modal analysis of discrete two-dimensional structures.

Chapter 5

Dynamic Analysis: Formulation and Implementation

The main objective of this chapter is to integrate various time-stepping algorithms, both linear and non-linear, into the HybriDFEM method to conduct numerical dynamic analyses.

Initially, the focus will be on formulating the damping matrix $\mathbf{C}$, a crucial component that governs the energy dissipation within the system. Three distinct methods will be implemented to address this aspect. Following this, a succinct overview of classical modal superposition will be provided. This method enables the computation of analytical solutions for linear systems, offering valuable insights for testing the numerical algorithms.

Subsequently, a series of time-stepping algorithms will be introduced, accompanied by their respective pseudocode. While numerous algorithms are available in the literature, only a subset of the most widely recognized has been selected for inclusion in this extension of the HybriDFEM method.

5.1 Damping Matrix

Damping is a fundamental concept in dynamics, describing the mechanism through which energy is gradually dissipated within a system, resulting in the attenuation of vibrations or oscillations. Consider the example of a swinging pendulum: in the absence of damping, the pendulum would perpetually oscillate with the same amplitude. However, with damping, the oscillations gradually diminish in magnitude until the pendulum eventually comes to rest. Similarly, civil engineering

structures, when subjected to seismic activity or other dynamic forces, exhibit analogous behaviour.

The subsequent section draws heavily from the dynamics of structures course taught by J. Almeida [36] and the book "Dynamics of Structures: Theory and Applications to Earthquake Engineering" by Anil K. Chopra [14].

As it will be seen in Section 5.2, when employing classical modal superposition, the damping matrix doesn't need to be fully specified as the equation can be uncoupled, requiring only the damping ratio ζ . This ratio represents the rate at which motion decays during free vibration. However, in the context of time-stepping methods, both linear and non-linear, such as those used in the HybridFEM method, a fully defined damping matrix is necessary.

For typical civil engineering structures like buildings, bridges, dams, nuclear power plants, and offshore structures, the damping ratio is generally less than 10%, classifying them as underdamped structures. Damping lowers the natural frequency and extends the natural period. However, these effects are negligible for underdamped structures, typically characterized by damping ratios below 20% (see Figure 5.1).

Stress Level	Type and Condition of Structure	Damping Ratio (%)
Working stress, no more than about $\frac{1}{2}$ yield point	Welded steel, prestressed concrete, well-reinforced concrete (only slight cracking)	2–3
	Reinforced concrete with considerable cracking	3–5
	Bolted and/or riveted steel, wood structures with nailed or bolted joints	5–7
At or just below yield point	Welded steel, prestressed concrete (without complete loss in prestress)	5–7
	Prestressed concrete with no prestress left	7–10
	Reinforced concrete	7–10
	Bolted and/or riveted steel, wood structures with bolted joints	10–15
	Wood structures with nailed joints	15–20

Source: N. M. Newmark, and W. J. Hall, Earthquake Spectra and Design, Earthquake Engineering Research Institute, Berkeley, Calif., 1982.

Figure 5.1: Recommended damping values [14].

The damping matrix for practical structures should not be computed from the structural dimensions, structural member sizes, and the damping of the structural materials used. One might think that it should be possible to determine the damping matrix for the structure from the damping properties of individual structural elements, just as the structural stiffness matrix is determined. However, it is impractical to determine the damping matrix in this manner because unlike the elastic modulus, which enters into the computation of stiffness, the damping properties of materials are not well established. Even if these properties were known, the resulting damping matrix would not account for a significant part of the energy dissipated in friction at steel connections, opening and closing of microcracks in concrete, stressing of non-structural elements—partition walls, mechanical equipment, fireproofing, etc.—friction between the structure itself and non-structural elements, and similar mechanisms, some of which are even difficult to identify. Thus, the damping matrix for a structure should be determined from its modal damping ratios, which account for all energy-dissipating mechanisms.

Classical damping is an appropriate idealization if similar damping mechanisms are distributed throughout the structure (e.g., a multistory building with a similar structural system and structural materials over its height). Linear systems that satisfy the following identity are said to possess classical damping:

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C} \quad (5.1.1)$$

If, however, the equation is not satisfied, the system is said to have non-classical damping. This type of damping is frequently employed in structures exhibiting inconsistencies in damping distribution, such as dams or models involving soil-structure interaction. In this extension of the HybriDFEM method, only classical damping implementations were coded, so the subject of non-classical damping was not further explored. However, it could be suitable for future exploration and development.

In the case of classical damping, the damped frequencies remain identical to those of the corresponding undamped system. When computing the natural frequencies and mode shapes via the solution of the matrix eigenvalue problem (Equation 3.2.7), damping effects were neglected. By following the same procedures outlined in Section 3.2 while accounting for damping, it becomes possible to compute the damped natural frequencies.

When treating non-linear systems, damping could theoretically be neglected, as some sources of energy dissipation are inherently accounted for within the system itself. However, even advanced models often struggle to include the inherent damping of members, which is characterized by irreversible phenomena related to internal friction and microcracking. Therefore, it remains advantageous to consider

a damping component to model these "unmodeled" sources of energy dissipation. To this end, a classical modal matrix can be used.

Three different methods were implemented in HybriDFEM to compute a classical damping matrix: the Rayleigh damping, Caughey damping and Superposition of Modal Damping Matrices.

5.1.1 Rayleigh Damping

Rayleigh damping is the sum of the mass-proportional damping and the stiffness-proportional damping defined as:

$$\mathbf{C} = a_0 \mathbf{M} + a_1 \mathbf{K} \quad (5.1.1.1)$$

where the constants a_0 and a_1 have units of s^{-1} and s , respectively. The matrix $\mathbf{C}$ is diagonal by virtue of the modal orthogonality properties; therefore, it is a classical damping matrix.

The damping ratio for the k -th mode of such a system is:

$$\zeta_k = \frac{a_0}{2} \frac{1}{\omega_k} + \frac{a_1}{2} \omega_k \quad (5.1.1.2)$$

The coefficients a_0 and a_1 can be determined from specified damping ratios ζ_i and ζ_j for the i -th and j -th modes, respectively.

Expressing the damping ratios for these two modes in matrix form leads to:

$$\frac{1}{2} \begin{bmatrix} 1/\omega_i & \omega_i \\ 1/\omega_j & \omega_j \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \zeta_i \\ \zeta_j \end{bmatrix} \quad (5.1.1.3)$$

These two algebraic equations can be solved to determine the coefficients a_0 and a_1 .

If both modes are assumed to have the same damping ratio ζ , which is reasonable based on experimental data, then:

$$a_0 = \zeta \frac{2\omega_i \omega_j}{\omega_i + \omega_j} \text{ and } a_1 = \zeta \frac{2}{\omega_i + \omega_j} \quad (5.1.1.4)$$

Knowing the mass matrix $\mathbf{M}$, the stiffness matrix $\mathbf{K}$, and the two coefficients a_0 and a_1 the damping matrix can be computed using Equation 5.1.1.1. In applying this procedure to a practical problem, the modes i and j with specified damping ratios should be chosen to ensure reasonable values for the damping ratios in all the modes contributing significantly to the response. Consider, for example, that five modes are to be included in the response analysis and roughly the same damping ratio ζ is desired for all modes. This ζ should be specified for the first mode and possibly for the fourth mode.

5.1.2 Caughey Damping

To specify values for damping ratios in more than two modes, a more general form of classical damping matrix, known as Caughey damping, should be used.

$$\mathbf{C} = \mathbf{M} \sum_{l=0}^{N-1} a_l [\mathbf{M}^{-1}\mathbf{K}]^l \quad (5.1.2.1)$$

where N is the number of degrees of freedom in the system and a_l are constants.

The first three terms of the series are:

$$a_0 \mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^0 = a_1 \mathbf{M} \quad a_1 \mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^1 = a_1 \mathbf{K} \quad a_2 \mathbf{M}(\mathbf{M}^{-1}\mathbf{K})^2 = a_2 \mathbf{K}\mathbf{M}^{-1}\mathbf{K} \quad (5.1.2.2)$$

If J is the number of modes, where the damping ratio is specified, the Caughey damping matrix is defined as:

$$\mathbf{C} = \mathbf{M} \sum_{l=0}^{J-1} a_l [\mathbf{M}^{-1}\mathbf{K}]^l \quad (5.1.2.3)$$

and the modal damping ratio ζ_n is given by:

$$\zeta_n = \frac{1}{2} \sum_{l=0}^{J-1} a_l \omega_n^{2l-1} \quad (5.1.2.4)$$

The coefficients a_l can be determined from the damping ratios specified in any J modes. Note that if the number of mode selected J is equal to 2, Caughey damping is equivalent to Rayleigh damping.

Knowing the mass matrix $\mathbf{M}$, the stiffness matrix $\mathbf{K}$ and the J coefficients a_l the damping matrix can be computed using Equation 5.1.2.3.

5.1.3 Superposition of Modal Damping Matrices

An alternative procedure to determine a classical damping matrix from modal damping ratios can be derived starting with:

$$\boldsymbol{\Phi}^T \mathbf{C} \boldsymbol{\Phi} = \mathbf{C}_g \quad (5.1.3.1)$$

where $\mathbf{C}_g$ is a diagonal matrix with the i -th diagonal element equal to the generalized modal damping:

$$C_i = \zeta_i (2M_i \omega_i) \quad (5.1.3.2)$$

with the damping ratio ζ_i and M_i the i -th diagonal element of the square matrix from Equation 3.2.13.

Equation 5.1.3.1 can be rewritten as:

$$\mathbf{C} = (\boldsymbol{\Phi}^T)^{-1} \mathbf{C}_g \boldsymbol{\Phi}^{-1} \quad (5.1.3.3)$$

Using the generalized modal mass matrix:

$$\boldsymbol{\Phi}^T \mathbf{M} \boldsymbol{\Phi} = \mathbf{M}_g \quad (5.1.3.4)$$

it can be shown that:

$$\boldsymbol{\Phi}^{-1} = (\mathbf{M}_g)^{-1} \boldsymbol{\Phi}^T \mathbf{M} \quad (\boldsymbol{\Phi}^T)^{-1} = \mathbf{M} \boldsymbol{\Phi} (\mathbf{M}_g)^{-1} \quad (5.1.3.5)$$

Equation 5.1.3.3 becomes:

$$\mathbf{C} = (\mathbf{M} \boldsymbol{\Phi} (\mathbf{M}_g)^{-1}) \mathbf{C}_g ((\mathbf{M}_g)^{-1} \boldsymbol{\Phi}^T \mathbf{M}) \quad (5.1.3.6)$$

Since $\mathbf{M}_g$ and $\mathbf{C}_g$ are diagonal matrices, Equation 5.1.3.6 can also be expressed as:

$$\mathbf{C} = \mathbf{M} \left(\sum_{i=1}^N \frac{2\zeta_i \omega_i}{M_i} \boldsymbol{\phi}_i \boldsymbol{\phi}_i^T \right) \mathbf{M} \quad (5.1.3.7)$$

5.2 Classical Modal Superposition

The following section is inspired from the book "Dynamics of Structures: Theory and Applications to Earthquake Engineering" by Anil K. Chopra [14].

For a classical damped multi degree of freedom (MDoF) linear system subjected to external forces, the governing differential equation is:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t) \quad (5.2.1)$$

The displacement vector $\mathbf{u}$ can also be expressed with the following form:

$$\mathbf{u} = \sum_{j=1}^N \boldsymbol{\phi}_j q_j = \boldsymbol{\Phi} \mathbf{q} \quad (5.2.2)$$

where q_j are scalar multipliers called modal coordinates, generalized coordinates or normal coordinates. Equation 5.2.1 can then be expressed as:

$$\mathbf{M}_g \ddot{\mathbf{q}} + \mathbf{C}_g \dot{\mathbf{q}} + \mathbf{K}_g \mathbf{q} = \mathbf{p}_g(t) \quad (5.2.3)$$

where $\mathbf{C}_g$ is, in the general case, a non-diagonal matrix that couples the set of equations. The external forces are defined as $\mathbf{p}_g(t) = \boldsymbol{\Phi}^T \mathbf{p}(t)$. The modal equations

will become uncoupled if the system has classical damping (see Section 5.1). For such systems, for the i -th modal coordinate, the following expression can be written:

$$M_i \ddot{q}_i(t) + C_i \dot{q}_i(t) + K_i q_i(t) = p_i(t) \quad (5.2.4)$$

Then by dividing by M_i :

$$\ddot{q}_i(t) + 2\zeta_i \omega_i \dot{q}_i(t) + \omega_i^2 q_i(t) = \frac{p_i(t)}{M_i} \quad (5.2.5)$$

where ζ_i is the damping ratio for the i -th mode. Such values can be directly chosen on experimental data and engineering judgement for each mode (see Figure 5.1). In other words, this allows avoiding using a damping matrix.

Once the modal coordinates $q_i(t)$ have been determined, the contribution of the i -th mode to the nodal displacements $\mathbf{u}(t)$ is given by:

$$\mathbf{u}_i(t) = \boldsymbol{\phi}_i q_i(t) \quad (5.2.6)$$

Making the superposition of these modal contributions, one obtains the total nodal displacements:

$$\mathbf{u}(t) = \sum_{i=1}^N \mathbf{u}_i(t) = \sum_{i=1}^N \boldsymbol{\phi}_i q_i(t) \quad (5.2.7)$$

This procedure is known as classical mode analysis, or classical modal superposition, or simply modal analysis. The linearity of the system is implicit in using the principle of superposition, and damping must be of the classical form.

Once the total displacements $\mathbf{u}(t)$ have been determined by dynamic analysis, the element forces $\mathbf{r}(t)$ required for the structural design or assessment can be determined by static analysis of the structure at each instant t . These forces can be easily computed by multiplying the nodal local displacements by the local stiffness matrix of each element. However, such approach works only for MDF systems with classical damping responding within their linearly elastic range. The uncoupling of modal equations is not possible if the system has non-classical damping or if it responds into the non-linear range. Additionally, the uncoupled modal equations can be solved in closed form if the external force $\mathbf{p}(t)$ doesn't vary arbitrarily with time, which is not always the case, such as during complex excitations like earthquake ground motions.

Numerical methods can be employed to solve these problems, handling both linear and non-linear issues with classical or non-classical damping. In HybriDFEM, a series of algorithms from the literature have been implemented to address these challenges, which will be explained in the following section.

5.3 Time-stepping Methods

The objective is to solve numerically the system of differential equations governing the response of MDF systems:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{P}_r(u) = \mathbf{p}(t) \quad (5.3.1)$$

with initial conditions:

$$\mathbf{u} = \mathbf{u}(0) \quad \dot{\mathbf{u}} = \dot{\mathbf{u}}(0) \quad (5.3.2)$$

at $t = 0$. The solution will provide the displacement vector $\mathbf{u}(t)$ as a function of time. The system is assumed to have linear viscous damping, but other forms of damping, including nonlinear damping, could be considered. However, this is rarely done due to the lack of information on damping characteristics, especially at large amplitudes of motion.

The timescale is divided into a series of time steps, usually of constant duration t . The excitation is given at discrete time instants $t_i = i\Delta t$; at t_i , denoted as time i , the excitation vector is $\mathbf{p}_i \equiv \mathbf{p}(t_i)$. The response will be determined at the same time instants and is denoted by $\mathbf{u}_i \equiv \mathbf{u}(t_i)$, $\dot{\mathbf{u}}_i \equiv \dot{\mathbf{u}}(t_i)$ and $\ddot{\mathbf{u}}_i \equiv \ddot{\mathbf{u}}(t_i)$.

Starting with the known response of the system at time i :

$$\mathbf{M}\ddot{\mathbf{u}}_i + \mathbf{C}\dot{\mathbf{u}}_i + (\mathbf{P}_r)_i = \mathbf{p}_i \quad (5.3.3)$$

Time-stepping methods enable to determine the response $\mathbf{u}_{i+1}$, $\dot{\mathbf{u}}_{i+1}$, and $\ddot{\mathbf{u}}_{i+1}$ of the system at time $i + 1$:

$$\mathbf{M}\ddot{\mathbf{u}}_{i+1} + \mathbf{C}\dot{\mathbf{u}}_{i+1} + (\mathbf{P}_r)_{i+1} = \mathbf{p}_{i+1} \quad (5.3.4)$$

Knowing the initial conditions at time $i = 0$, the procedure can be used to compute the response at any given time.

The numerical procedure requires three matrix equations to determine the three unknown vectors $\mathbf{u}_{i+1}$, $\dot{\mathbf{u}}_{i+1}$, and $\ddot{\mathbf{u}}_{i+1}$. Two of these equations are derived from either finite difference equations for the velocity and acceleration vectors or from an assumption on how the acceleration varies during a time step. The third is Equation 5.3.1 at a selected time instant. If it is the current time i , the method of integration is said to be an explicit method. If the time $i + 1$ at the end of the time step is used, the method is known as an implicit method.

For a numerical procedure to be useful, it should (1) converge to the exact solution as t decreases, (2) be stable in the presence of numerical round-off errors, and (3) be accurate (i.e., the computational errors should be within an acceptable limit).

Stability criteria are critical considerations in MDoF systems. Conditionally stable procedures can be somewhat effective for the analysis of linear MDoF systems with few DoFs. However, for linear systems with many DoFs and for nonlinear systems, unconditionally stable procedures are generally necessary.

5.3.1 Implemented Algorithms

Following, a series of algorithms from the literature will be presented. Each algorithm will be briefly described, and its respective pseudocode will be provided. For further details, interested readers are encouraged to refer to the cited papers.

Central Difference Method

Based on a finite difference approximation of the time derivatives of displacement [14], the Central Difference Method is an explicit time integration scheme. This means that the solution $\mathbf{u}_{i+1}$ at time $i + 1$ is determined only from the equilibrium condition, Equation 5.3.1 at time i .

Algorithm 1: Central Difference Method, Linear systems [14]

1. Initial calculations
 - 1.1. Impose initial displacements $\mathbf{u}_0$, velocities $\dot{\mathbf{u}}_0$ and external forces $\mathbf{p}_0$
 - 1.2. Solve $\mathbf{M}\ddot{\mathbf{u}}_0 + \mathbf{C}\dot{\mathbf{u}}_0 + \mathbf{K}\mathbf{u}_0 = \mathbf{p}_0$ to compute $\ddot{\mathbf{u}}_0$
 - 1.3. $\mathbf{u}_{-1} = \mathbf{u}_0 - \Delta t \dot{\mathbf{u}}_0 + \frac{(\Delta t)^2}{2} \ddot{\mathbf{u}}_0$
 - 1.4. Select Δt
 - 1.5. $\mathbf{A} = \frac{1}{2(\Delta t)^2} \mathbf{M} - \frac{1}{2\Delta t} \mathbf{C}$, $\mathbf{B} = \mathbf{K} - \frac{2}{(\Delta t)^2} \mathbf{M}$
 - 1.6. $\hat{\mathbf{K}} = \frac{1}{(\Delta t)^2} \mathbf{M} + \frac{1}{2\Delta t} \mathbf{C}$
2. Calculations for each time step i
 - 2.1. $\hat{\mathbf{p}}_i = \mathbf{p}_i - \mathbf{A}\mathbf{u}_{i-1} - \mathbf{B}\mathbf{u}_i$
 - 2.2. Solve $\hat{\mathbf{K}}\mathbf{u}_{i+1} = \hat{\mathbf{p}}_i$ to compute $\mathbf{u}_{i+1}$
 - 2.3. $\dot{\mathbf{u}}_{i+1} = \frac{1}{2\Delta t}(\mathbf{u}_{i+1} - \mathbf{u}_{i-1})$
 - 2.4. $\ddot{\mathbf{u}}_{i+1} = \frac{1}{(\Delta t)^2}(\mathbf{u}_{i+1} - 2\mathbf{u}_i + \mathbf{u}_{i-1})$
3. Repetition for the next time step. Replace i by $i + 1$ and implement steps 2.1 to 2.4 for the next time step.

The method has conditional stability, which means that using a time step Δt too big, causes the response to blow up and increase indefinitely, giving meaningless results. The specific requirement for stability is:

$$\frac{\Delta t}{T_n} \leq \frac{1}{\pi} \quad (5.3.1.1)$$

with $T_n = \pi/\omega_n$ the period of the highest mode.

In HybriDFEM, the linear version of the method is implemented using the Central Difference algorithm (see Algorithm 1). The nonlinear version is implemented using the nonlinear Central Difference algorithm (see Algorithm 2).

Algorithm 2: Central Difference Method, Non-Linear systems [14]

1. Initial calculations
 - 1.1. Impose initial displacements $\mathbf{u}_0$, velocities $\dot{\mathbf{u}}_0$ and external forces $\mathbf{p}_0$
 - 1.2. State determination: $\mathbf{P}_r(\mathbf{u}_0)$ and $\mathbf{K}_t(\mathbf{u}_0)$
 - 1.3. Solve $\mathbf{M}\ddot{\mathbf{u}}_0 + \mathbf{C}\dot{\mathbf{u}}_0 + \mathbf{P}_r(\mathbf{u}_0) = \mathbf{p}_0$ to compute $\ddot{\mathbf{u}}_0$
 - 1.4. $\mathbf{u}_{-1} = \mathbf{u}_0 - \Delta t \dot{\mathbf{u}}_0 + \frac{(\Delta t)^2}{2} \ddot{\mathbf{u}}_0$
 - 1.5. Select Δt
 - 1.6. $\mathbf{A} = \frac{1}{2(\Delta t)^2} \mathbf{M} - \frac{1}{2\Delta t} \mathbf{C}$, $\mathbf{B} = \frac{2}{(\Delta t)^2} \mathbf{M}$
 - 1.7. $\hat{\mathbf{K}} = \frac{1}{(\Delta t)^2} \mathbf{M} + \frac{1}{2\Delta t} \mathbf{C}$
2. Calculations for each time step i
 - 2.1. State determination: $\mathbf{P}_r(\mathbf{u}_i)$ and $\mathbf{K}_t(\mathbf{u}_i)$
 - 2.2. $\hat{\mathbf{p}}_i = \mathbf{p}_i - \mathbf{A}\mathbf{u}_{i-1} + \mathbf{B}\mathbf{u}_i - \mathbf{P}_r(\mathbf{u}_i)$
 - 2.3. Solve $\hat{\mathbf{K}}\mathbf{u}_{i+1} = \hat{\mathbf{p}}_i$ to compute $\mathbf{u}_{i+1}$
 - 2.4. $\dot{\mathbf{u}}_{i+1} = \frac{1}{2\Delta t}(\mathbf{u}_{i+1} - \mathbf{u}_{i-1})$
 - 2.5. $\ddot{\mathbf{u}}_{i+1} = \frac{1}{(\Delta t)^2}(\mathbf{u}_{i+1} - 2\mathbf{u}_i + \mathbf{u}_{i-1})$
3. Repetition for the next time step. Replace i by $i + 1$ and implement steps 2.1 to 2.5 for the next time step.

Constant Average Acceleration Method

Making the assumption that the variation of acceleration over a time step is constant, the Constant Average Acceleration Method is an implicit time integration scheme. This means that the solution $\mathbf{u}_{i+1}$ at time $i + 1$ is determined only from the equilibrium condition, Equation 5.3.1 at time $i + 1$.

The method has unconditional stability, which means that the method is stable for any time step Δt , no matter how large. However, selecting excessively large values for Δt may result in accuracy errors.

In HybriDFEM, the linear version of the method is implemented using the Newmark algorithm (see Algorithm 3), with $\gamma = 1/2$ and $\beta = 1/4$. The nonlinear version is implemented using the nonlinear Newmark algorithm (see Algorithm 4), with $\gamma = 1/2$ and $\beta = 1/4$.

Algorithm 3: Newmark Method, Linear systems [14]

1. Initial calculations
 - 1.1. Impose initial displacements $\mathbf{u}_0$, velocities $\dot{\mathbf{u}}_0$ and external forces $\mathbf{p}_0$
 - 1.2. Solve $\mathbf{M}\ddot{\mathbf{u}}_0 + \mathbf{C}\dot{\mathbf{u}}_0 + \mathbf{K}\mathbf{u}_0 = \mathbf{p}_0$ to compute $\ddot{\mathbf{u}}_0$
 - 1.3. Select Δt
 - 1.4. $\mathbf{A}_1 = \frac{1}{2\beta(\Delta t)^2} \mathbf{M} + \frac{\gamma}{\beta\Delta t} \mathbf{C}$, $\mathbf{A}_2 = \frac{1}{\beta\Delta t} \mathbf{M} + \left(\frac{\gamma}{\beta} - 1\right) \mathbf{C}$ and
 $\mathbf{A}_3 = \left(\frac{1}{2\beta} - 1\right) \mathbf{M} + \Delta t \left(\frac{\gamma}{2\beta} - 1\right) \mathbf{C}$
 - 1.5. $\hat{\mathbf{K}} = \mathbf{K} + \mathbf{A}_1$
2. Calculations for each time step, $i = 0, 1, 2, \dots$
 - 2.1. $\hat{\mathbf{p}}_{i+1} = \mathbf{p}_{i+1} + \mathbf{A}_1\mathbf{u}_i + \mathbf{A}_2\dot{\mathbf{u}}_i + \mathbf{A}_3\ddot{\mathbf{u}}_i$
 - 2.2. Solve $\hat{\mathbf{K}}\mathbf{u}_{i+1} = \hat{\mathbf{p}}_{i+1}$ to compute $\mathbf{u}_{i+1}$
 - 2.3. $\dot{\mathbf{u}}_{i+1} = \frac{\gamma}{\beta\Delta t}(\mathbf{u}_{i+1} - \mathbf{u}_i) + \left(1 - \frac{\gamma}{\beta}\right)\dot{\mathbf{u}}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{\mathbf{u}}_i$
 - 2.4. $\ddot{\mathbf{u}}_{i+1} = \frac{1}{\beta(\Delta t)^2}(\mathbf{u}_{i+1} - \mathbf{u}_i) - \frac{1}{\beta\Delta t}\dot{\mathbf{u}}_i - \left(\frac{1}{2\beta} - 1\right)\ddot{\mathbf{u}}_i$
3. Repetition for the next time step. Replace i by $i + 1$ and implement steps 2.1 to 2.4 for the next time step.

Linear Acceleration Method

Making the assumption that the variation of acceleration over a time step varies linearly, the Linear Acceleration Method is an implicit time integration scheme.

The method has conditional stability, and is stable if:

$$\frac{\Delta t}{T_n} \leq 0.551 \quad (5.3.1.2)$$

with $T_n = \pi/\omega_n$ the period of the highest mode.

For a given time step that is much smaller than this stability limit, the linear acceleration method is more accurate than the average acceleration method. Therefore, it is especially useful for linear systems because the t chosen to obtain accurate response in the highest mode included would satisfy the stability requirements.

In HybriDFEM, the linear version of the method is implemented using the Newmark algorithm (see Algorithm 3), with $\gamma = 1/2$ and $\beta = 1/6$. The nonlinear version is implemented using the nonlinear Newmark algorithm (see Algorithm 4), with $\gamma = 1/2$ and $\beta = 1/6$.

Newmark Method

Developed in 1959 by N. M. Newmark [14], the Newmark Method is an implicit time integration method.

The parameters β and γ define the variation of acceleration over a time step and determine the stability and accuracy characteristics of the method.

The method has conditional stability, and is stable for linear systems if:

$$\frac{\Delta t}{T_n} \leq \frac{1}{\pi\sqrt{2}} \frac{1}{\sqrt{\gamma - 2\beta}} \quad (5.3.1.3)$$

with $T_n = \pi/\omega_n$ the period of the highest mode.

The two special cases of Newmark's method that are commonly used are:

1. $\gamma = 1/2, \beta = 1/4$, which gives the constant average acceleration method,
2. $\gamma = 1/2, \beta = 1/6$, which gives the linear acceleration method.

Algorithm 4: Newmark Method, Non-Linear systems [14]

1. Initial calculations
 - 1.1. Impose initial displacements $\mathbf{u}_0$, velocities $\dot{\mathbf{u}}_0$ and external forces $\mathbf{p}_0$
 - 1.2. State determination: $\mathbf{P}_r(\mathbf{u}_0)$ and $\mathbf{K}_t(\mathbf{u}_0)$
 - 1.3. Solve $\mathbf{M}\ddot{\mathbf{u}}_0 + \mathbf{C}\dot{\mathbf{u}}_0 + \mathbf{P}_r(\mathbf{u}_0) = \mathbf{p}_0$ to compute $\ddot{\mathbf{u}}_0$
 - 1.4. Select Δt
 - 1.5. $\mathbf{A}_1 = \frac{1}{2\beta(\Delta t)^2} \mathbf{M} + \frac{\gamma}{\beta\Delta t} \mathbf{C}$, $\mathbf{A}_2 = \frac{1}{\beta\Delta t} \mathbf{M} + \left(\frac{\gamma}{\beta} - 1\right) \mathbf{C}$
 and $\mathbf{A}_3 = \left(\frac{1}{2\beta} - 1\right) \mathbf{M} + \Delta t\left(\frac{\gamma}{2\beta} - 1\right) \mathbf{C}$
2. Calculations for each time step, $i = 0, 1, 2, \dots$
 - 2.1. $\hat{\mathbf{p}}_{i+1} = \mathbf{p}_{i+1} + \mathbf{A}_1\mathbf{u}_i + \mathbf{A}_2\dot{\mathbf{u}}_i + \mathbf{A}_3\ddot{\mathbf{u}}_i$
 - 2.2. Initialize $j = 0$, $\mathbf{u}_{i+1}^j = \mathbf{u}_i$
 - 2.3. State determination: $\mathbf{P}_r(\mathbf{u}_{i+1}^j)$ and $\mathbf{K}_t(\mathbf{u}_{i+1}^j)$
3. Calculations for each time step, $j = 0, 1, 2, \dots$
 - 3.1. Evaluate residual: $\hat{\mathbf{R}}_{i+1}^j = \hat{\mathbf{p}}_{i+1} - \mathbf{P}_r(\mathbf{u}_{i+1}^j) - \mathbf{A}_1\mathbf{u}_{i+1}^j$
 - 3.2. Check convergence
 - 3.3. Evaluate tangent matrix: $\hat{\mathbf{K}}_t(\mathbf{u}_{i+1}^j) = \mathbf{K}_t(\mathbf{u}_{i+1}^j) + \mathbf{A}_1$
 - 3.4. Solve and update displacements: $\mathbf{u}_{i+1}^{j+1} = \mathbf{u}_{i+1}^j + \hat{\mathbf{R}}_{i+1}^j / \hat{\mathbf{K}}_t(\mathbf{u}_{i+1}^j)$
 - 3.5. State determination: $\mathbf{P}_r(\mathbf{u}_{i+1}^{j+1})$ and $\mathbf{K}_t(\mathbf{u}_{i+1}^{j+1})$
4. Replace j by $j + 1$ and repeat steps 3.1 to 3.5, until convergence; denote final value as $\mathbf{u}_{i+1}$.
5. Calculations for velocity and acceleration
 - 5.1. $\dot{\mathbf{u}}_{i+1} = \frac{\gamma}{\beta\Delta t}(\mathbf{u}_{i+1} - \mathbf{u}_i) + \left(1 - \frac{\gamma}{\beta}\right)\dot{\mathbf{u}}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{\mathbf{u}}_i$
 - 5.2. $\ddot{\mathbf{u}}_{i+1} = \frac{1}{\beta(\Delta t)^2}(\mathbf{u}_{i+1} - \mathbf{u}_i) - \frac{1}{\beta\Delta t}\dot{\mathbf{u}}_i - \left(\frac{1}{2\beta} - 1\right)\ddot{\mathbf{u}}_i$
6. Repetition for the next time step. Replace i by $i + 1$ and implement steps 2 to 5 for the next time step.

Wilson's Method

Developed in 1973 by K. J. Bathe and E. L. Wilson [4], the Wilson- θ method is an implicit time integration method principally used in slowly oscillating systems. The method assumes that the acceleration varies linearly within the time interval $[t, t + \theta\Delta t]$ with $\theta \geq 1$.

In linear problems, the method is unconditionally stable for $\theta \geq 1.37$. If $\theta = 1$, the method is equal to the linear acceleration method. For values higher than one, numerical damping is introduced, which decreases the amplitude of the response and suppresses any spurious participation of higher modes, making it a very powerful method in MDoF systems. However, it is not possible to obtain both zero dissipation and an unconditionally stable algorithm simultaneously using the Wilson method.

Algorithm 5: Wilson's Method, Linear systems [4]

1. Initial calculations
 - 1.1. Impose initial displacements $\mathbf{u}_0$, velocities $\dot{\mathbf{u}}_0$ and external forces $\mathbf{p}_0$
 - 1.2. Solve $\mathbf{M}\ddot{\mathbf{u}}_0 + \mathbf{C}\dot{\mathbf{u}}_0 + \mathbf{K}\mathbf{u}_0 = \mathbf{p}_0$ to compute $\ddot{\mathbf{u}}_0$
 - 1.3. Select Δt
 - 1.4. $\mathbf{A}_1 = \frac{6}{\theta\Delta t} \mathbf{M} + 3 \mathbf{C}$, $\mathbf{A}_2 = 3 \mathbf{M} + \frac{\theta\Delta t}{2} \mathbf{C}$
 - 1.5. $\hat{\mathbf{K}} = \mathbf{K} + \frac{6}{(\theta\Delta t)^2} \mathbf{M} + \frac{3}{\theta\Delta t} \mathbf{C}$
2. Calculations for each time step, $i = 0, 1, 2, \dots$
 - 2.1. $\Delta\hat{\mathbf{p}}_{i+1} = (\theta - 1)(\mathbf{p}_{i+2} - \mathbf{p}_{i+1}) + (\mathbf{p}_{i+1} - \mathbf{p}_i) + \mathbf{A}_1\dot{\mathbf{u}}_i + \mathbf{A}_2\ddot{\mathbf{u}}_i$
 - 2.2. Solve $\hat{\mathbf{K}}\Delta\hat{\mathbf{u}}_{i+1} = \Delta\hat{\mathbf{p}}_{i+1}$ to compute $\Delta\hat{\mathbf{u}}_{i+1}$
 - 2.3. $\Delta\ddot{\mathbf{u}}_{i+1} = \left(\frac{6}{(\theta\Delta t)^2} \Delta\hat{\mathbf{u}}_{i+1} - \frac{6}{\theta\Delta t} \dot{\mathbf{u}}_i - 3\ddot{\mathbf{u}}_i \right) / \theta$
 - 2.4. $\Delta\dot{\mathbf{u}}_{i+1} = \Delta t \ddot{\mathbf{u}}_i + \frac{\Delta t}{2} \Delta\ddot{\mathbf{u}}_{i+1}$
 - 2.5. $\Delta\mathbf{u}_{i+1} = \Delta t \dot{\mathbf{u}}_i + \frac{(\Delta t)^2}{2} \ddot{\mathbf{u}}_i + \frac{(\Delta t)^2}{6} \Delta\ddot{\mathbf{u}}_{i+1}$
 - 2.6. $\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta\mathbf{u}_{i+1}$
 - 2.7. $\dot{\mathbf{u}}_{i+1} = \dot{\mathbf{u}}_i + \Delta\dot{\mathbf{u}}_{i+1}$
 - 2.8. $\ddot{\mathbf{u}}_{i+1} = \ddot{\mathbf{u}}_i + \Delta\ddot{\mathbf{u}}_{i+1}$
3. Repetition for the next time step. Replace i by $i + 1$ and implement steps 2.1 to 2.8 for the next time step.

In HybriDFEM, the linear version of the method is implemented using Wilson algorithm (see Algorithm 5). The only user-defined parameter is θ . No non-linear version is implemented.

Hilber-Hughes-Taylor Method

Developed in 1977 [24, 25, 26, 27], the Hilber-Hughes-Taylor method is an implicit time integration method, belonging to the family of alpha methods. It represents only a minor modification of the Newmark family. Like many commonly used algorithms, it allows for unconditional stability in linear cases and achieves second-order accuracy.

Similar to Wilson's method, alpha methods incorporate some form of numerical dissipation to suppress any spurious participation of higher modes. However, alpha methods provide much better control over the amount of dissipation. They allow for an unconditionally stable algorithm even with zero dissipation, something not achievable with Wilson's method.

Like the Newmark family, the algorithm includes the parameters β and γ , which define the variation of acceleration over a time step. An additional parameter, α , is introduced to regulate numerical damping.

In the linear case, the algorithm is unconditionally stable if:

$$\gamma = \frac{1 + 2\alpha}{2}, \quad \beta = \frac{(1 + \alpha)^2}{4} \quad \text{and} \quad \alpha \in [0, 1/3] \quad (5.3.1.4)$$

For $\alpha = 0$, the algorithm is equivalent to the Newmark method discussed previously. Increasing α introduces numerical damping into the response.

In HybriDFEM, the linear version of the method is implemented using the Generalized- α algorithm (see Algorithm 6). The only user-defined parameter is α_f which represents α from Equation 5.3.1.4. The parameter α_m is equal to 0, while β and γ are determined using Equation 5.3.1.4. The nonlinear version is implemented using the nonlinear Hilber-Hughes-Taylor algorithm, with β , γ , and α specified by the user (see Algorithm 7).

Wood-Bossak-Zienkiewicz Method

Developed in 1980 [26, 27, 51], just a couple of years after the Hilber-Hughes-Taylor method, the Wood-Bossak-Zienkiewicz method is also an implicit time integration scheme, belonging to the family of alpha methods.

Similar to the Hilber-Hughes-Taylor method, it represents only a minor modification of the Newmark family. It allows for unconditional stability, achieves second-order accuracy, and introduces positive artificial damping, thanks to the α parameter.

The algorithm is unconditionally stable if:

$$\gamma \geq \frac{1 - 2\alpha}{2}, \quad \beta \geq \frac{\gamma}{2} \geq \frac{1}{4} \quad \text{and} \quad \alpha \leq \frac{1}{2} \quad (5.3.1.5)$$

For $\alpha = 0$, the algorithm is equivalent to the Newmark method discussed previously.

In HybriDFEM, the linear version of the method is implemented using the Generalized- α algorithm (see Algorithm 6). The parameters β , γ and $\alpha_m = \alpha$ are determined by the user. The parameter α_f is equal to 0. No non-linear version is implemented.

The Generalized- α Method

Developed in 1995 [26, 27, 52], the Generalized- α Method is an implicit time integration method and one of the latest modifications within the alpha-type methods. It represents a fusion of the Hilber-Hughes-Taylor and Wood-Bossak-Zienkiewicz methods, offering unconditional stability, second-order accuracy, and minimized low-frequency dissipation.

Similar to the Newmark family, the algorithm employs the β and γ parameters, which define the variation of acceleration over a time step. The parameter α_m embodies the numerical damping from Wood-Bossak-Zienkiewicz's method, while α_f represents that from Hilber-Hughes-Taylor's method.

The algorithm achieves unconditional stability if:

$$\gamma = \frac{1}{2} - \alpha_m + \alpha_f, \quad \beta = \frac{1}{4}(1 - \alpha_m + \alpha_f)^2, \quad \alpha_m = \frac{2\mu - 1}{\mu + 1}, \quad \alpha_f = \frac{\mu}{\mu + 1} \quad (5.3.1.6)$$

where μ is a user-specified algorithmic dissipation control parameter which can range from 0 to 1. If $\mu = 1$, this algorithm becomes a non-dissipative algorithm, whereas $\mu = 0$ gives the maximum algorithmic dissipation within its family.

The controllable algorithmic dissipation and the second-order accuracy for dissipative cases have been considered as the main advantages of the generalized- α method.

For $\alpha_m = 0$, the algorithm is equivalent to the Hilber-Hughes-Taylor's Method.
For $\alpha_f = 0$, the algorithm is equivalent to the Wood-Bossak-Zienkiewicz Method.

In HybriDFEM, the linear version of the method is implemented using the Generalized- α algorithm (see Algorithm 6). The parameters α_f , β , γ and α_m are defined by Equation 5.3.1.6. The only user-defined parameter is μ . No non-linear version is implemented.

Algorithm 6: Generalized- α Method, Linear systems [26]

1. Initial calculations
 - 1.1. Impose initial displacements $\mathbf{u}_0$, velocities $\dot{\mathbf{u}}_0$ and external forces $\mathbf{p}_0$
 - 1.2. Solve $\mathbf{M}\ddot{\mathbf{u}}_0 + \mathbf{C}\dot{\mathbf{u}}_0 + \mathbf{K}\mathbf{u}_0 = \mathbf{p}_0$ to compute $\ddot{\mathbf{u}}_0$
 - 1.3. Select Δt
 - 1.4. $\mathbf{A}_1 = \frac{1 - \alpha_m}{2\beta(\Delta t)^2} \mathbf{M} + \frac{\gamma(1 - \alpha_f)}{\beta\Delta t} \mathbf{C}$, $\mathbf{A}_2 = \frac{1 - \alpha_m}{\beta\Delta t} \mathbf{M} + \left(\frac{\gamma(1 - \alpha_f)}{\beta} - 1\right) \mathbf{C}$
and $\mathbf{A}_3 = \left(\frac{1 - \alpha_m}{2\beta} - 1\right) \mathbf{M} + \Delta t(1 - \alpha_f)\left(\frac{\gamma}{2\beta} - 1\right) \mathbf{C}$
 - 1.5. $\hat{\mathbf{K}} = (1 - \alpha_f)\mathbf{K} + \mathbf{A}_1$
2. Calculations for each time step, $i = 0, 1, 2, \dots$
 - 2.1. $\hat{\mathbf{p}}_{i+1} = \mathbf{p}_{i+1} + \mathbf{A}_1\mathbf{u}_i + \mathbf{A}_2\dot{\mathbf{u}}_i + \mathbf{A}_3\ddot{\mathbf{u}}_i - \alpha_f\mathbf{K}\mathbf{u}_i$
 - 2.2. Solve $\hat{\mathbf{K}}\mathbf{u}_{i+1} = \hat{\mathbf{p}}_{i+1}$ to compute $\mathbf{u}_{i+1}$
 - 2.3. $\dot{\mathbf{u}}_{i+1} = \frac{\gamma}{\beta\Delta t}(\mathbf{u}_{i+1} - \mathbf{u}_i) + \left(1 - \frac{\gamma}{\beta}\right)\dot{\mathbf{u}}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{\mathbf{u}}_i$
 - 2.4. $\ddot{\mathbf{u}}_{i+1} = \frac{1}{\beta(\Delta t)^2}(\mathbf{u}_{i+1} - \mathbf{u}_i) - \frac{1}{\beta\Delta t}\dot{\mathbf{u}}_i - \left(\frac{1}{2\beta} - 1\right)\ddot{\mathbf{u}}_i$
3. Repetition for the next time step. Replace i by $i + 1$ and implement steps 2.1 to 2.4 for the next time step.

Algorithm 7: Hughes-Hilber-Taylor Method, Non-Linear systems [26]

1. Initial calculations
 - 1.1. Impose initial displacements $\mathbf{u}_0$, velocities $\dot{\mathbf{u}}_0$ and external forces $\mathbf{p}_0$
 - 1.2. State determination: $\mathbf{P}_r(\mathbf{u}_0)$ and $\mathbf{K}_t(\mathbf{u}_0)$
 - 1.3. Solve $\mathbf{M}\ddot{\mathbf{u}}_0 + \mathbf{C}\dot{\mathbf{u}}_0 + \mathbf{P}_r(\mathbf{u}_0) = \mathbf{p}_0$ to compute $\ddot{\mathbf{u}}_0$
 - 1.4. Select Δt
 - 1.5. $\mathbf{A}_1 = \frac{1}{2\beta(\Delta t)^2} \mathbf{M} + \frac{\gamma(1-\alpha_f)}{\beta\Delta t} \mathbf{C}$, $\mathbf{A}_2 = \frac{1}{\beta\Delta t} \mathbf{M} + \left(\frac{\gamma(1-\alpha_f)}{\beta} - 1\right) \mathbf{C}$
 and $\mathbf{A}_3 = \left(\frac{1}{2\beta} - 1\right) \mathbf{M} + \Delta t(1-\alpha_f)\left(\frac{\gamma}{2\beta} - 1\right) \mathbf{C}$
2. Calculations for each time step, $i = 0, 1, 2, \dots$
 - 2.1. $\hat{\mathbf{p}}_{i+1} = \mathbf{p}_{i+1} + \mathbf{A}_1 \mathbf{u}_i + \mathbf{A}_2 \dot{\mathbf{u}}_i + \mathbf{A}_3 \ddot{\mathbf{u}}_i - \alpha_f \mathbf{P}_r(\mathbf{u}_i)$
 - 2.2. Initialize $j = 0$, $\mathbf{u}_{i+1}^j = \mathbf{u}_i$
 - 2.3. State determination: $\mathbf{P}_r(\mathbf{u}_{i+1}^j)$ and $\mathbf{K}_t(\mathbf{u}_{i+1}^j)$
3. Calculations for each time step, $j = 0, 1, 2, \dots$
 - 3.1. Evaluate residual: $\hat{\mathbf{R}}_{i+1}^j = \hat{\mathbf{p}}_{i+1} - \mathbf{P}_r(\mathbf{u}_{i+1}^j) - \mathbf{A}_1 \mathbf{u}_{i+1}^j$
 - 3.2. Check convergence
 - 3.3. Evaluate tangent matrix: $\hat{\mathbf{K}}_t(\mathbf{u}_{i+1}^j) = \mathbf{K}_t(\mathbf{u}_{i+1}^j) + \mathbf{A}_1$
 - 3.4. Solve and update displacements: $\mathbf{u}_{i+1}^{j+1} = \mathbf{u}_{i+1}^j + \hat{\mathbf{R}}_{i+1}^j / \hat{\mathbf{K}}_t(\mathbf{u}_{i+1}^j)$
 - 3.5. State determination: $\mathbf{P}_r(\mathbf{u}_{i+1}^{j+1})$ and $\mathbf{K}_t(\mathbf{u}_{i+1}^{j+1})$
4. Replace j by $j + 1$ and repeat steps 3.1 to 3.5, until convergence; denote final value as $\mathbf{u}_{i+1}$.
5. Calculations for velocity and acceleration
 - 5.1. $\dot{\mathbf{u}}_{i+1} = \frac{\gamma}{\beta\Delta t}(\mathbf{u}_{i+1} - \mathbf{u}_i) + \left(1 - \frac{\gamma}{\beta}\right)\dot{\mathbf{u}}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{\mathbf{u}}_i$
 - 5.2. $\ddot{\mathbf{u}}_{i+1} = \frac{1}{\beta(\Delta t)^2}(\mathbf{u}_{i+1} - \mathbf{u}_i) - \frac{1}{\beta\Delta t}\dot{\mathbf{u}}_i - \left(\frac{1}{2\beta} - 1\right)\ddot{\mathbf{u}}_i$
6. Repetition for the next time step. Replace i by $i + 1$ and implement steps 2 to 5 for the next time step.

Chapter 6

Dynamic Analysis: Tests and Validations

The objective of this chapter is to validate the HybriDFEM formulation and its various implemented algorithms using benchmarks sourced from existing literature. The validation process includes testing both linear and nonlinear scenarios to comprehensively evaluate the performance and accuracy of HybriDFEM.

6.1 Modal Superposition and Linear Time-stepping Methods

Linear dynamic systems are frequently used in dynamic analysis due to their mathematical simplicity compared to nonlinear systems. They often permit analytical solutions, offering insights into structure behaviour under various conditions. Moreover, they serve as initial approximations for structure behaviour, providing a starting point for understanding responses to loads and disturbances.

Since HybriDFEM has already undergone rigorous testing and validation of its core components—such as the computation of natural frequencies, mode shapes, mass and stiffness matrices, and its implemented algorithms are widely tested in their respective papers, a significant level of reliability can already be confidently associated with HybriDFEM in the context of linear systems. However, conducting further testing and verification of its performance remains valuable. Such examinations can offer additional insights and understanding. This section details the additional testing and verification undertaken to further evaluate HybriDFEM's performance.

Three benchmarks will be tested. The first two draw inspiration from examples found in [14] and involve comparing some of the implemented HybriDFEM time-stepping algorithms with the analytical results from classical modal superposition, both neglecting and considering damping effects. The third benchmark will evaluate a simple column left in free vibration, analysing the impact of varying time steps on method precision and stability.

6.1.1 Undamped Two Free-of-Freedom System

This benchmark is adapted from an example of "Dynamics of Structures: Theory and Applications to Earthquake Engineering" by Anil K. Chopra [14]. It features a two-degree-of-freedom system, as illustrated in Figure 6.1, subject to a harmonic force excitation $p_1(t) = p_0 \sin(\omega t)$ applied to the mass m_1 .

The two masses m_1 and m_2 are both equal to $m = 2000$ kg, and are connected by springs with stiffness $k = 200 \times 10^3$ kN/m. The parameters of the applied force are $\omega = 10$ rad/s and $p_0 = 1.66 \times 10^4$ kN.

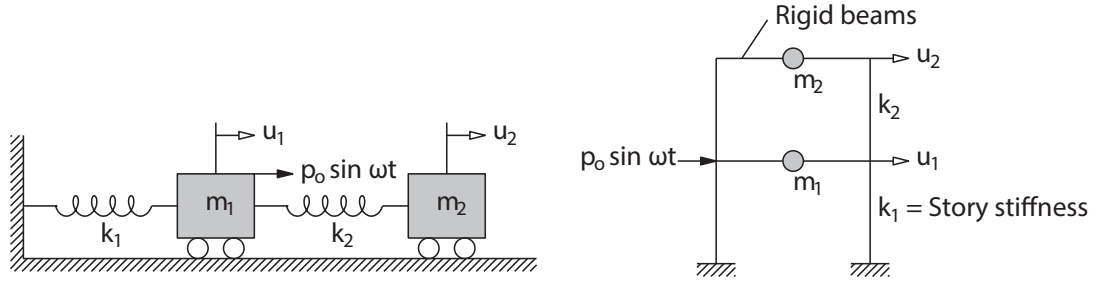


Figure 6.1: Two-degree-of-freedom systems.

The goal of the benchmark is to compute the displacement of the two blocks over a period $T = 2$ s using the implemented linear time stepping methods with a time step $\Delta t = 0.001$ s and to compare it with the analytical solution computed using the classical modal superposition.

Classical modal superposition

Considering Equation 5.2.1, the equations of motion of the systems can be written as:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} p_0 \\ 0 \end{bmatrix} \sin \omega t \quad (6.1.1.1)$$

Natural frequencies and mode shapes are easily computed by solving the eigenvalue problem from Equation 3.2.8:

$$\omega_1 = \sqrt{-\frac{(\sqrt{5}-3)k}{2m}} = 195.4 \text{ rad/s}, \quad \omega_2 = \sqrt{\frac{(\sqrt{5}+3)k}{2m}} = 511.7 \text{ rad/s} \quad (6.1.1.2)$$

$$\phi_1 = \begin{bmatrix} 0.618 & 1 \end{bmatrix}^T \quad \phi_2 = \begin{bmatrix} 1 & -0.618 \end{bmatrix}^T \quad (6.1.1.3)$$

As this is a linear system, the equation of motion can be uncoupled, as it was done in Equation 5.2.4.

$$M_i \ddot{q}_i(t) + K_i q_i(t) = P_{i,0} \sin \omega t \quad (6.1.1.4)$$

Solving this equation, the modal coordinates can be obtained:

$$q_i(t) = \frac{P_{i,0}}{K_i} \mathcal{C}_i \sin \omega t, \quad \mathcal{C}_i = \frac{1}{1 - (\omega/\omega_i)^2} \quad (6.1.1.5)$$

Finally, using Equation 5.2.2, the total displacement can be obtained:

$$\mathbf{u}(t) = \mathbf{u}_1(t) + \mathbf{u}_2(t) = \phi_1 q_1(t) + \phi_2 q_2(t) \quad (6.1.1.6)$$

Note that this development is computing the steady-state of the response, and it is valid if there is no initial displacement or velocity.

HybriDFEM time-stepping methods

Figure 6.2 and Figure 6.3 report the comparison between the displacement computed with HybriDFEM and with the classical modal superposition. Three different algorithms were tested:

- The Central Difference Method (CDM)
- The Constant Average Acceleration Method (CAA)
- The Linear Acceleration Method (LA)

The numerical solutions appear to accurately approximate the analytical solution in both figures. However, upon closer inspection, it becomes evident that the numerical solution exhibits slight oscillations around the analytical solution, rather than precisely computing the correct value, resulting in minor discrepancies. This discrepancy arises from the influence of multiple modes on the response. Using an algorithm with numerical damping, such as the Hilber-Hughes-Taylor method with a damping parameter of $\alpha = 0.1$ (see Figure 6.4), mitigates these oscillations and yields a response more closely aligned with the analytical solution after half a second. The slight difference in amplitude can be solved by decreasing the time step to obtain an increased precision. Overall, HybriDFEM demonstrates an accurate approximation of the response and can be deemed reliable for non-damped cases.

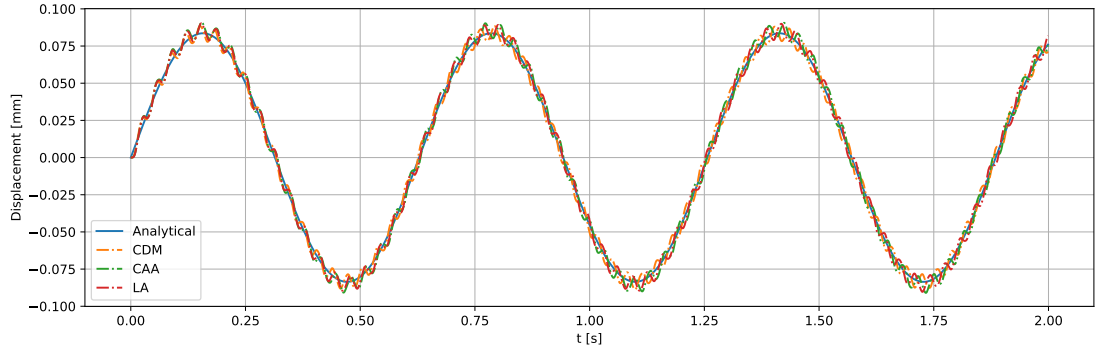


Figure 6.2: Displacement of the first block in the horizontal direction $u_1(t)$.

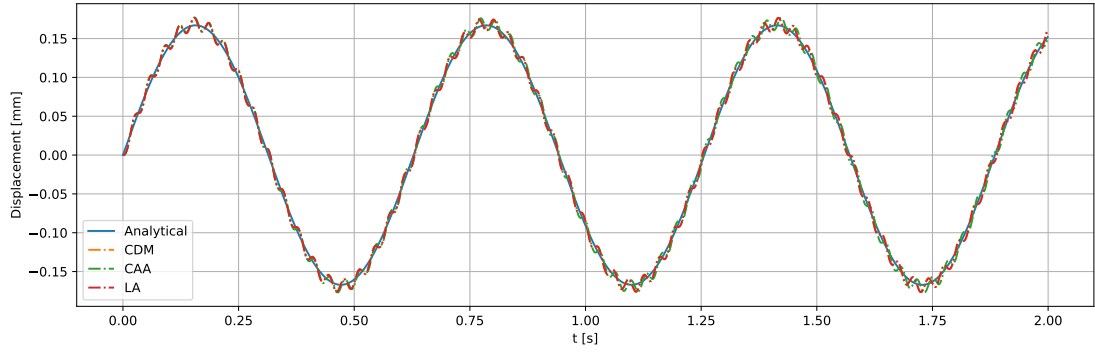


Figure 6.3: Displacement of the second block in the horizontal direction $u_2(t)$.

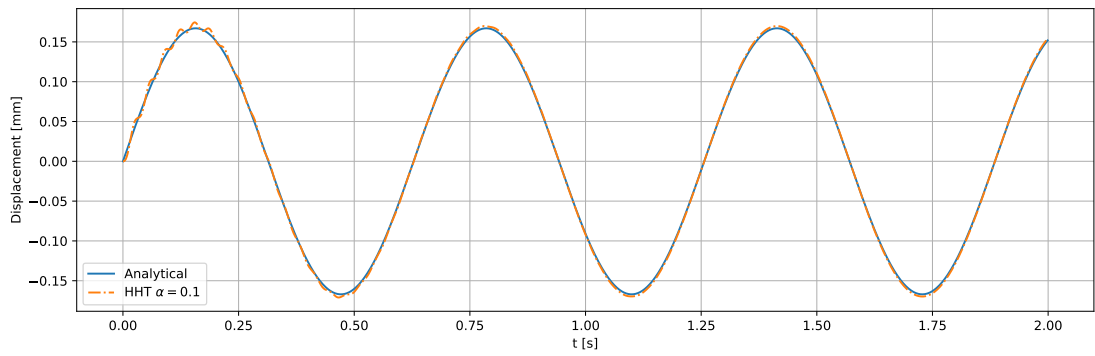


Figure 6.4: Displacement of the first block in the horizontal direction $u_1(t)$.

6.1.2 Damped Two Degree-of-Freedom System

This benchmark is adapted from an example of "Dynamics of Structures: Theory and Applications to Earthquake Engineering" by Anil K. Chopra [14]. It features the same two-degree-of-freedom system from the previous benchmark, as illustrated in Figure 6.1, but now considering also damping effects. The modal damping ratio is $\zeta = 0.2$, while the mass m , stiffness k , ω and p_0 are the same from before.

The objective of the benchmark is to compute the displacement of the two blocks over a period $T = 2$ s using the implemented linear time stepping methods with a time step $\Delta t = 0.001$ s and to compare it with the analytical solution computed using the classical modal superposition. The key distinction from the previous benchmark is the inclusion of damping effects in the analysis.

Classical modal superposition

Mode shapes and natural frequencies are the same as the previous benchmark, as damping doesn't influence their computation for small values of ζ .

As this is a linear system, the equation of motion can be uncoupled, as it was done in Equation 5.2.4. This time, the damping term C_i is present.

$$M_i \ddot{q}_i(t) + C_i \dot{q}_i(t) + K_i q_i(t) = P_{i,0} \sin \omega t \quad (6.1.2.1)$$

Solving this equation, the modal coordinates can be obtained:

$$q_i(t) = \frac{P_{i,0}}{K_i} (C_i \sin \omega t + \mathcal{D}_i \cos \omega t) \quad (6.1.2.2)$$

$$C_i = \frac{1 - (\omega/\omega_i)^2}{[1 - (\omega/\omega_i)^2]^2 + (2\zeta_i \omega/\omega_i)^2}, \quad \mathcal{D}_i = \frac{-2\zeta_i \omega/\omega_i}{[1 - (\omega/\omega_i)^2]^2 + (2\zeta_i \omega/\omega_i)^2} \quad (6.1.2.3)$$

Finally, using Equation 5.2.2, the total displacement can be obtained:

$$\mathbf{u}(t) = \mathbf{u}_1(t) + \mathbf{u}_2(t) = \phi_1 q_1(t) + \phi_2 q_2(t) \quad (6.1.2.4)$$

Note that this development is computing the steady-state of the response, and it is valid if there is no initial displacement or velocity.

HybriDFEM time-stepping methods

Figure 6.5 and Figure 6.6 report the comparison between the displacement computed with HybriDFEM and with the classical modal superposition. The same three algorithms from the previous benchmark were tested. In the case of HybriDFEM

the Superposition of Modal Damping Matrices method (explained in Section 5.1.3) was used to compute the damping matrix, using $\zeta = 0.2$.

The numerical solutions demonstrate a close approximation to the analytical solution in both figures. Notably, compared with the previous benchmark, the amplitudes in the damped and undamped cases exhibit significant similarities, with the exception being the absence of oscillations around the exact solution in the damped case. Explicit damping, akin to the HHT method, effectively suppresses these oscillations and provides a more precise approximation. In summary, HybriDFEM showcases an accurate representation of the response and can be considered reliable for damped cases.

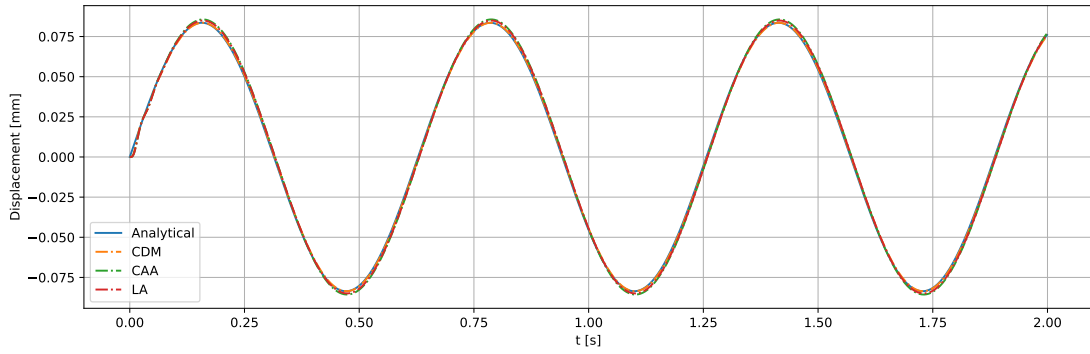


Figure 6.5: Displacement of the first block in the horizontal direction $u_1(t)$.

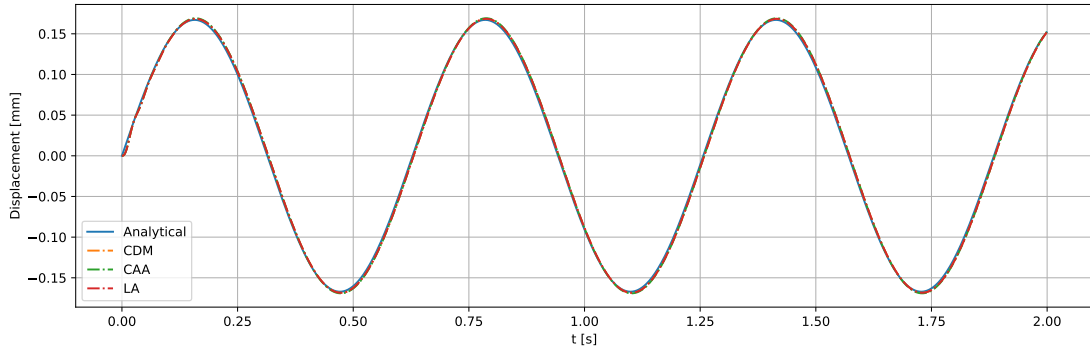


Figure 6.6: Displacement of the second block in the horizontal direction $u_2(t)$.

6.1.3 Time Step Sensitivity Analysis

Many different methods were implemented in HybriDFEM, each with their stability and precision. The following benchmark aims at analysing the influence of the time step Δt relative to the precision and stability of the methods, as well as its influence on the numerical damping introduced by some methods like the one proposed by Hilber.

The shape of the column modelled in this example is inspired from the 9th column of the eastern flank of the temple of Apollo at Bassae, Greece, modelled in a paper by I. Bouckaert et al. [6] (see Figure 6.7a). It is composed of 8 drums and has a total height of 5.95 m, a base width of 1.11 m and a top width of 0.9 m. The contact between the blocks is a linear-elastic law. In HybriDFEM the contact is simulated with 10 CP, with an axial and tangential stiffness $k_n = k_s = 10^5$ kN/m respectively.

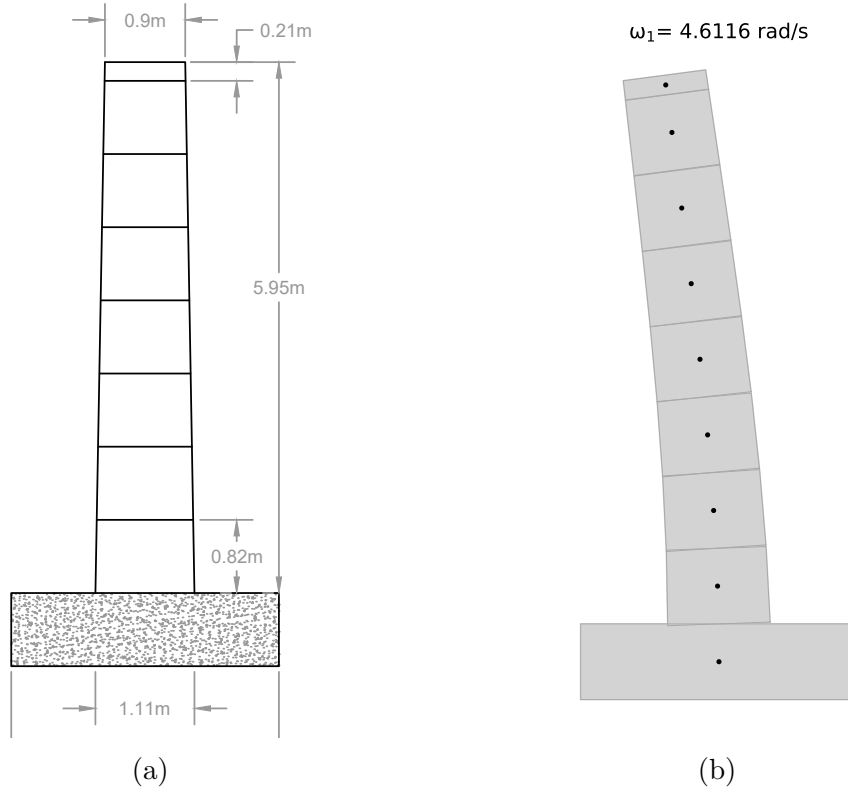


Figure 6.7: (a) Geometry of the column [6] and (b) fundamental mode shape computed with HybriDFEM.

The benchmark consists of computing the horizontal displacement of the column over a period $T = 3$ s using the various implemented time-stepping methods. Four different time steps Δt for each algorithm were used to show its impact on the precision of the results. However, the full analysis of only two of these times steps, $\Delta t = 0.01$ s and $\Delta t = 0.001$ is reported. Eight cases are tested:

- The Central Difference Method (CDM)
- The Constant Average Acceleration Method (CAA)
- The Linear Acceleration Method (LA)
- The Wilson- θ Method, with $\theta = 1.4$ (WLS)
- The Hilber-Hughes-Taylor Method, with $\alpha = 0.1$ (HHT1)
- The Hilber-Hughes-Taylor Method, with $\alpha = 0.3$ (HHT2)
- The Wood-Bossak-Zienkiewicz Method, with $\alpha = 0.1, \gamma = 0.6, \beta = 0.3025$ (WBZ1)
- The Wood-Bossak-Zienkiewicz Method, with $\alpha = -0.1, \gamma = 0.6, \beta = 0.3025$ (WBZ2)

The fundamental frequency of the column is $\omega_1 = 4.611$ rad/s. The initial column displacement is set to the fundamental mode shape (see Figure 6.7b), and then the column is left in free vibration, without any external force applied.

In Figure 6.8 are reported the results of the different methods with a time step $\Delta t = 0.01$. The abscissa reports the elapsed time while the ordinate the horizontal displacement of the highest drum. The CDM and LA method are not reported, as their stability condition are not respected and their response blows up.

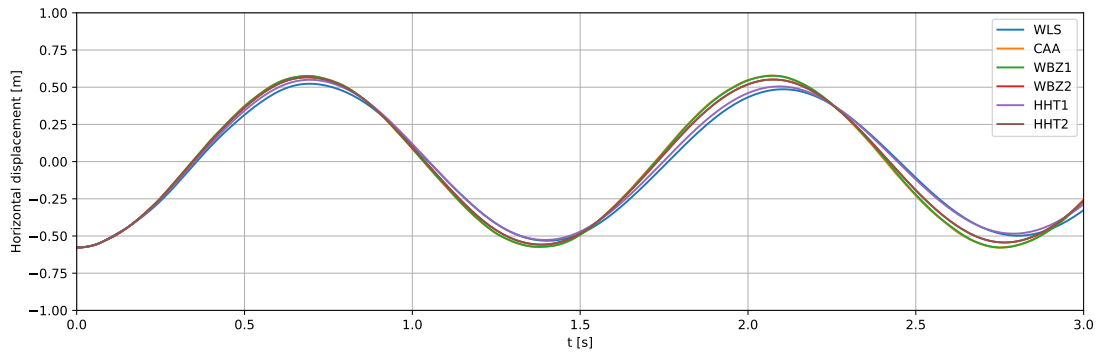


Figure 6.8: Horizontal displacement on top of the highest drum computed with HybriDFEM with $\Delta t = 0.01$ s.

The tested methods exhibit generally comparable results. As expected, some of the method employing numerical damping (such as HHT1, HHT2, WBZ2, and Wilson- θ) demonstrate a decrease in amplitude. Notably, in the depicted Figure, Wilson- θ displays the most pronounced reduction in amplitude, with HHT1 following closely behind. The magnitude of this reduction is influenced by the specific values chosen for the methods' parameters α or θ .

It is important to notice that the parameters selected for WBZ1 enable the method not only to produce results that align with those of CAA, but also dampen the response of higher modes, thereby being a strict upgrade of the CAA method. This observation was also noted in the paper by W. L. Wood et al. [51].

In Figure 6.9 are reported the results of the different methods with a time step $\Delta t = 0.001$. The abscissa reports the elapsed time while the ordinate the horizontal displacement of the highest drum.

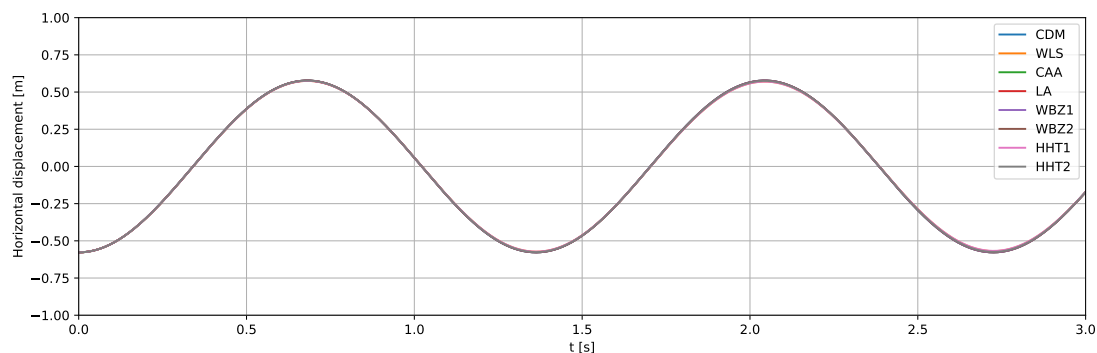


Figure 6.9: Horizontal displacement on top of the highest drum computed with HybriDFEM with $\Delta t = 0.001$ s.

The responses of all the methods are extremely close between each other, making it difficult to distinguish the individual methods. It is important to note that the reduction in amplitude presented before, fully disappears for very small time steps Δt . This phenomenon is something expected, as it is discussed in the paper by H. Hilber et al. [25]. Figure 6.10 comes directly from the article, and shows the reduction of the damping factor for very small ratios $\Delta t/T$.

The exact precision of the methods can be readily determined by computing the inverse of the distance between two peaks in the response. This value corresponds to the frequency of the response, and the closer it aligns with the fundamental frequency, the higher the precision is.

In Table 6.1 are reported the relative error of all the different time steps for each tested method. This error corresponds to the difference between the numerical and

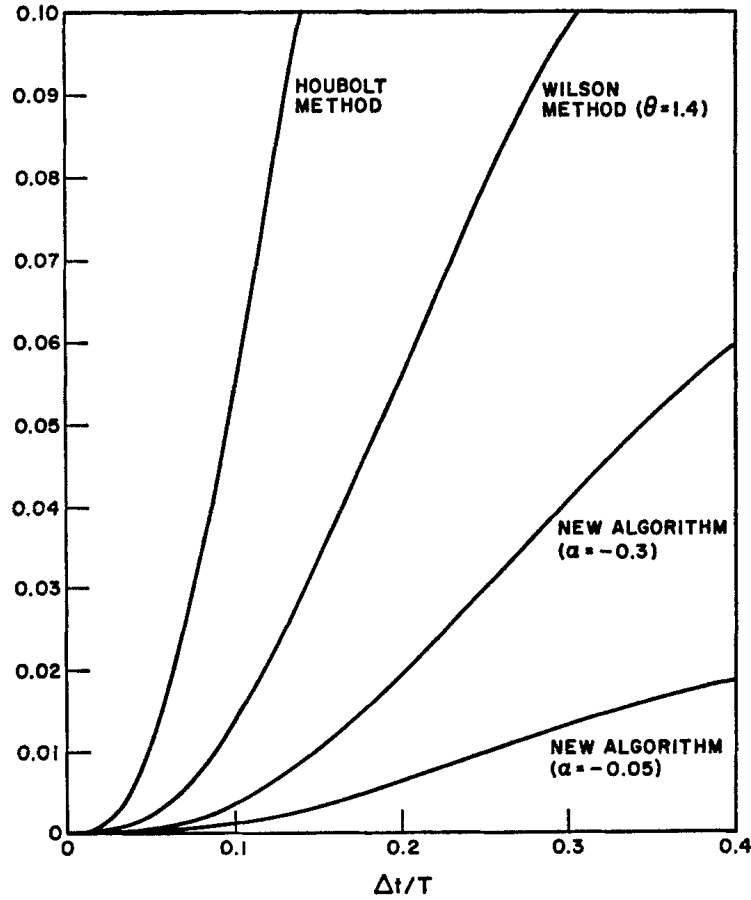


Figure 6.10: Damping ratios vs $\Delta t/T$ for Hilber-Hughes-Taylor, Houbolt and Wilson schemes [25].

natural frequency, divided by the latter.

With a large time step, such as $\Delta t = 0.05$ s, conditionally stable methods like LA or CDM tend to diverge, yielding results that are unusable. Conversely, unconditionally stable methods typically produce results, but with significant errors.

By reducing the time step, the stability conditions of certain methods, such as LA, can be met, resulting in acceptable outcomes. Meanwhile, unconditionally stable methods offer more precise solutions. However, opting for a finer time step inevitably increases computational time. Thus, a balance between precision and efficiency is crucial for accurate modelling.

It's evident that conditionally stable methods, while capable of providing precise results, necessitate very small time steps. Even in this simple example, the CDM

Δt	CDM	LA	CAA	WLS	HHT1	HHT2	WBZ1	WBZ2
0.05 s	∞	∞	-13.39	-43.4	-19.85	-26.35	29.69	-26.35
0.01 s	∞	∞	-1.21	-1.27	-1.98	-1.98	-1.98	-1.98
0.005 s	∞	-0.18	-0.18	-0.55	-0.91	-0.91	-0.91	-0.91
0.001 s	0.035	-0.038	-0.038	-0.038	-0.038	-0.038	-0.038	-0.038

Table 6.1: Relative errors in % for the different tested methods relative to the time step Δt .

method already requires at least $\Delta t = 0.001$ s. For more complex simulations involving numerous degrees of freedom, unconditionally stable methods should be prioritized.

Comments and solutions for the displacement of other drums are not detailed here, as similar conclusions can be drawn. Overall, HybriDFEM appears proficient at accurately computing displacements of a linear system across the various implemented methods.

6.1.4 Section Summary and Reflections

In summary, HybriDFEM demonstrates robust performance and computes results coherent with the analytical methods when modelling MDoF linear systems.

The response of a damped and an undamped two-DoF system in HybriDFEM was compared with the analytical solution obtained via classical modal superposition. The CDM, CAA, LA, and HHT algorithms were evaluated, all yielding satisfactory results. However, in the undamped case, some methods exhibited slight oscillations around the analytical response due to the influence of multiple modes on the system's behaviour. The HHT method, with its inherent numerical damping, proved particularly effective in mitigating these oscillations, offering a better approximation. Damped systems, by virtue of their damping, did not exhibit this issue.

Furthermore, the influence of the time step Δt was investigated across various algorithms. Conditionally stable methods such as LA or CDM required a very small time step to prevent an explosion in the response. While effective for small MDoF systems, these methods become impractical as the number of blocks increases. Unconditionally stable methods like HHT, WBZ or CAA should be prioritized in such cases. Additionally, the influence of numerical damping and its dependence on the time step was discussed.

6.2 Non-Linear Time-stepping Methods

Non-linear dynamic systems exhibit a significantly higher level of complexity in analysis compared to linear systems. This complexity arises from the various sources of non-linearity and the necessity of convergence algorithms. Despite their importance, few benchmarks could be found in the literature to truly validate HybriDFEM.

Three benchmarks were tested. The first two focus on a SDoF system, where the displacement over time was computed using various implemented algorithms. The first benchmark corresponds to an undamped system, while the second addresses the damped case. In the case of SDoF systems, non-linear effects can be somewhat identified, and material behavior can be more easily tested compared to high DoF systems, where multiple modes influence the result and the response is inherently harder to understand. The analysis of a MDoF system is detailed in Appendix A for further insights. The last benchmark, inspired by the paper by I. Bouckaert et al. [6], delves into a much more complex case, analysing the failure of a column subjected to a step impulse. This case study serves as an examination of HybriDFEM's ability to handle real-world scenarios with complex non-linear behaviour.

6.2.1 Undamped Single Degree-of-Freedom System

The benchmark features a single-degree-of-freedom system with a mass $m = 2000$ kg, as illustrated in Figure 6.11. This system is subjected to a harmonic force excitation $p(t) = (p_0 t) \sin(\omega t)$ where the force increases over time until reaching 20000 kN. After 1 second, the force is abruptly stopped, initiating free vibration within the system (see Figure 6.12). This excitation allows the observation of the system's behaviour within and beyond the yield strength of the material, encompassing responses to both harmonic force and free vibration.

The natural frequency of the system is $\omega_1 = 316.23$ rad/s, while the parameters of the applied force are $\omega = 100$ rad/s and $p_0 = 5 \times 10^4$ kN. No explicit damping effects are present in the system.

Two distinct material laws were tested: an elastic-perfectly plastic material and a strain-hardening material. Both material laws exhibit residual deformations.

The goal of the benchmark is to compute the displacement of the two blocks over a period $T = 1.5$ s using the implemented non-linear time stepping methods with a time step $\Delta t = 0.0001$ s. The analysis aims to identify and examine the various non-linear effects present within the system, thereby facilitating the validation of HybriDFEM.

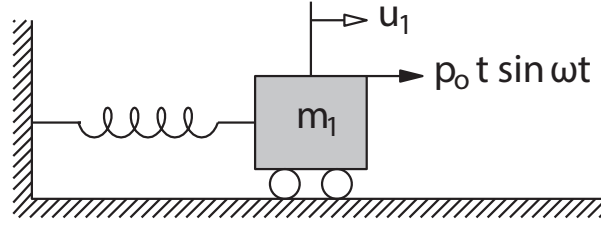


Figure 6.11: Single-degree-of-freedom system.

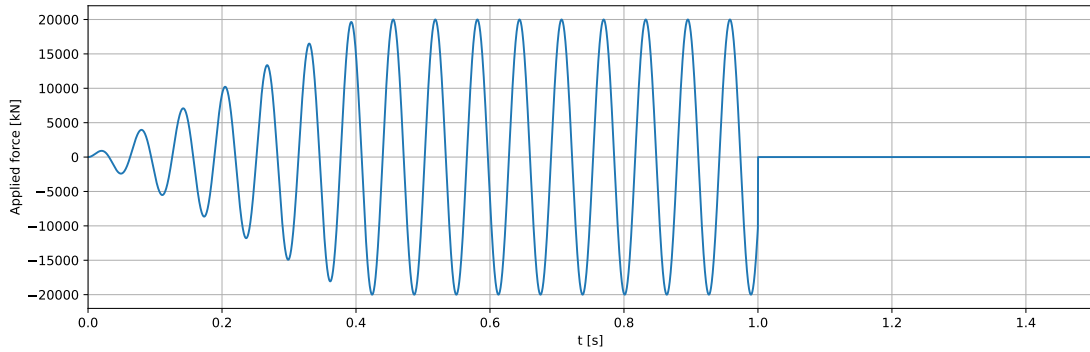


Figure 6.12: Applied force to the system $p(t) = (p_0 t) \sin(\omega t)$.

An elastic–perfectly plastic material

The stress-strain relation of the tested material is depicted in Figure 6.13. The material has a Young’s modulus of $E = 200$ MPa and a yield strength of $f_y = 10$ Mpa. Once yield strength f_y is reached, the material behaves plastically, maintaining a constant stress f_y .

Figure 6.14 presents a comparison of the displacement computed with HybriDFEM for an elastic–perfectly plastic material using different non-linear algorithms. The three algorithms tested are:

- The Central Difference Method (CDM)
- The Constant Average Acceleration Method (CAA)
- The Hilber-Hughes-Taylor Method, with $\alpha = 0.3$, $\gamma = 0.8$ and $\beta = 0.422$ (HHT)

Additionally, the figure includes the response of a linear material with $E = 200$ MPa. This comparison serves to verify the consistency of results before surpassing the yield strength f_y of the plastic material.

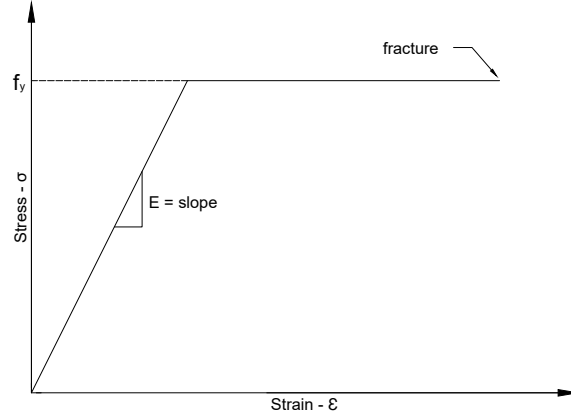


Figure 6.13: The stress-strain relation for an elastic-perfectly plastic material.

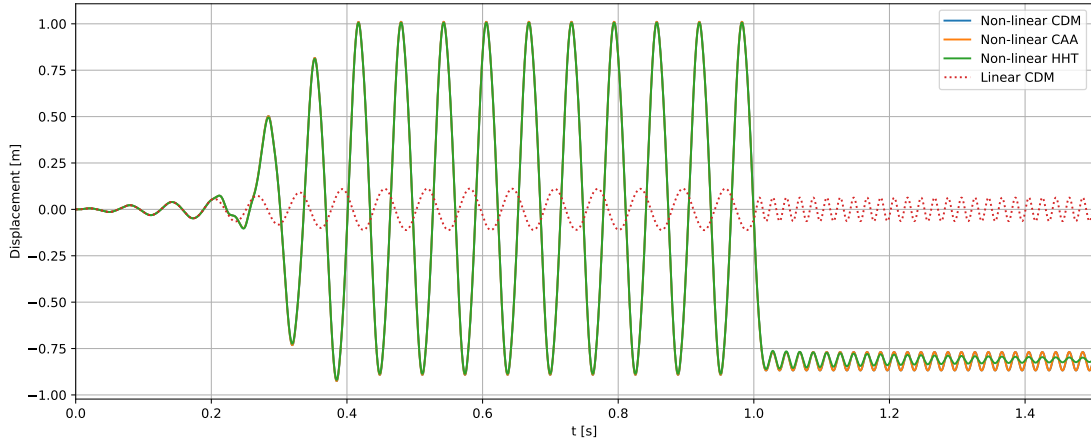


Figure 6.14: Displacement-time relation computed with HybriDFEM.

All numerical solutions exhibit similar responses before surpassing the yield strength, behaving like a linear material. However, once surpassed at $t = 0.2026$ s, the non-linear material's amplitude increases while its frequency slightly decreases. Similar to the linear case discussed in Section 6.1.1, the responses of all three algorithms are extremely similar. However, upon close inspection, a slight decrease in amplitude can be noticed for the HHT method. This is caused by the its numerical damping , which is very small given the very small time step Δt .

Upon the removal of the force, the system transitions into free vibration. As expected, CDM and CAA continue oscillating, while the HHT response dampens. A residual plastic deformation of 0.29 m is clearly visible across all three methods.

Figure 6.15 reports the internal force-displacement relation for all three methods. Consistent with the aforementioned observations, CDM and CAA exhibit comparable responses, while HHT presents a slight decrease in displacement. Notably, the shape of the relation closely resembles the elastic–perfectly plastic law depicted in Figure 6.13, affirming the correct behaviour of HybriDFEM in modelling non-linear Single Degree of Freedom systems.

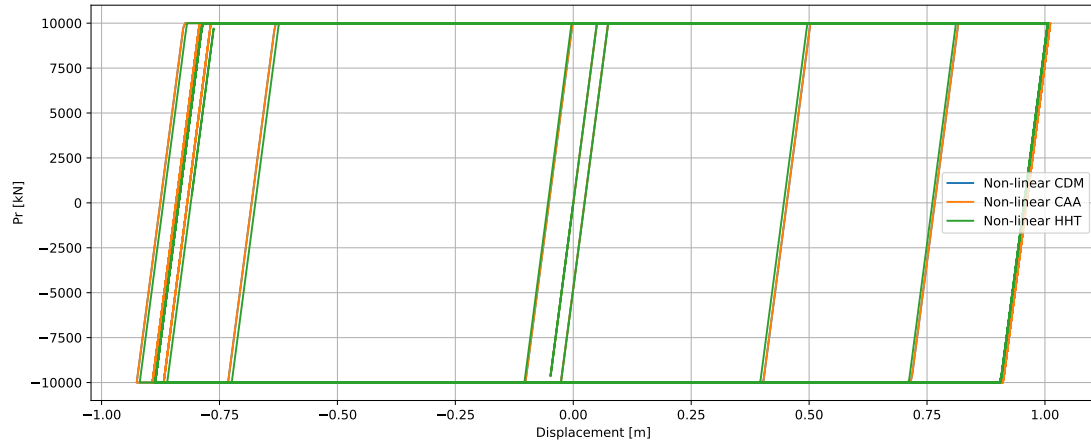


Figure 6.15: Internal forces-displacement relation computed with HybriDFEM.

A strain hardening material

The stress-strain graph of the tested material is depicted in Figure 6.16. The material has a Young's modulus of $E = 200$ MPa and a yield strength of $f_y = 10$ Mpa. Upon surpassing the yield strength limit, the material exhibits strain hardening, with the Young's modulus decreasing by a factor of $\alpha = 0.2$, becoming $E_t = \alpha E$.

Figure 6.17 presents a comparison of the displacement computed with the different non-linear algorithms implemented in HybriDFEM, for a strain hardening material. The three tested algorithms are the same from the previous material: CDM, CAA and HHT. The figure also includes the response of a linear material with $E = 200$ MPa.

As expected, all numerical solutions exhibit similar responses before surpassing the yield strength, behaving like a linear material. Surpassed the limit, all three modes seem to give similar results, similarly to the elastic–perfectly plastic material. A very slight decrease of the displacement is still visible for HHT.

Interestingly, after a certain duration, the response reverts to a linear pattern, with some residual plastic deformation. This phenomenon occurs when the material

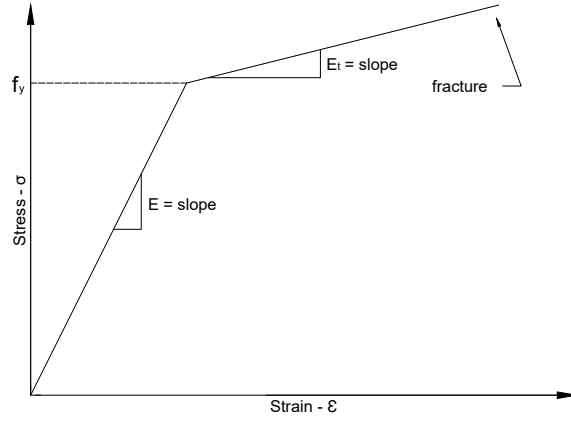


Figure 6.16: The stress-strain relation for a strain hardening material.

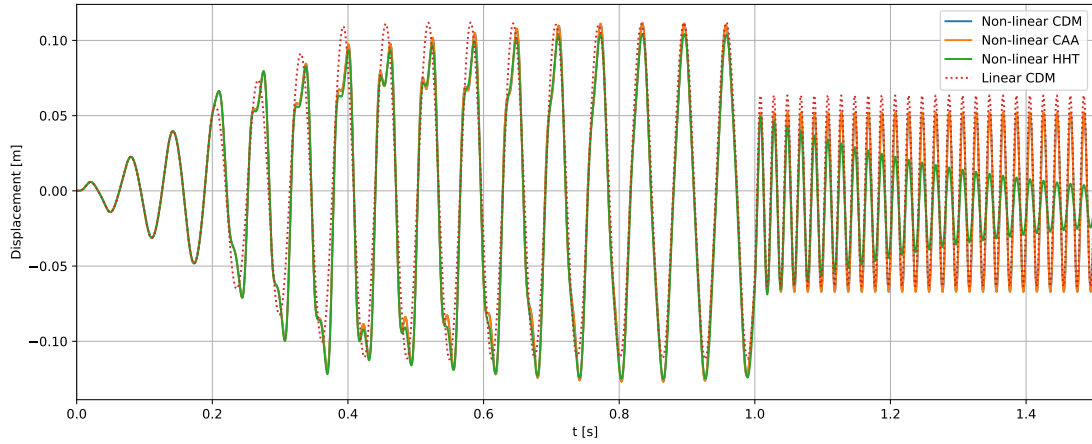


Figure 6.17: Displacement-time relation computed with HybridFEM.

transitions into the strain-hardening region and subsequently discharges, effectively increasing the yield strength. Consequently, as the applied force stabilizes, successive loadings will no longer surpass the yield strength limit, and the material will behave linearly. Once the force is removed, the system enters free vibration. As expected, CDM and CAA continue to oscillate, while the HHT response dampens. A residual plastic deformation of -0.015 m can be observed for all three methods.

Figure 6.18 reports the internal force-displacement relation for all three methods. Consistent with the aforementioned observations, CDM, CAA, and HHT all exhibit comparable responses, with HHT having slightly smaller displacements due to numerical damping. Notably, the shape of the relation closely resembles the strain hardening material law depicted in Figure 6.16, confirming again the correct

behaviour of HybriDFEM in modelling non-linear Single Degree of Freedom systems.

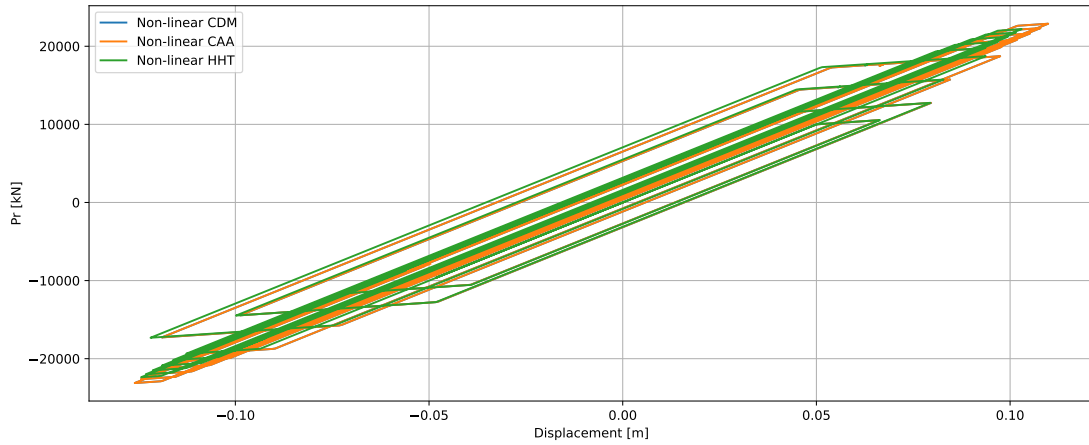


Figure 6.18: Internal forces-displacement relation computed with HybriDFEM.

6.2.2 Damped Single Degree-of-Freedom System

The benchmark features a single-degree-of-freedom system with a mass $m = 2000$ kg, as illustrated in Figure A.14. This system is subjected to the same harmonic force excitation $p(t)$ from the previous benchmark (see Figure 6.12). The natural frequency is the same as the previous benchmark, but this time explicit damping effects are present in the system. They are considered by using a damping matrix computed with the Superposition of Modal Damping Matrices method (explained in Section 5.1.3), using $\zeta = 0.05$.

Only an elastic-perfectly plastic material with residual deformations will be tested in this case.

The goal of the benchmark is to compute the displacement of the two blocks over a period $T = 1.5$ s using the implemented non-linear time stepping methods with a time step $\Delta t = 0.0001$ s. The analysis aims to identify and examine the various non-linear effects present within the system, thereby facilitating the validation of HybriDFEM.

An elastic-perfectly plastic material

The stress-strain relation of the tested material is the same as the previous benchmark and is depicted in Figure 6.13. The material has a Young's modulus of $E = 200$ MPa and a yield strength of $f_y = 10$ Mpa.

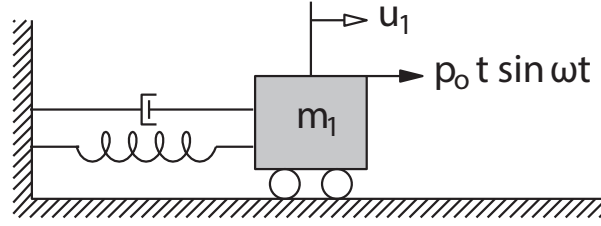


Figure 6.19: Single-degree-of-freedom system.

Figure 6.20 presents a comparison of the displacement computed with the different non-linear algorithms implemented in HybriDFEM, for an elastic–perfectly plastic material considering damping effects. The three tested algorithms are the same from the previous benchmark: CDM, CAA and HHT.

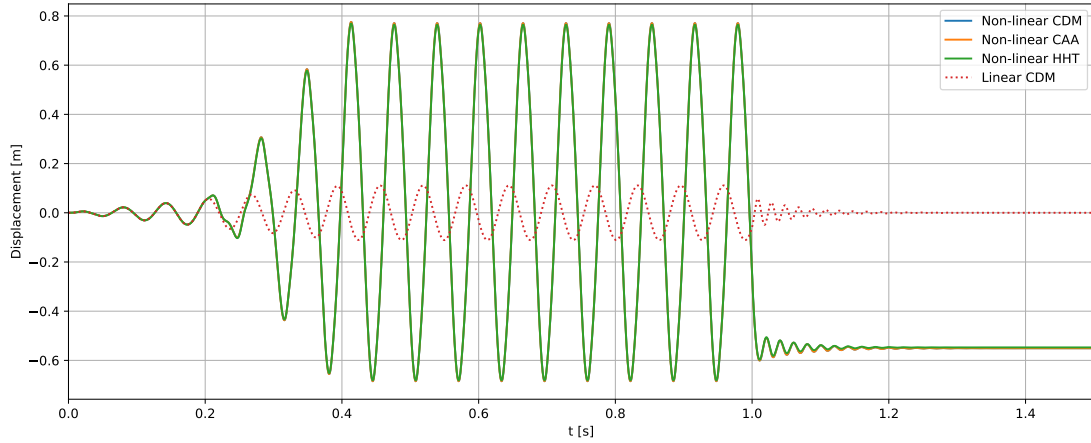


Figure 6.20: Displacement-time relation computed with HybriDFEM.

Very similar comments to the previous benchmark can also be made here. All numerical solutions exhibit similar responses before surpassing the yield strength, behaving like linear material. However, once surpassed at $t = 0.2026$ s the non-linear material's amplitude increases while its frequency decreases. CDM and CAA give exactly the same response, while HHT gives a slightly smaller amplitude, caused by the numerical damping. The biggest difference with the previous benchmark is that, here, the amplitude is 20% smaller.

Figure 6.21 reports the internal force-displacement relation for all three methods. Consistent with the upper mentioned observations, the CDM and CAA exhibit comparable responses, while HHT presents a slight decrease of the displacement.

Also in these cases, the shape of the relation closely resembles the elastic–perfectly plastic law depicted in Figure 6.13, thereby affirming the correct behaviour of HybriDFEM.

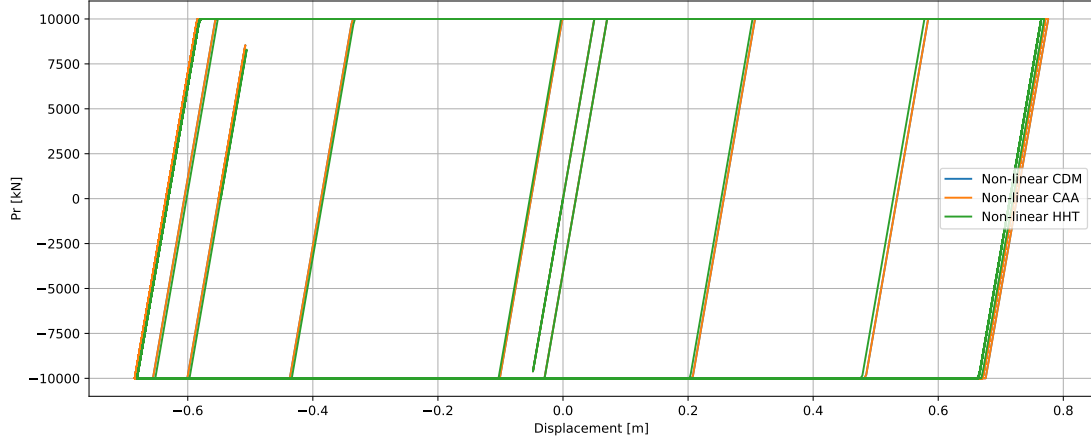


Figure 6.21: Internal forces-displacement relation computed with HybriDFEM.

A strain-hardening material with explicit damping was also tested. However, due to the similarity of the observations with the previously discussed case, it was excluded from the paper. Nevertheless, the test reaffirmed the capability and accuracy of HybriDFEM in computing non-linear SDoF systems.

6.2.3 Large Displacement Response of a URM Column

The following benchmark is based on the paper by I. Bouckaert et al. [6], which aims to capture the rocking mechanism of masonry structures using the open-source software LMGC90. The paper investigates the dynamic behaviour of an unreinforced masonry column (shown in Figure 6.22a) subjected to a two-sided horizontal base impulse acceleration, represented in Figure 6.22b. A parametric study was carried out on the column by increasing the acceleration amplitude a for a range of impulse duration t_p varying between 0.1 and 1.0 s. This study aimed to determine the failure domain of the column, identifying the pairs of values (a, t_p) that leads to the column's collapse. The results of the study are compared with the one of Dimitri et al. [17] and reported in Figure 6.23.

The numerical example consists of a column composed of 8 drums, which are modelled as rigid blocks in HybriDFEM. The structure has a total height of 5.95 m, a base width of 1.11 m and a top width of 0.9 m as visible in Figure 6.22a. The blocks have a density of $\rho = 2650 \text{ kg/m}^3$ and are connected to each other by 2 CPs, positioned at the corner of the blocks.

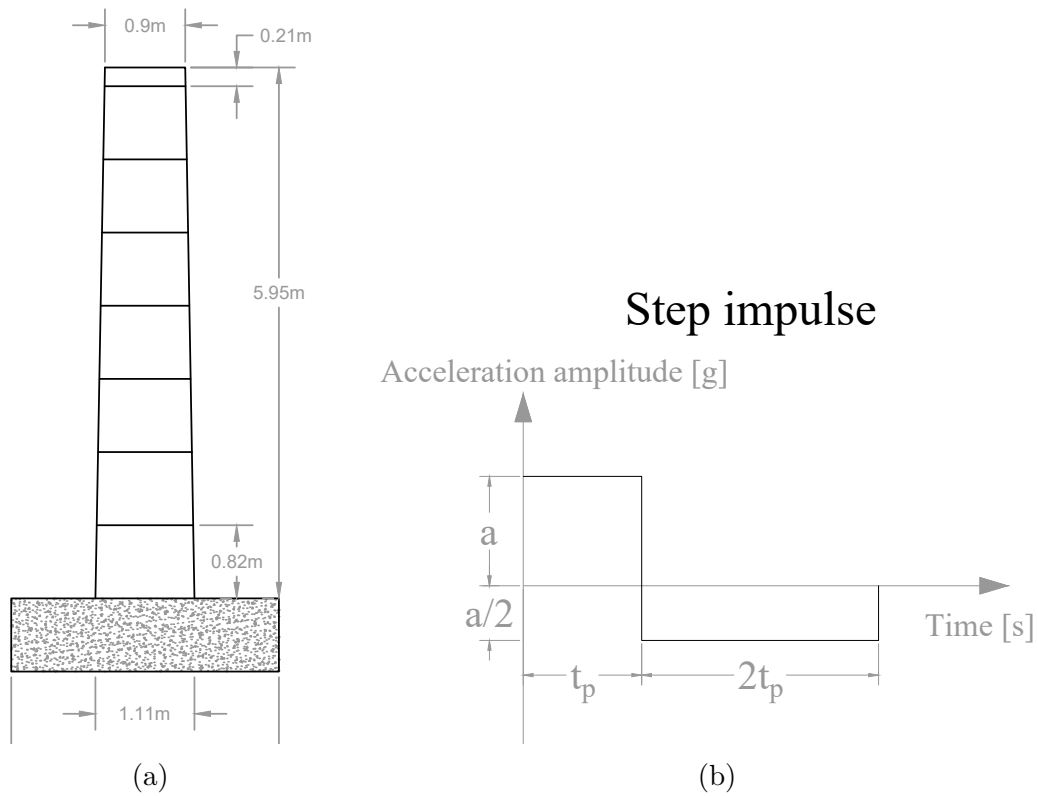


Figure 6.22: Geometry of the column (a) [6] and acceleration step impulse applied to the base of the column (b) [6].

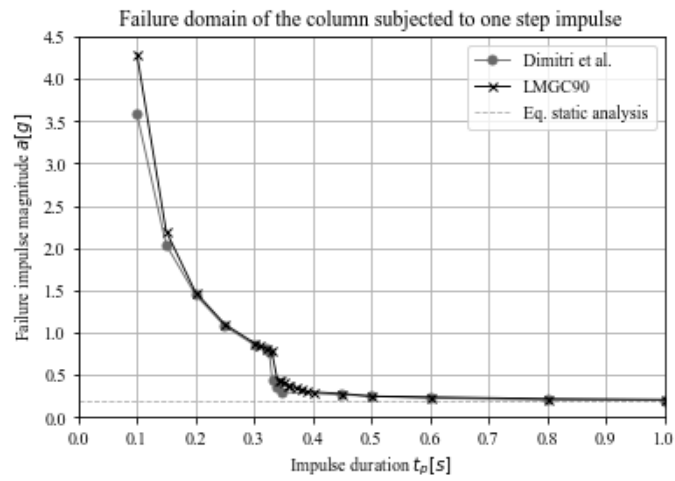


Figure 6.23: Failure domain of the column subjected to the step impulse [6].

In I. Bouckaert et al. [6], the contact between the blocks is modelled using a Coulomb friction law with a friction angle of 35° , equivalent to a coefficient of friction $\mu = 0.7$. In HybriDFEM, the coefficient was set to $\mu = \infty$ for simplicity. Given that the analysed collapse mechanisms are primarily due to the tilting of the column rather than sliding, friction effects can be considered negligible, and the results should not differ significantly.

To obtain the failure domain, the acceleration amplitude a was progressively increased until the structure failed. The failure is identified visually when at least one block detaches from the main body and fall over. From Figure 6.23, two cases were analysed, corresponding to $t_p = 0.2$ s and $t_p = 0.8$ s. All the dynamic analyses were performed using the Hilber-Hughes-Taylor method, with an $\alpha = 0.3$ and a time step $\Delta t = 0.0001$.

No information was provided on the rigidity of the contact law, so it was determined by computing the failure domain for $t_p = 0.8$ s, considering the column as a single rigid block. Using a shear and axial stiffness $k_s = k_n = 500$ MPa on the contact face, the failure of the column was attained at $a = 0.2$, which is coherent with Figure 6.23. However, the real model consists of 8 blocks, resulting in more springs and inherently, a greater flexibility. To compensate for this increase in flexibility, the stiffness was multiplied by the number of blocks, reaching $k_s = k_n = 4$ GPa.

It was observed in this phase that more flexible models, tend to collapse with a smaller acceleration amplitude a , and thus with a less significant impulse. This observation aligns with expectations, as flexible structures tend to bend more, thus reaching the point at which gravity transition from a balancing force to a collapsing force with a smaller impulse.

Considering HybriDFEM simulation with the column composed of 8 rigid blocks, for a $t_p = 0.8$ s, the failure is reached with $a = 0.207$. The result is very close to I. Bouckaert et al. results, which is equal to 0.2, the slight difference could be adjusted by tuning better the stiffness of the system, making it slightly more flexible. The failure mode obtained is depicted in Figure 6.24c, which closely resembles the one reported by I. Bouckaert et al. in Figure 6.24f. In this mode, the full column tilts to the right. Additionally, Figure 6.24 includes several other frames from the simulation.

Considering HybriDFEM simulation with the column composed of 8 rigid blocks, for a $t_p = 0.2$ s, the failure is reached with $a = 1.51$. The result is, again, very close to I. Bouckaert et al. results, which is equal to 1.48-1.5, the slight difference is likely due to the same previously mention reasons. The failure mode obtained is depicted in Figure 6.25d, which closely resembles the one reported by I. Bouckaert et al. in Figure 6.25f. In this mode, the column tilts to the side and only a portion of

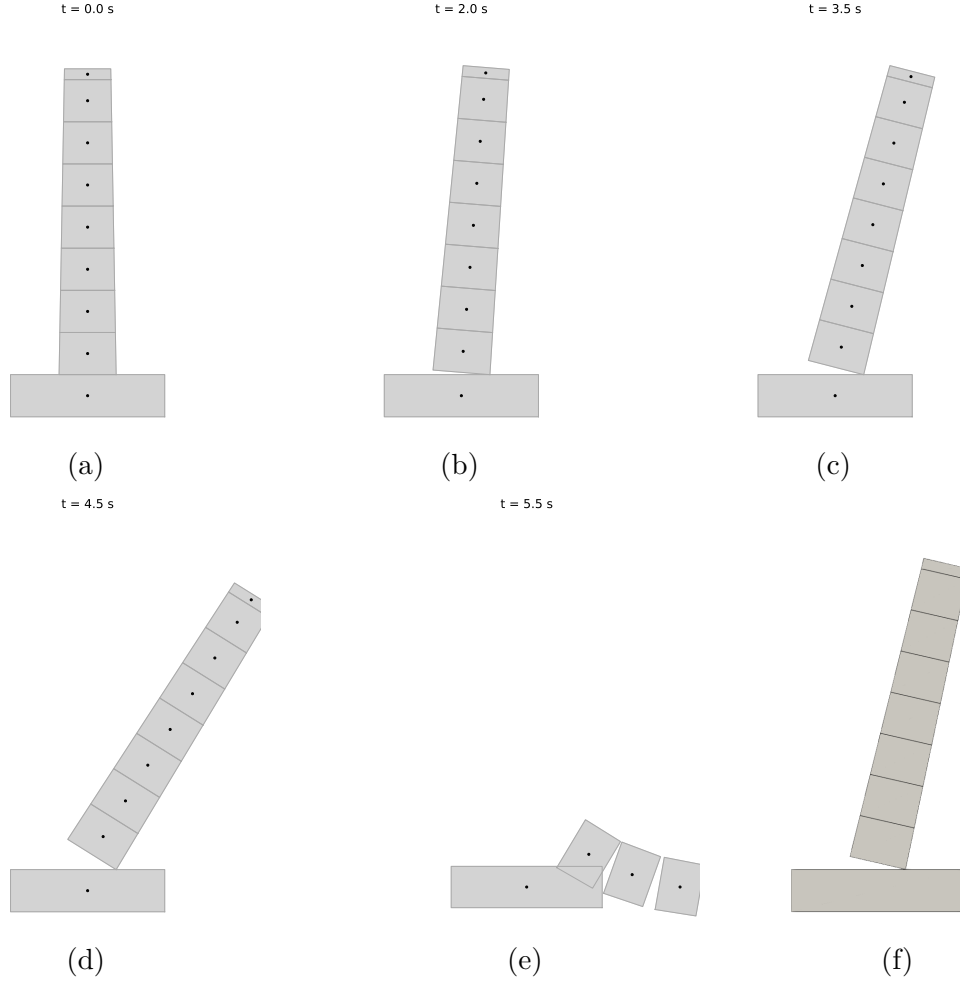


Figure 6.24: Different frames from the HybriDFEM simulation (a, b, c, d, e) and the expected failure mode for $t_p = 0.8$ and $a = 0.2$ (f) [6].

the blocks is involved in the failure mechanism. Additionally, Figure 6.25 includes several other frames from the simulation.

During the various simulations, it was noted that HybriDFEM, when employing a Coulomb friction law, which is a no-tension law, occasionally encounters convergence difficulties. When the stiffness of a CP becomes 0, so in the instance where a block start detaching from another, the local stiffness matrix is badly conditioned. During the assembly of the global matrix, the bad condition can propagate to the global matrix, making it very difficult to be inverted, and blocking HybriDFEM convergence to a solution.

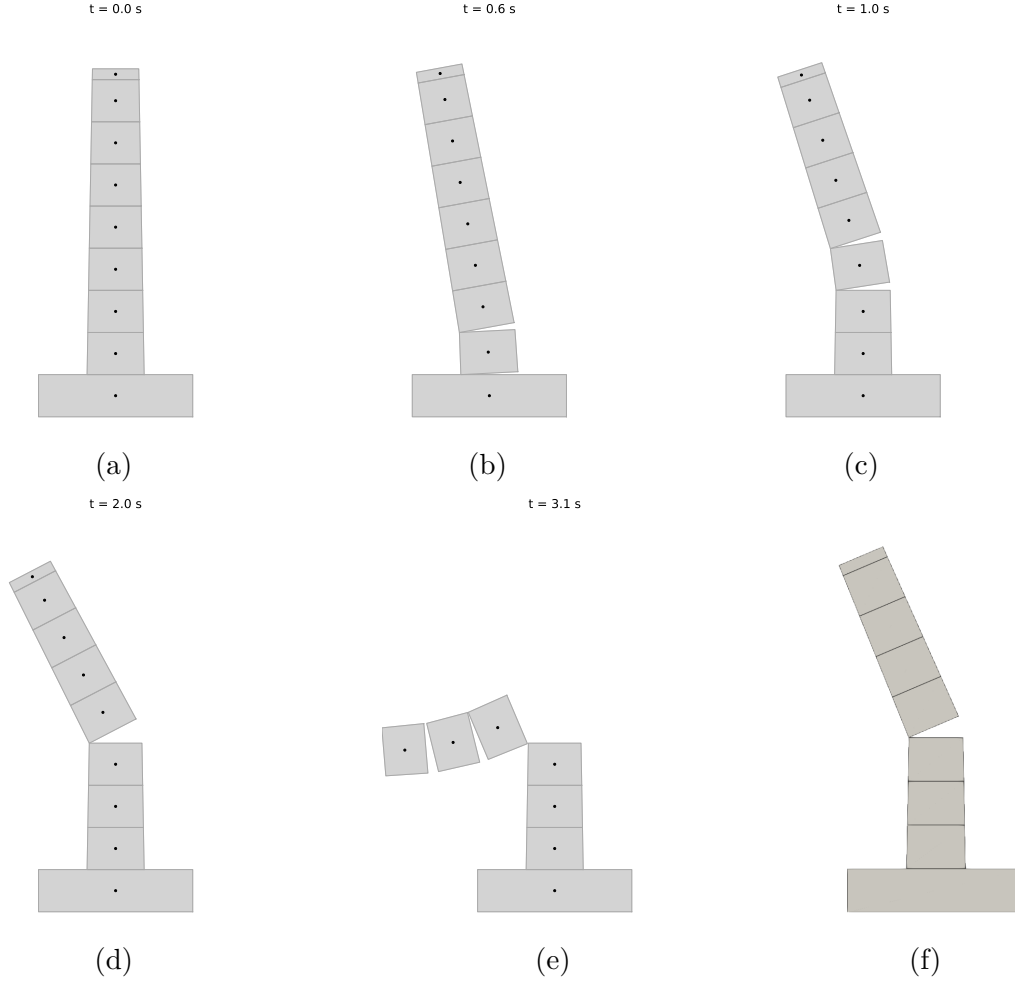


Figure 6.25: Different frames from the HybriDFEM simulation (a, b, c, d, e) and the expected failure mode for $t_p = 0.2$ and $a = 1.5$ (f) [6].

However, it was observed that maintaining the elastic behaviour in shear, while using the same material, significantly improved the algorithm's performance, leading to a more consistent convergence of the method, although causing some differences in the response. This hypothesis can be considered valid as the problem is governed by bending, so assigning small shear stiffness k_s won't change the solution, but allow HybriDFEM to converge. This can be confirmed by the internal force in the spring, which, although not zero, is extremely weak, being approximately 10^{-8} times smaller than other internal forces. However, future works should focus on adapting the method to this type of contact law, to model more accurately the problem.

6.2.4 Section Summary and Reflections

In summary, HybriDFEM demonstrates robust performance in the case of non-linear systems. While these validations provide a strong initial foundation, the breadth of the subject suggests that further exploration is necessary for comprehensive validation. It is imperative to not only conduct additional testing but also adapt the method to better suit specific non-linear problems.

The response of a damped and an undamped SDoF systems subjected to growing harmonic force excitation was computed using HybriDFEM. The transition from elastic to plastic (or hardening) zones was discussed, along with the behaviour of the response during free vibration. By examining the internal force-displacement relationship, the shape of the stress-strain relation could be identified, confirming the accuracy and correctness of the HybriDFEM method. For damped systems or algorithms experiencing numerical damping, a damping of the response was observed during the free vibration phase.

A URM column subjected to a step impulse was analysed, revealing two expected failure modes. Remarkably, there was a striking similarity between the failure parameters computed with HybriDFEM and those reported in the paper. However, it was observed that the no-tension contact law occasionally hindered the convergence of the method. Although a partial solution was identified to mitigate this issue, the subject should be further explored. Therefore, future research efforts should prioritize adapting the method to more accurately accommodate this type of contact law.

Chapter 7

Conclusion

7.1 Conclusion

The Hybrid Discrete-Finite Element Method (HybriDFEM) offers an innovative approach to structural analysis. It effectively handles discontinuities in structures, akin to the Discrete Element Method, while also offering the possibility of linking the method with finite elements for potential hybrid applications. This greatly mitigates the typical high computational demand of DEM, offering a precise yet efficient model.

The goal of this master's thesis was to extend the HybriDFEM method to dynamic analysis, further enhancing its capabilities and positioning it closer to other commercially used methods.

A formulation for computing the mass matrix as well as the natural frequencies and mode shapes within the model was developed, representing a significant advancement given the limited information and researches available on modal superposition for discrete systems.

Numerous numerical time-stepping algorithms, like the Central Difference Method, the Newmark- β or the Generalized- α just to name a few, were implemented, enabling HybriDFEM to compute the response of both linear and non-linear, damped and undamped systems, as well as incorporating elements such as numerical damping. This is, another, very promising advancement, as it opens the doors on dynamic analysis, allowing computing the response of masonry walls or any other structure, when subjected to dynamic forces, like earthquakes, waves, currents or powerful winds.

All implemented functions were rigorously tested to assess their correctness and performance. Natural frequencies and mode shapes were evaluated for both continuous and discrete materials, ranging from simple cases like beams to more complex structures such as arches and walls. Special attention was given to testing beam elements due to their prevalence in the construction field. The various time-stepping methods were also tested, yielding very promising results. In instances where full validation was not feasible due to some discrepancies with the tested benchmark, special care was taken to justify these differences or identify sources of error.

In summary, HybriDFEM, despite its novelty, is a highly promising model. Thanks to the work presented in this thesis, it is now capable of computing dynamic problems, bringing it one step closer to competing with existing models.

7.2 Future Developments

Given the broad scope of the subject and the novelty of the method, numerous future developments are possible. To maintain consistency with this thesis, only advancements related to dynamic analysis will be discussed.

Firstly, despite the extensive benchmarking performed, more complex and specific cases should be tested. Examples include tapered beams, where the height evolves non-linearly, composite beams, reinforced concrete elements, and orthotropic materials like wood. Additionally, testing on a larger scale could be very insightful. For example, analyzing a real façade with a reinforced concrete frame and masonry walls, similar to those found in seismic zones. Further, delving into the modelling of cracks, such as cracked reinforced beams or cracked masonry façades, could be explored. This could be further enriched by performing dynamic analysis to identify potential collapses or failure mechanisms. Moreover, while the different algorithms were only tested for harmonic or constant forces, subjecting a structure to real earthquake recordings would be interesting to evaluate how HybriDFEM performs under such conditions.

The mass matrix could be further generalized by developing a formulation capable of computing the mass and rotational inertia for "circular" elements.

The damping matrix implementation could also be extended by considering non-classical damping or by modeling the damping as a series of dash-pots at the contact face level.

All the implemented methods could benefit from space optimization, particularly in managing matrix operations. Stiffness, mass, and damping matrices all are sparse matrices, which means that a big part of the matrix is composed of zeros.

Numerous techniques described in the literature offer ways to better define the matrices and fasten these operations. One straightforward optimization involves storing the mass matrix, which, in the current implementation, is always diagonal, in a single vector. This would reduce the memory needs from $O(N^2)$ to $O(N)$.

Finally, the current setup saves all simulation data solely in the RAM, requiring a complete restart of the simulation if one wants to revisit the simulations at a later time. To address this issue, elements such as the displacement and internal forces vectors could be saved in CSV files for future retrieval. This approach would also reduce RAM usage by accessing only the necessary elements when displaying a particular graph.

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Appendices

Appendix A

Further testing

This annex primarily focuses on additional tests conducted to evaluate the performance of HybriDFEM. As the structure and outcomes closely resemble examples from the main text, they are presented here as annexes.

A.1 Clamped Euler-Bernoulli Beam

This appendix is a complementary example for section 4.3.1. The numerical example consists of a clamped beam (Figure A.1) of length $L = 5$ m and rectangular cross-section of dimension $h \times b = 0.4$ m $\times$ 0.2 m. It is composed of a homogeneous linear elastic material with $E = 210$ GPa, $\rho = 7850$ kg/m³ and $\nu = 0.3$. Shear deformations and rotational inertias are neglected according to the Euler-Bernoulli beam hypothesis.

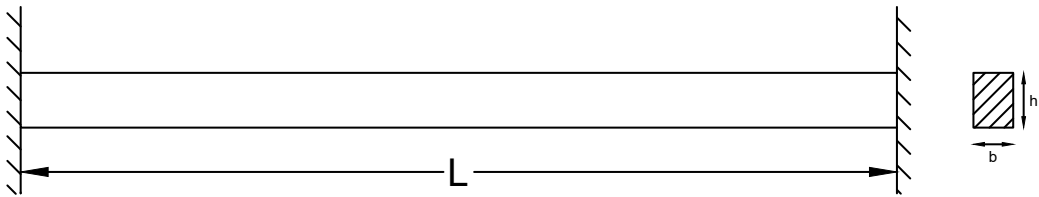


Figure A.1: Example of clamped beam.

The benchmark consists of computing the natural frequencies and mode shapes for such beam using HybriDFEM with different levels of discretization, both at the block and CP level.

Comparisons between the first ten natural frequencies obtained with HybriDFEM and the analytical solution (Equation 4.3.1.1) are shown in Figure A.2, with the mode number on the abscissa and the percentage error on the ordinate, computed as the difference between the numerical and analytical frequencies divided by the latter.

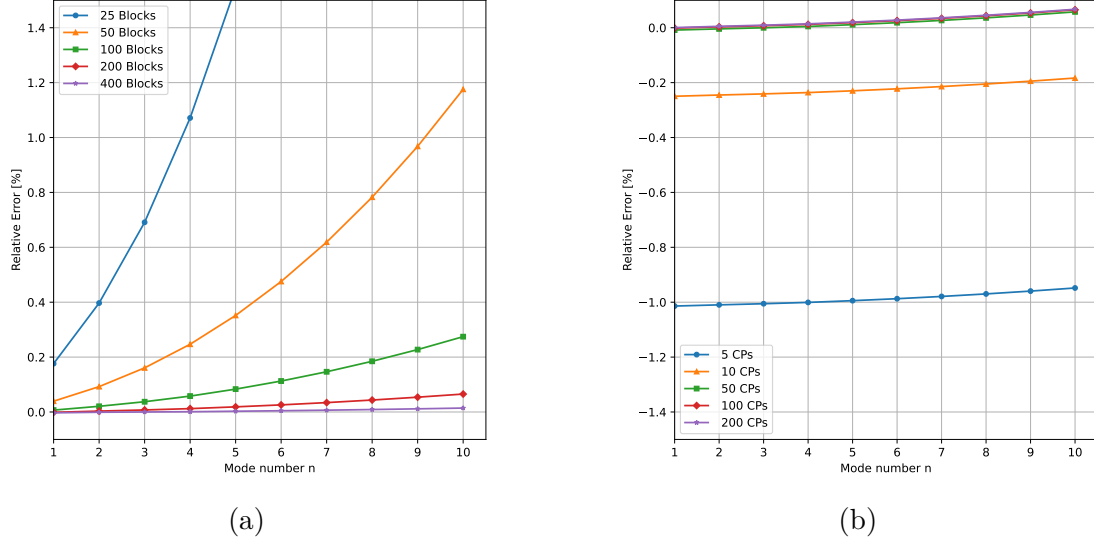


Figure A.2: Influence of the block (a) and CP (b) discretization on the first 10 frequencies. On (a) the CP number is fixed at 100, while on (b) the number of blocks is fixed at 200.

For Figure A.2a, computed with 100 CPs, the error of the finest discretization, composed of 400 blocks, reaches -0.003% for the first frequency, while 0.014% for the tenth. The error for the coarser discretization, composed of 25 blocks, reaches 0.18% for the first mode, while 5.91% for the tenth.

For Figure A.2b, computed with 200 blocks, the error of the finest discretization, composed of 200 CPs, reaches 0.000% for the first frequency, while 0.067% for the tenth. The error for the coarser discretization, composed of 25 blocks, reaches -1.014% for the first mode, and -0.948% for the tenth.

Both types of discretizations show a clear convergence to the analytical solution. Observations similar to those for the simply supported beam apply here as well. Notably, in the case of CPs discretization, the errors for both simply supported and clamped beams are relatively similar. In contrast, with block discretization, the

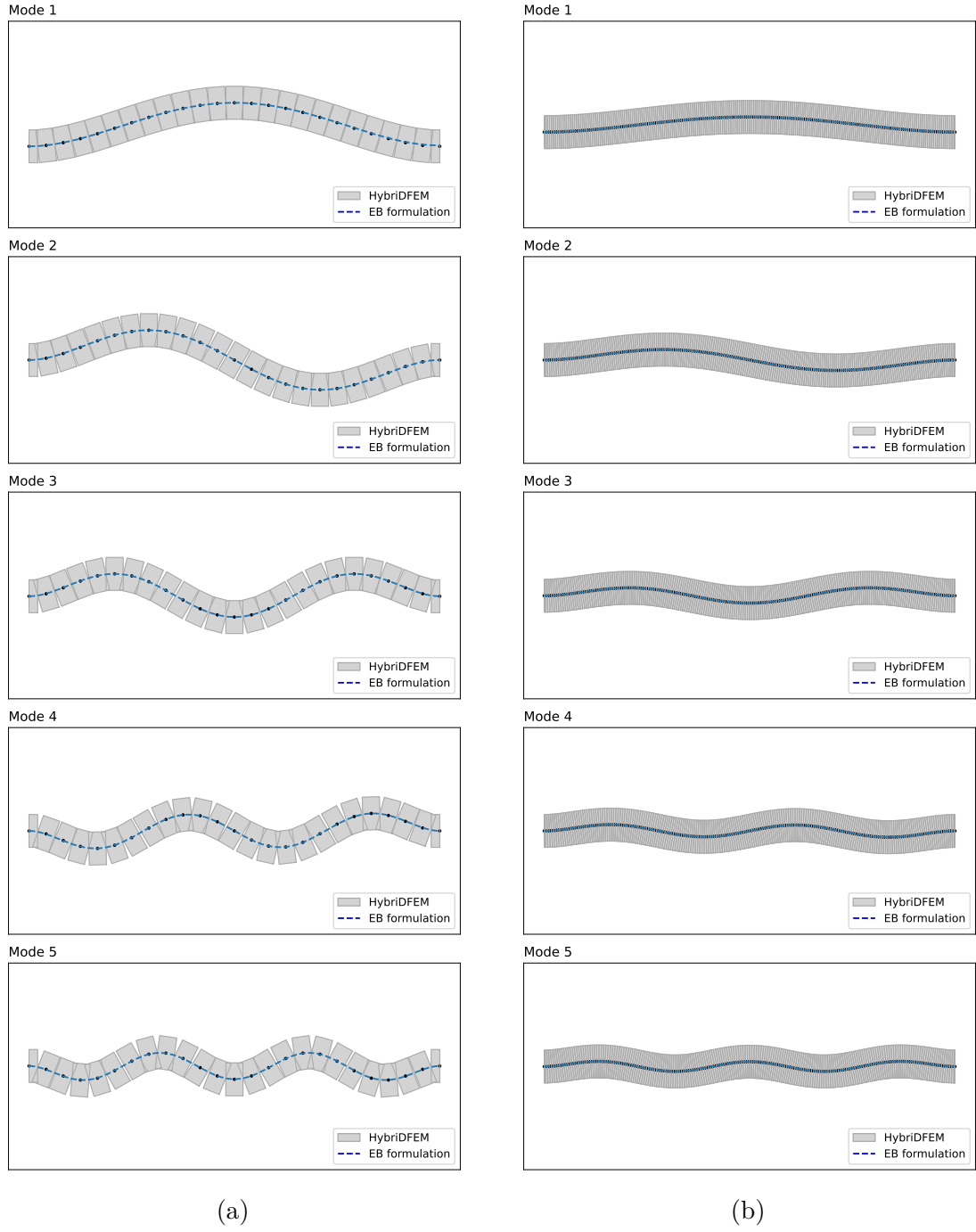


Figure A.3: Comparison between HybriDFEM and Euler Bernoulli first 5 mode shapes. A coarse (a) and fine (b) block discretization were used, with 100 CP for both cases.

clamped beam converges more slowly towards the analytical solution. This suggests that for this beam type, the mass distribution probably plays a more significant role. The difference could also be attributed to the increased complexity of the boundary conditions. As shown in Table 4.6, the frequency equation for a clamped beam is much more complex than that for a simply supported beam. Consequently, a finer block discretization is necessary to accurately represent the model and correctly approximate the natural frequencies.

The first five mode shapes obtained with HybriDFEM are compared with the analytical solution from Table 4.6 and are visible on Figure A.3. The number of CP was kept at 100 as they don't influence the mode shapes. It's visible on Figure A.3a that, even with a coarse block discretization of 25 blocks, HybriDFEM fits really well with the analytical solution. Using a finer block discretization, as the one from Figure A.3b, where 200 blocks were used, will cause an increase in the number of points on the curve which will allow for a better approximation of the shape.

A.2 Clamped Timoshenko-Ehrenfest Beam

This appendix is a complementary example for Section 4.3.2. As it was explained in the thesis, no analytical formula was given, so the geometric and material data are the same as the example case in Cazzani et al. [13]. The numerical example consists of a clamped beam (Figure A.4) of length $L = 2$ m and rectangular cross-section of dimension $h \times b = 0.1$ m $\times$ 0.1 m. It is composed of a homogeneous linear elastic material with $E = 260$ GPa, $\rho = 8000$ kg/m³ and $\nu = 0.3$. Shear deformations and rotational inertias are neglected according to the Euler-Bernoulli beam hypothesis.

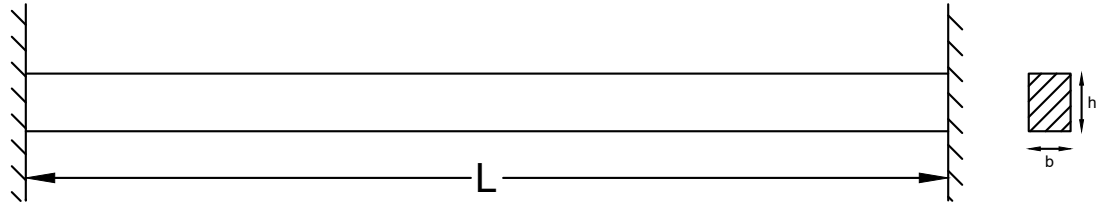


Figure A.4: Example of clamped beam.

The benchmark consists of computing frequencies and mode shapes for such beam using HybriDFEM with different levels of discretization, both at the block and CP level. The beam is computed with the shear correction factor for a rectangular

section established by Goens (1975) [21]:

$$\kappa = \frac{5}{6} \quad (\text{A.2.1})$$

The first ten natural frequencies obtained with HybriDFEM are compared with the analytical solution from Equation 4.3.1.1, as shown in Figure A.5. Additionally, the same comparison for the natural frequencies 20 to 29 is reported in Figure A.6.

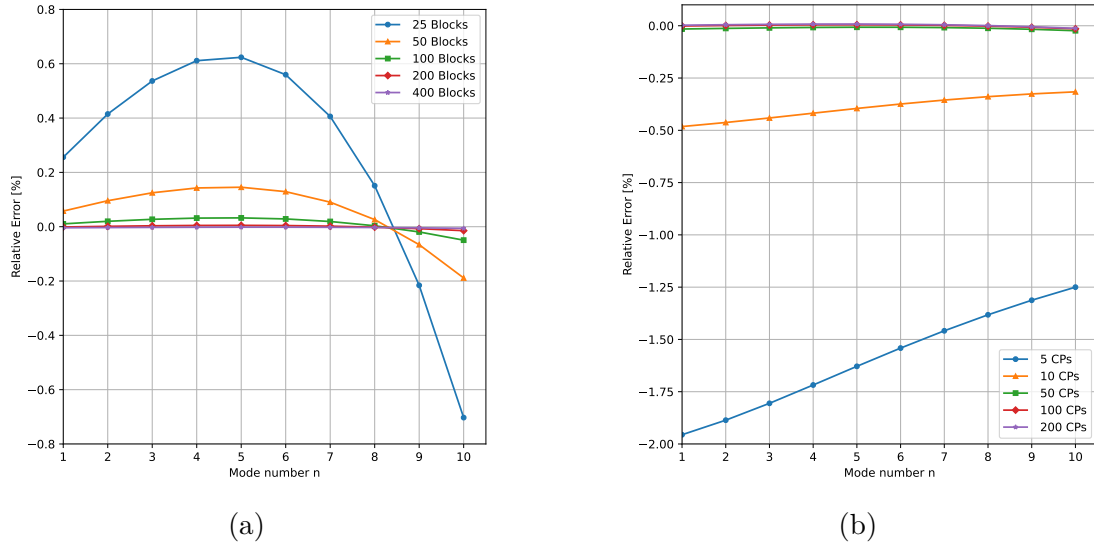
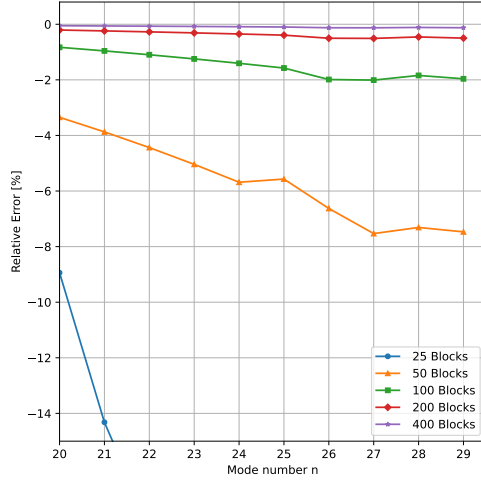


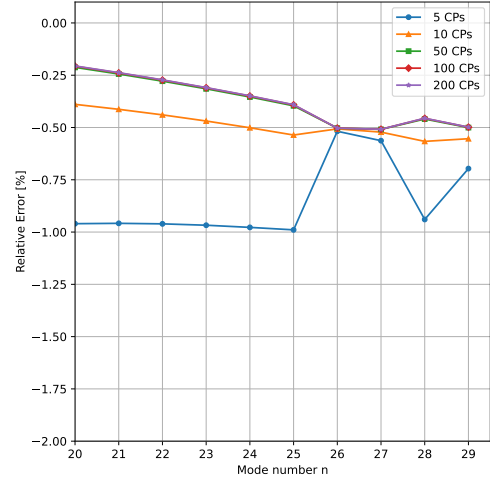
Figure A.5: Influence of the block (a) and CP (b) discretization on the first 10 frequencies. On (a) the CP number is fixed at 100, while on (b) the number of blocks is fixed at 200.

For Figure A.5a, computed with 100 CPs, the error of the finest discretization, composed of 400 blocks, reaches -0.003% for the first frequency, while -0.005% for the tenth. The error for the coarser discretization, composed of 25 blocks, reaches 0.256% for the first mode, while -0.702% for the tenth. The shape is clearly different from what was observed in a simply supported beam (see Figure). This is probably caused by the lack of frequencies from the shear-propagation spectrum. This is confirmed by observing Figure A.6a, where the shape resembles more that of simply supported beams. As expected, the relative error becomes extremely high for coarser discretizations but remains somewhat acceptable for finer ones.

For Figure A.5b, computed with 200 blocks, the error of the finest discretization, composed of 200 CPs, reaches 0.002% for the first frequency, while -0.012% for the tenth. The error for the coarser discretization, composed of 25 blocks, reaches -1.955% for the first mode, and -1.25% for the tenth.



(a)



(b)

Figure A.6: Influence of the block (a) and CP (b) discretization on higher frequencies. On (a) the CP number is fixed at 100, while on (b) the number of blocks is fixed at 200.

The transition frequency in this case is $\tilde{\omega} = 111.803$ rad/s, which is located between the 25th and 26th modes and is not part of the spectrum. It could be argued that the behaviour of the 26th mode resembles that of a transition frequency, as it appears to have the same natural frequency for any CP discretization when observed by the naked eye. However, this is not true. For a simply supported beam, the difference was on the order of 10^{-16} %, while here it is just 0.01% between a 5 and 10 CP discretization. Additionally, by observing the mode shapes (see Figure A.7), the mode is clearly not a pure-shear vibration mode.

For both types of discretizations, convergence to the analytical solution is clearly visible. Similar observations to those made for the simply supported beam can be applied. However, a direct comparison between the two is not entirely possible, as the geometric and material data are different.

Figures A.7a and A.7b show the transverse displacement of mode shapes 1 to 5 and 26 to 30, computed with HybriDFEM (100 blocks $\times$ 200 CPs). However, it is quite difficult to compare these results with the analytical solution due to the small scale.

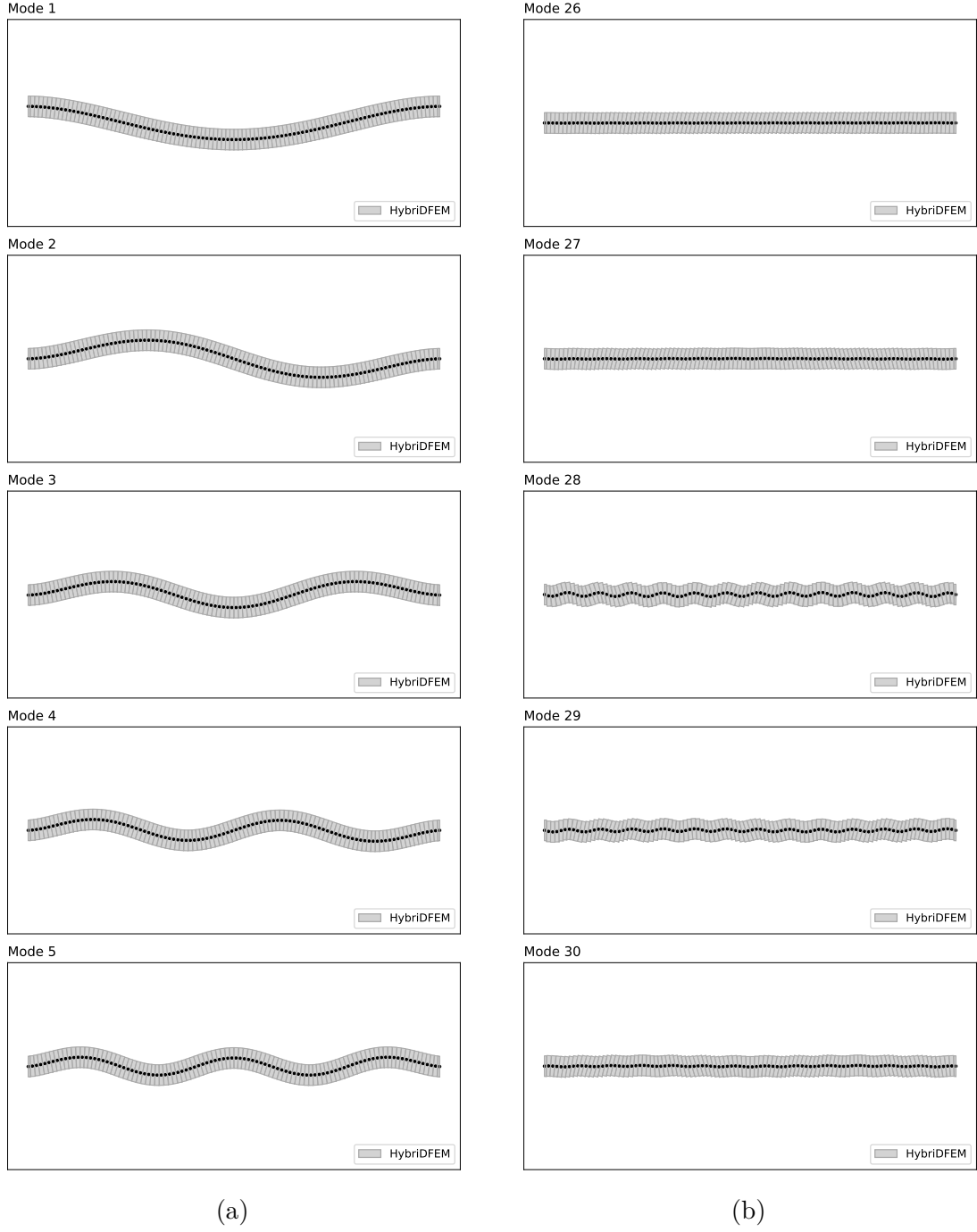
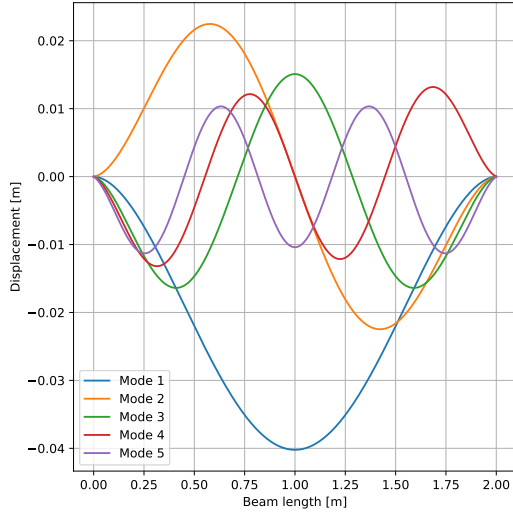
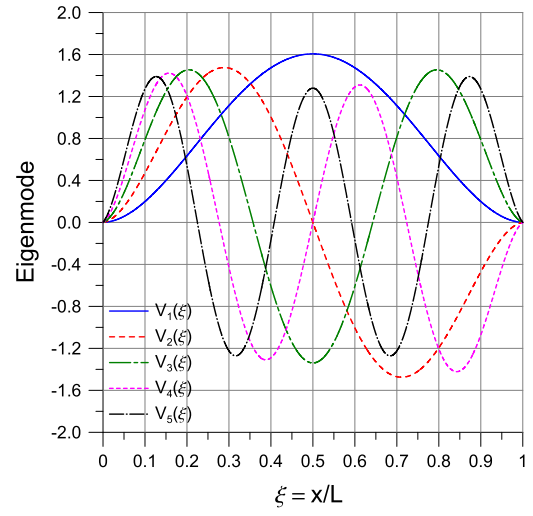


Figure A.7: HybriDFEM mode shapes (a) 1 to 5 and (b) 26 to 30. 100 blocks $\times$ 200 CPs.

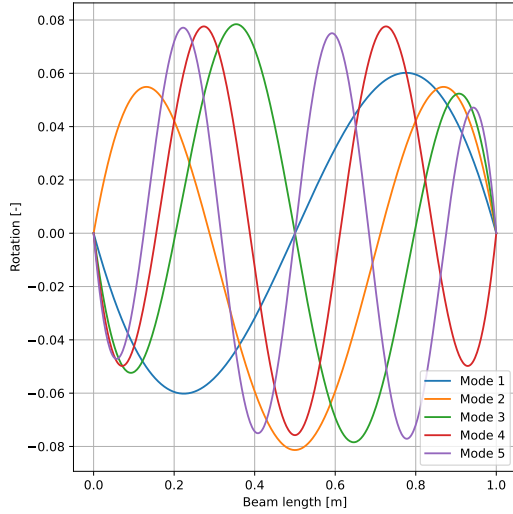


(a)

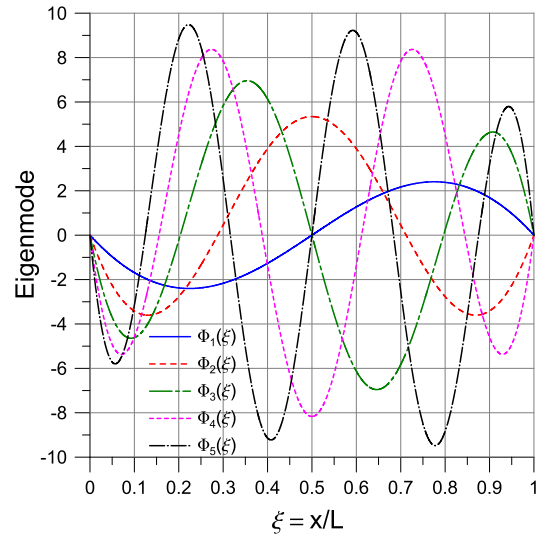


(b)

Figure A.8: Comparison between HybriDFEM and Timoshenko transverse displacement of the first five mode shapes. 100 blocks $\times$ 200 CPs.

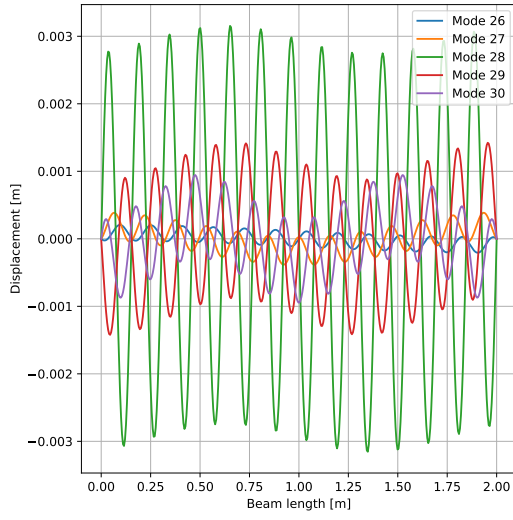


(a)

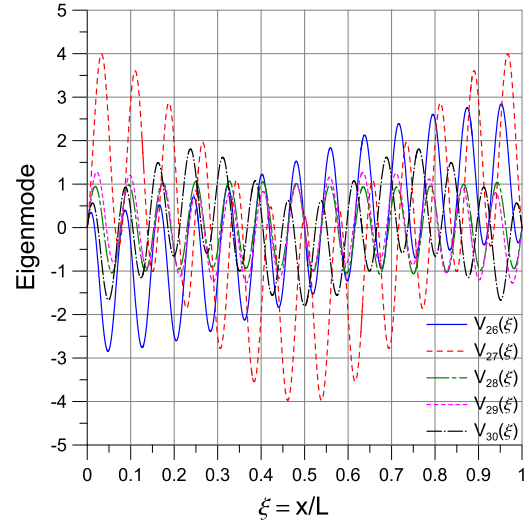


(b)

Figure A.9: Comparison between HybriDFEM and Timoshenko rotation of the first five mode shapes. 100 blocks $\times$ 200 CPs.

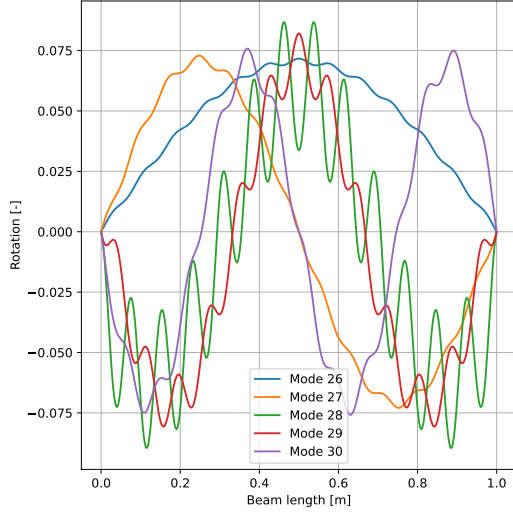


(a)

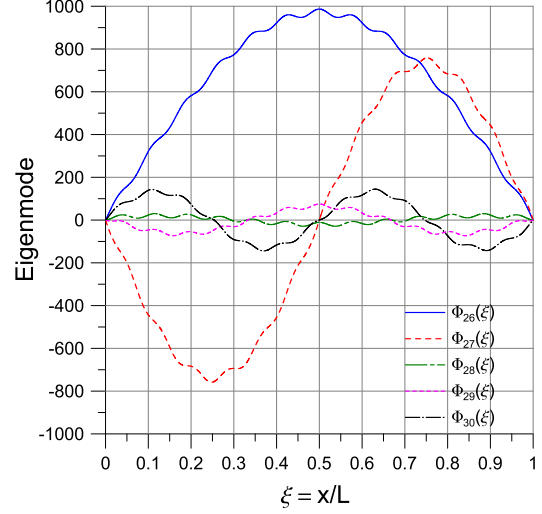


(b)

Figure A.10: Comparison between HybriDFEM and Timoshenko transverse displacement of mode shapes 26 to 30. 100 blocks $\times$ 200 CPs.



(a)



(b)

Figure A.11: Comparison between HybriDFEM and Timoshenko rotation of mode shapes 26 to 30. 100 blocks $\times$ 200 CPs.

Figures A.8, A.9, A.10 and A.11 provide a clearer comparison between the HybriDFEM and analytical mode shapes. Both the transverse displacement and rotation are shown for modes 1 to 5 and 26 to 30. The HybriDFEM mode shapes appear to match the analytical solutions quite well. However, this is somewhat difficult to observe due to the different amplitudes. Given the absence of detailed information in the paper by Cazzani et al., a precise comparison, like the one made for the simply supported beam, cannot be made.

As it was mentioned before, unlike the simply supported case, modes from the bending-dominated spectrum do not appear in the shear-dominated spectrum.

A.3 Simply Supported Neves Beam

This appendix is a complementary example for section 4.3.4. The numerical example consists of a simply supported beam of length $L = 8$ m and rectangular cross-section of dimension $h \times b = 0.22$ m $\times$ 0.06 m. It is composed of a homogeneous linear elastic material with $E = 210$ GPa and $\rho = 7800$ kg/m³. Shear deformations are neglected, while the rotational inertias are not.

The benchmark consists of comparing HybriDFEM (with 100 CPs) and Neves model. Both beams are analysed for different block discretizations. Following a similar approach to Neves' paper, the results are compared to the Euler-Bernoulli beam theory.

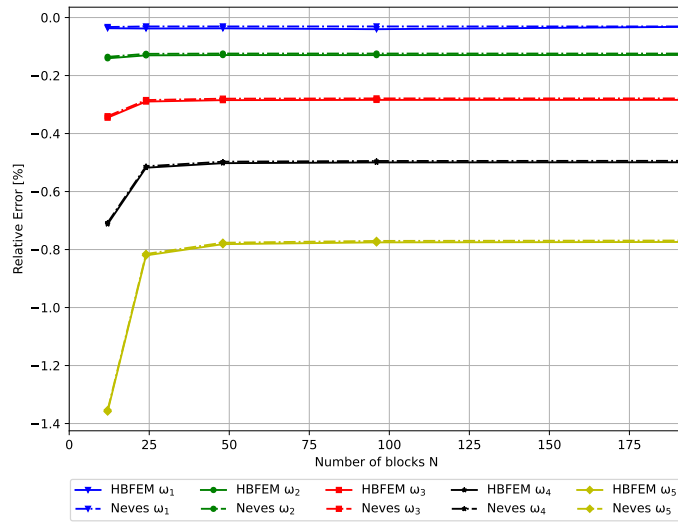


Figure A.12: Comparison between Neves and HybriDFEM models for different block discretizations.

The results are visible in Figure A.12. The abscissa reports the block discretization while the ordinate the percentage error, corresponding to the difference between the numerical and analytical frequency, divided by the latter.

Even though the hypotheses are slightly different, the HybriDFEM beam and Neves' model yield exactly the same results. This is likely because the only differences are located at the supports, unlike in the cantilever case.

The error of the finest discretization, 384 blocks, reaches -0.032% for the first frequency and -0.972% for the fifth. The error for the coarser discretization, composed of 24 blocks, reaches -1.35% for the first mode and -0.77% for the fifth.

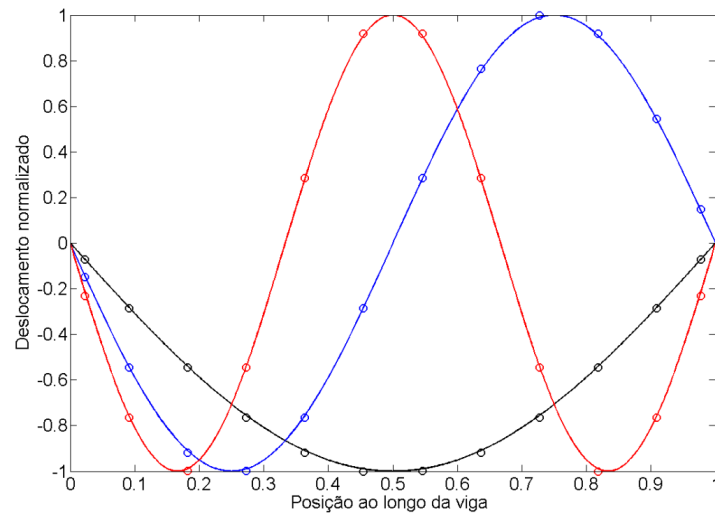


Figure A.13: Comparison between NEVES formulation and Euler Bernoulli first 3 mode shapes. A coarse block discretization was used.

○ Numerical first mode — Euler-Bernoulli first mode
 ○ Numerical second mode — Euler-Bernoulli second mode
 ○ Numerical third mode — Euler-Bernoulli third mode

In Section 4.3.1, HybriDFEM was compared with the Euler-Bernoulli analytical mode shapes and produced very good results. Neves' formulation for simply supported beams, unlike for cantilever beams, also yields good results. This is likely due to the accurate definition of the beam length and the absence of the mass overestimation that was present in the cantilever case.

A.4 Undamped Two-Degree-of-Freedom Non-Linear System

This appendix is a complementary example for Section 6.2.

The benchmark features a two-degree-of-freedom system with a mass $m = 2000$ kg, as illustrated in Figure A.14. The first mass m_1 is subjected to a harmonic force excitation $p(t) = (p_0 t) \sin(\omega t)$ where the force increases over time until reaching 20000 kN. After 1 second, the force is abruptly halted, initiating free vibration within the system. This excitation allows the observation of the system's behaviour within and beyond the yield strength of the material, encompassing responses to both harmonic force and free vibration, as illustrated in Figure 6.12.

The natural frequency of the system are $\omega = 195.43, 511.66$ rad/s, while the parameters of the applied force are $\omega = 100$ rad/s and $p_0 = 5 \times 10^4$ kN. No explicit damping effects are present in the system.

Two distinct material laws will undergo testing: an elastic–perfectly plastic material and a strain-hardening material. Both material laws exhibit residual deformations.

The goal of the benchmark is to compute the displacement of the two blocks over a period $T = 1.5$ s using the implemented linear time stepping methods with a time step $\Delta t = 0.0001$ s. The analysis aims to identify and examine the various non-linear effects present within the system, thereby facilitating the validation of HybriDFEM.

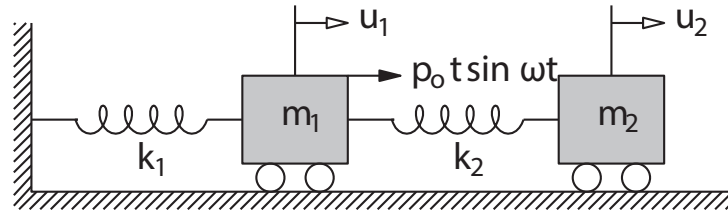


Figure A.14: Two-degree-of-freedom system.

An elastic–perfectly plastic material

The stress-strain relation of the tested material is depicted in Figure 6.13. The material has a Young's modulus of $E = 200$ MPa and a yield strength of $f_y = 10$ Mpa. Once yield strength f_y is reached, the material behaves as plastic, flowing at a constant stress f_y .

Figures A.15 and A.16, presents a comparison of the displacement computed with HybriDFEM for an elastic–perfectly plastic material using different non-linear algorithms. The three algorithms subjected to testing are CDM, CAA and HHT with $\alpha = 0.3$, $\gamma = 0.8$ and $\beta = 0.422$. Additionally, the figure includes the response of a linear material with $E = 200MPa$.

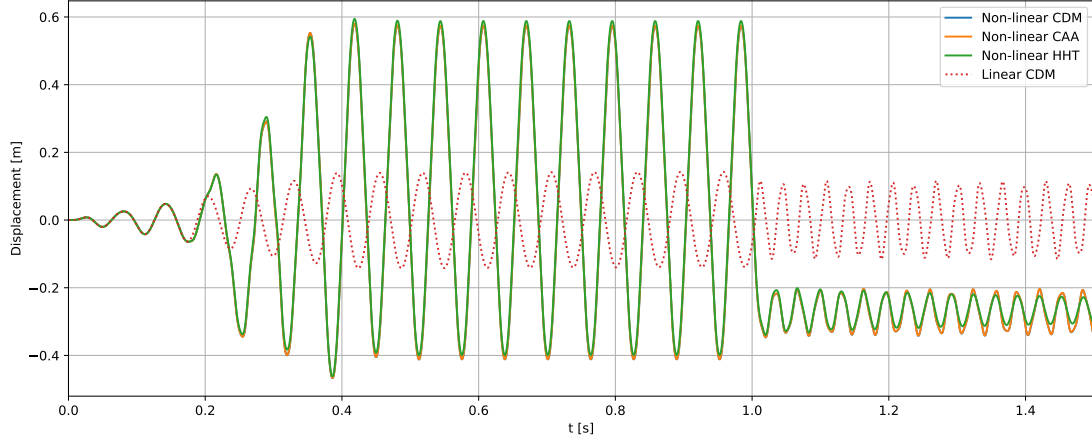


Figure A.15: Displacement of the first block in the horizontal direction $u_1(t)$.

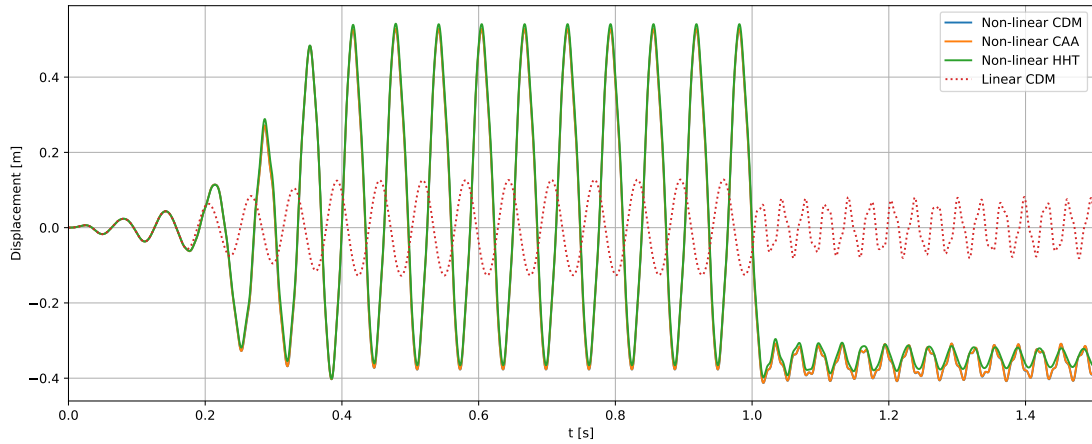


Figure A.16: Displacement of the second block in the horizontal direction $u_2(t)$.

All numerical solutions exhibit similar responses before surpassing the yield strength, behaving akin to a linear material. However, once surpassed, the non-linear material's amplitude increases while its frequency decreases. The CDM and CAA methods give similar results, while the HHT, although having the same frequency, has a slightly smaller amplitude, caused by its numerical damping. As the time step Δt is rather small, this decrease is very small.

Upon removal of the force, the system transitions into free vibration. As expected, CDM and CAA continue oscillating, while the HHT response dampens. A residual plastic deformation of ≈ 0.4 m is clearly visible across all three methods.

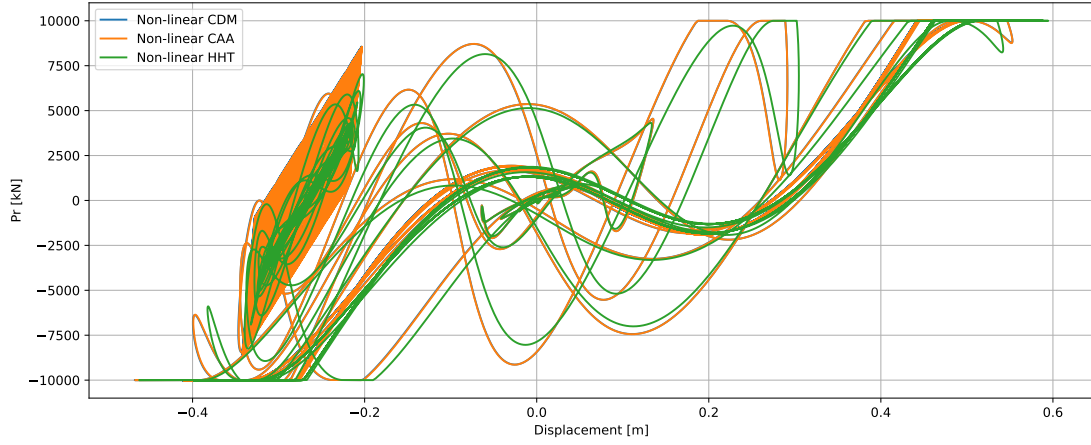


Figure A.17: Internal forces-displacement relation computed with HybriDFEM relative to the first mass.

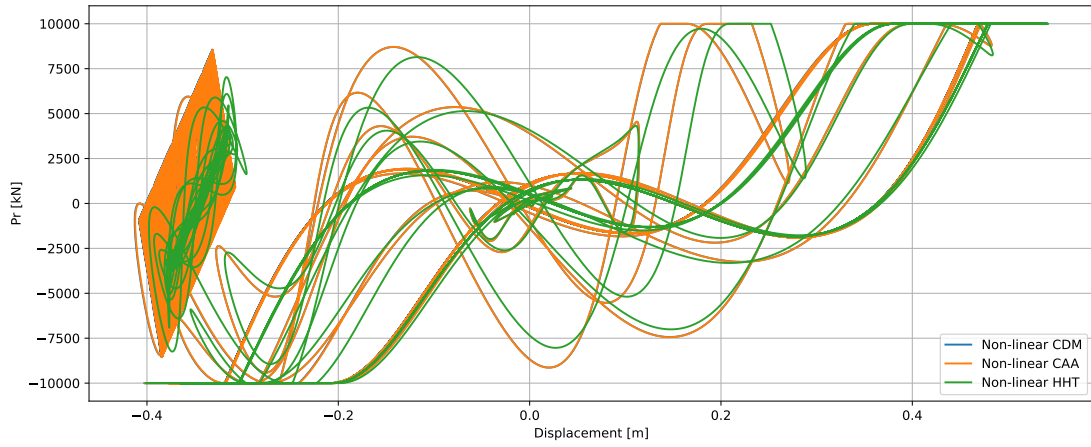


Figure A.18: Internal forces-displacement relation computed with HybriDFEM relative to the second mass.

Figures A.17 and A.18 show the internal force-displacement relationship for all three methods. Consistent with the aforementioned observations, the CDM and CAA exhibit comparable responses. However, the graph becomes hard to interpret as some sinusoidal shapes appear. Nonetheless, certain elements can be identified. The force P_r does not exceed 10,000 kN, which is the maximum value allowed by

the material law. A diamond shape appears on the graph, resulting from the system being left in free vibration. The HHT method, for example, does not exhibit this shape but rather shows somewhat curve lines that seem to converge to a straight line due to the damping of the displacement.

As the previous graph does not provide enough information to assess the validity of the implementation, the dynamic equilibrium can be tested. The following expression must be valid for each time:

$$P_n(t) = P_{r,b}(t) + m_1 a_1(t) + m_2 a_2(t) \quad (\text{A.4.1})$$

where P_n is the applied force to the system, $P_{r,b}$ is the internal force at the support, m and a are the mass and acceleration of the block, respectively. Figure A.19 clearly shows that this expression is satisfied.

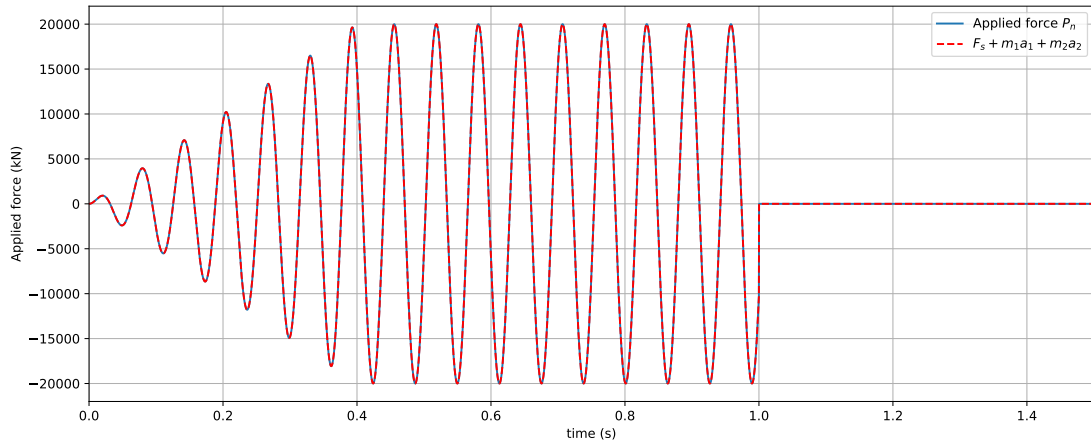


Figure A.19: Dynamic equilibrium of the system.

A strain hardening material

The stress-strain graph of the tested material is depicted in Figure 6.16. The material has a Young's modulus of $E = 200$ MPa and a yield strength of $f_y = 10$ Mpa. Upon surpassing the yield strength limit, the material's Young's modulus decreases at a decrease angle of $\alpha = 0.2$, such that the Young's modulus becomes $E_t = \alpha E$.

Figures A.20 and A.21 presents a comparison of the displacement computed with HybriDFEM for a strain hardening material using different non-linear algorithms. The three algorithms subjected to testing are the same from the previous material:

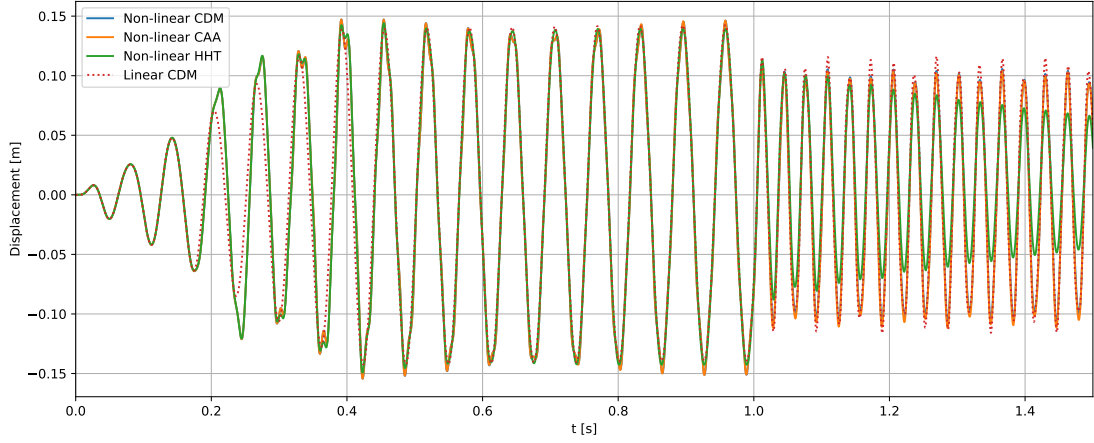


Figure A.20: Displacement of the first block in the horizontal direction $u_1(t)$.

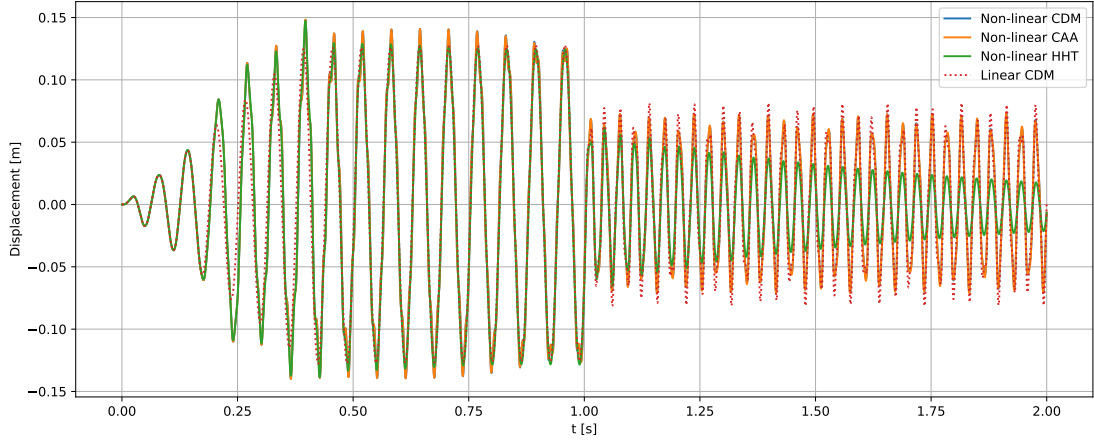


Figure A.21: Displacement of the second block in the horizontal direction $u_2(t)$.

CDM, CAA and HHT. Additionally, the figure includes the response of a linear material with $E = 200MPa$.

As expected, all numerical solutions exhibit similar responses before surpassing the yield strength, behaving akin to a linear material. Surpassed the limit, all three modes seem to give similar results, similarly to the elastic–perfectly plastic material. A very slight decrease of the displacement is still visible for HHT.

As for a SDoF system, after a certain duration, the response reverts to a linear pattern, with residual plastic deformation. This phenomenon occurs as the material transitions into the strain-hardening region and subsequently discharges, effectively increasing the yield strength of the material. Consequently, as the applied force

stabilizes, successive loadings no longer surpasses the yield strength limit. Once the force is removed, the system is left in free vibration. As it could be expected, CDM and CAA, continue to oscillate, while the HHT response dampens. Although it is difficult to notice, a very slight residual plastic deformation can be observed for all three methods.

Figures A.17 and A.18 show the internal force-displacement relationship for all three methods. In this case, it is quite difficult to correctly explain what is happening. A diamond shape appears on the graph, resulting from the system being left in free vibration. The HHT method, for example, does not exhibit this shape but rather shows somewhat straight lines due to the damping of the displacement.

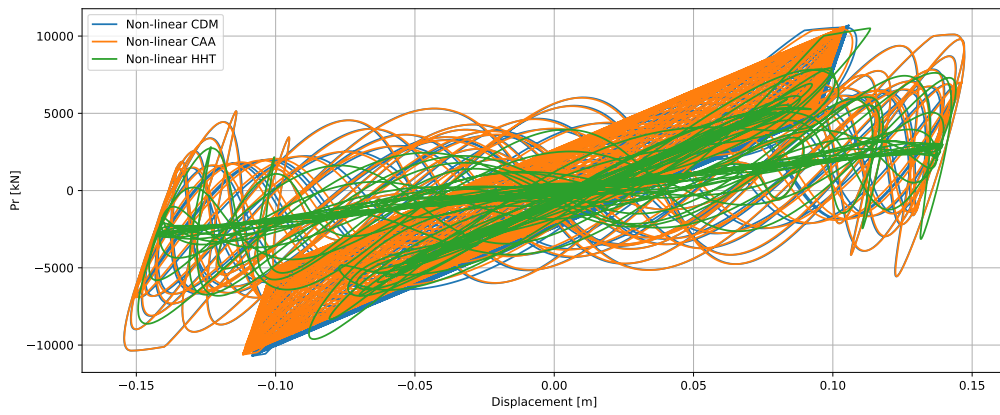


Figure A.22: Internal forces-displacement relation computed with HybriDFEM relative to the first mass.

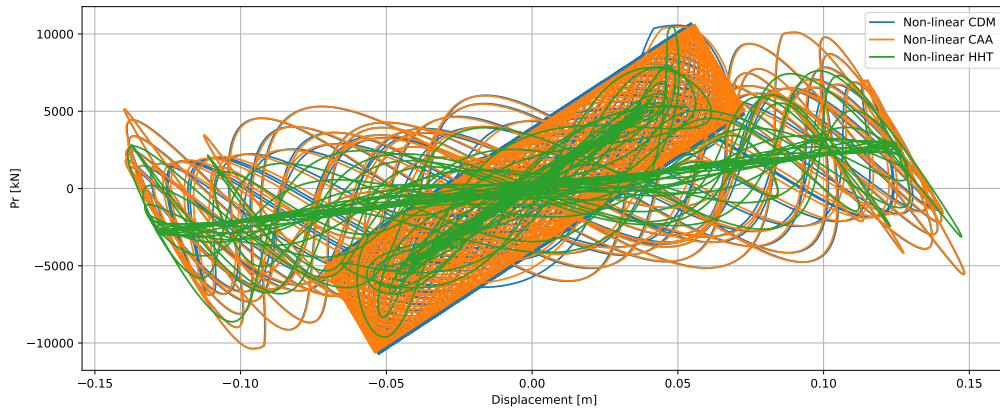


Figure A.23: Internal forces-displacement relation computed with HybriDFEM relative to the second mass.

As the previous graph does not provide enough information to assess the validity of the implementation, the dynamic equilibrium can be tested. The following expression must be valid for each time t :

$$P_n(t) = P_{r,b}(t) + m_1 a_1(t) + m_2 a_2(t) \quad (\text{A.4.2})$$

where P_n is the applied force to the system, $P_{r,b}$ is the internal force at the support, m and a are the mass and acceleration of the block, respectively. Figure A.24 clearly shows that this expression is satisfied.

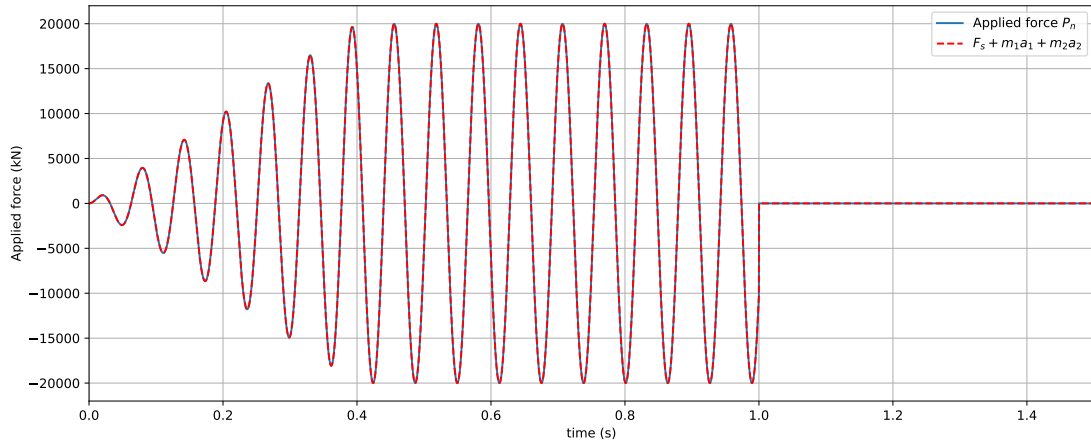


Figure A.24: Dynamic equilibrium of the system.

Appendix B

Graphical User Interface

During the development of the linear and not-linear algorithms phase, some efforts were also put in realizing a simple graphical interface to better visualize and show the results. The interface is very bare bone and is missing some features, but it allows to better visualize the response computed by HybriDFEM. It is based on the tkinter python package, which allow creating interactive interfaces.

Figures B.1, B.2, B.3, B.4, B.5 and B.6 reports some examples.

On the left side of the interface, three buttons allow users to control the simulation: start, pause, and reset. A red marker on the structure indicates the selected node, which users can specify using the white box near the node label. Furthermore, users can customize the display on the right graph by selecting horizontal displacement, vertical displacement, or rotation. The left structure is animated to provide a dynamic visualization of the simulation results, enhancing the user's understanding of the structural behaviour.

Future developments could enhance the user experience by incorporating features such as visualizing internal forces, implementing zoom functionality for both the graph and structure views, enabling precise instance selection t within the simulation, and integrating a 'save as .gif' button for convenient export of simulation animations. While the current focus remains on refining formulation and computations, improving the user interface and experience is equally paramount. A user-friendly interface enhances accessibility and usability, boosting the program quality.

Moreover, optimizing performance could involve saving all simulation data to a .csv file instead of relying solely on computer RAM. This approach offers several advantages, including improved performance on lower-spec machines, reduced

memory usage by storing only the necessary data, and avoid the need to restart the simulation to revisit results. By implementing these enhancements, the simulation tool can provide a smoother and more efficient experience for users.

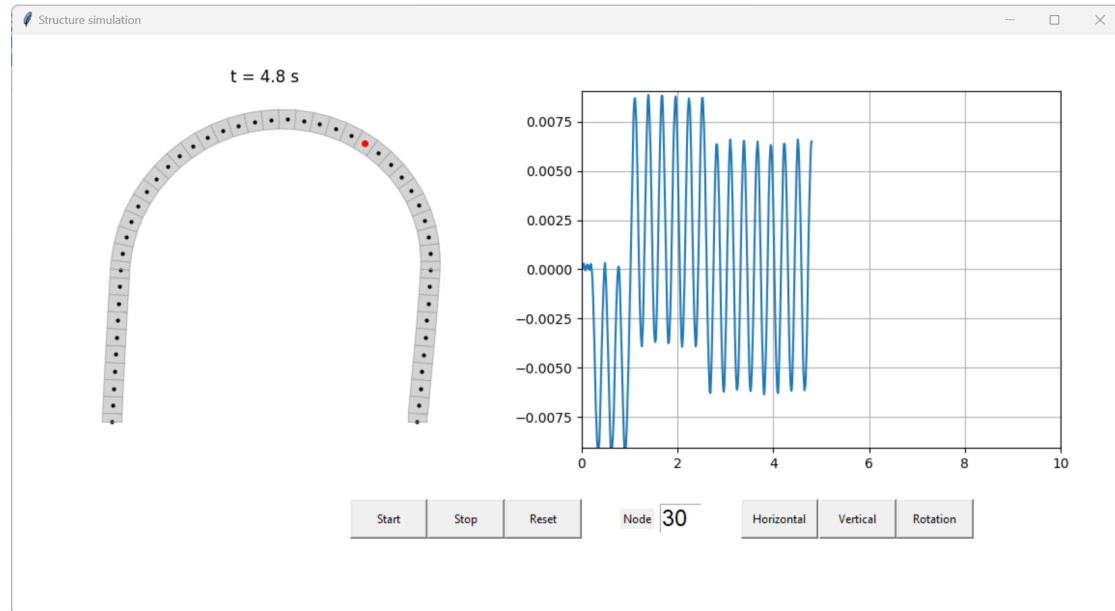


Figure B.1: Horizontal displacement of an arch frame.

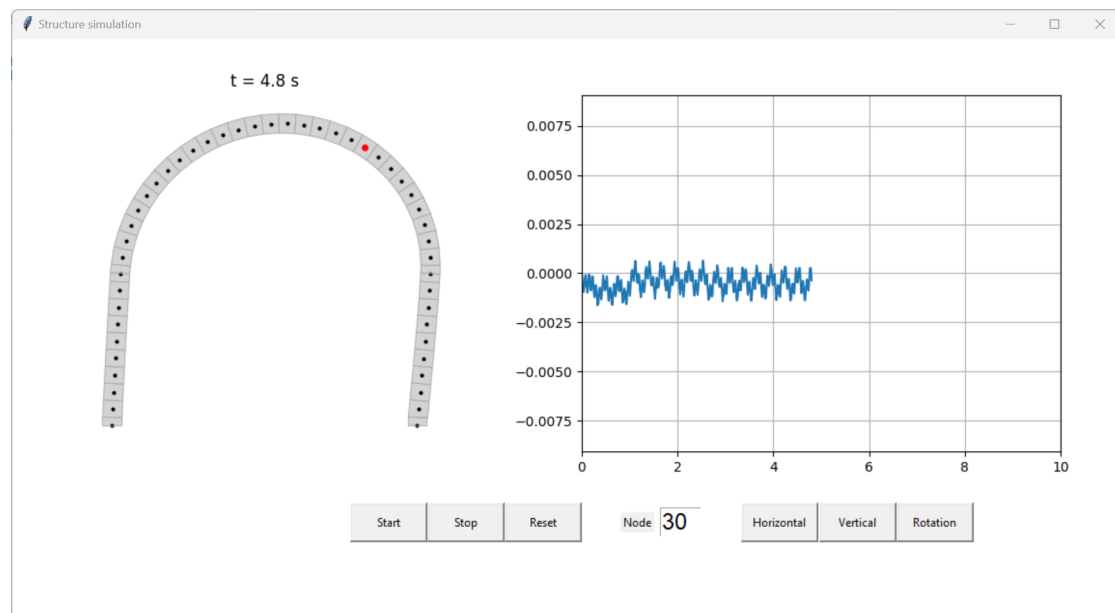


Figure B.2: Vertical displacement of an arch frame.

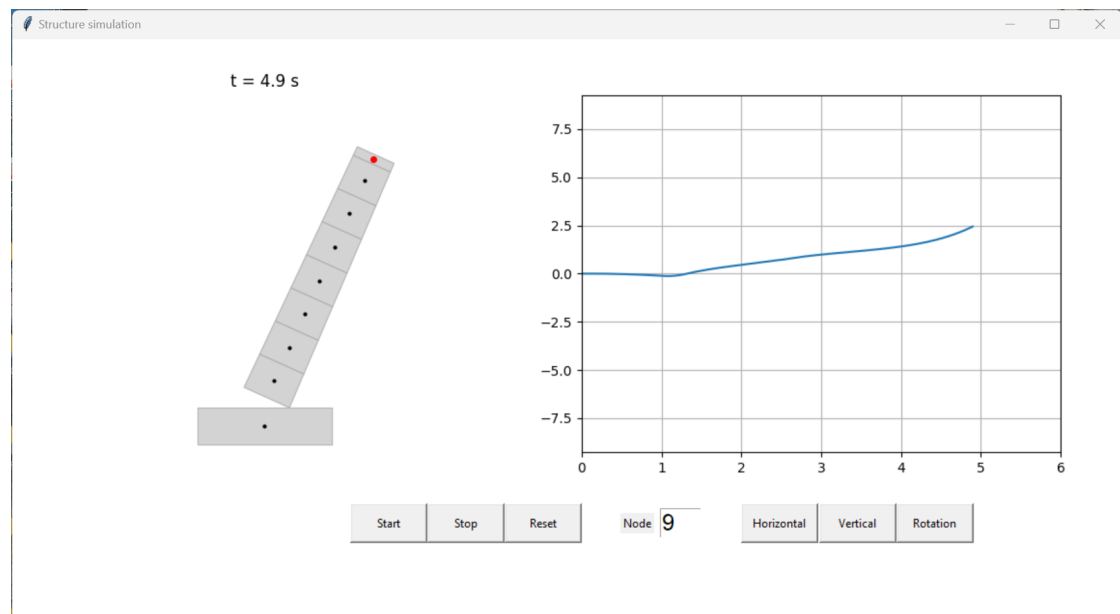


Figure B.3: Horizontal displacement at the top of the column.

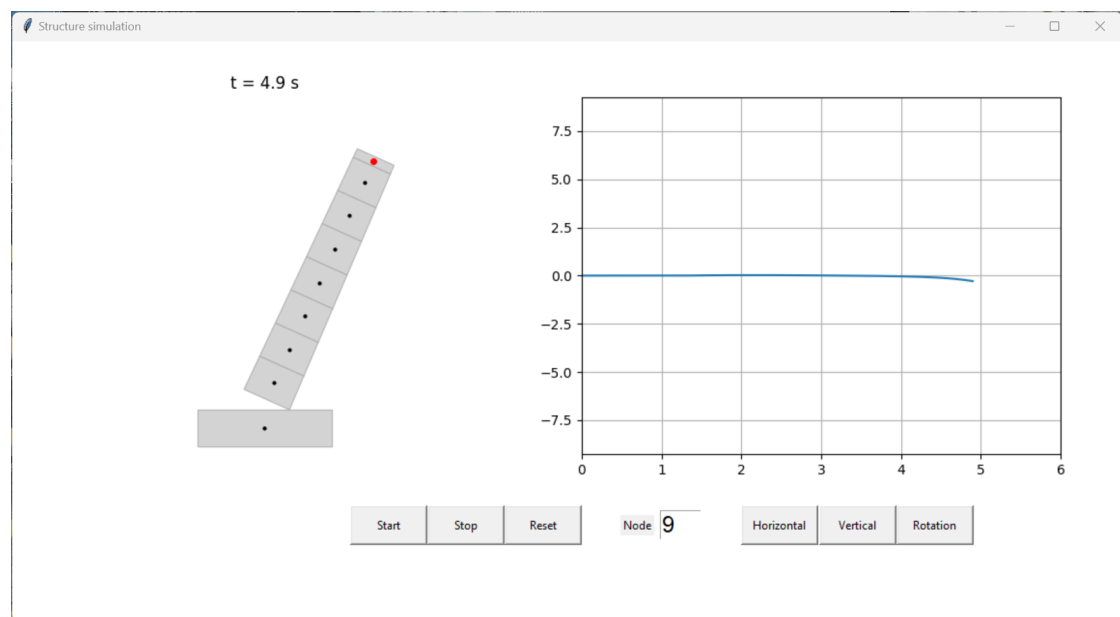


Figure B.4: Vertical displacement at the top of the column.

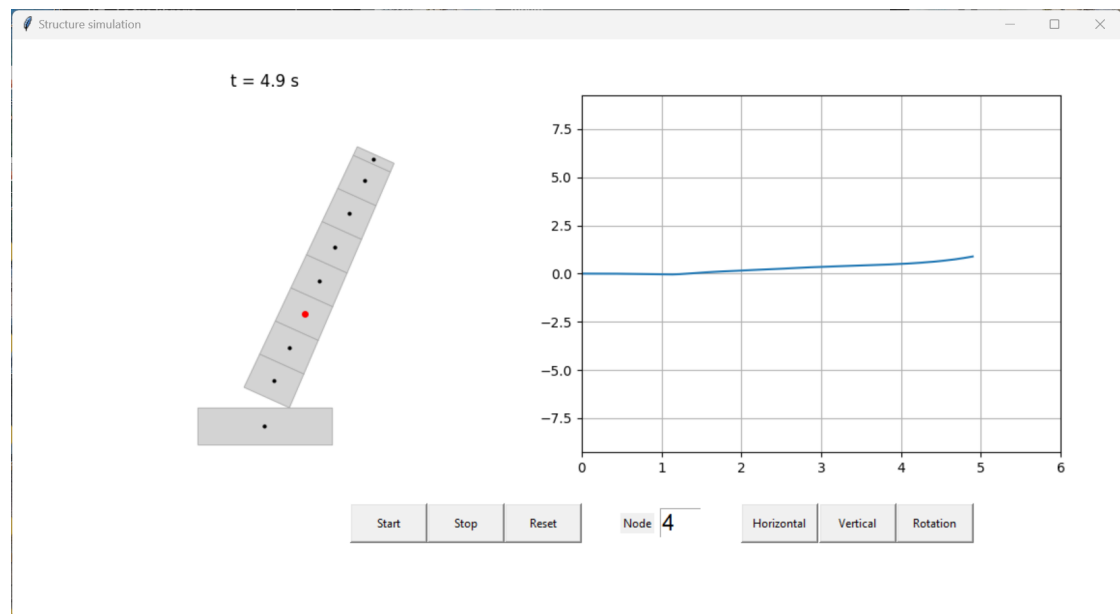


Figure B.5: Horizontal displacement at the middle of the column.

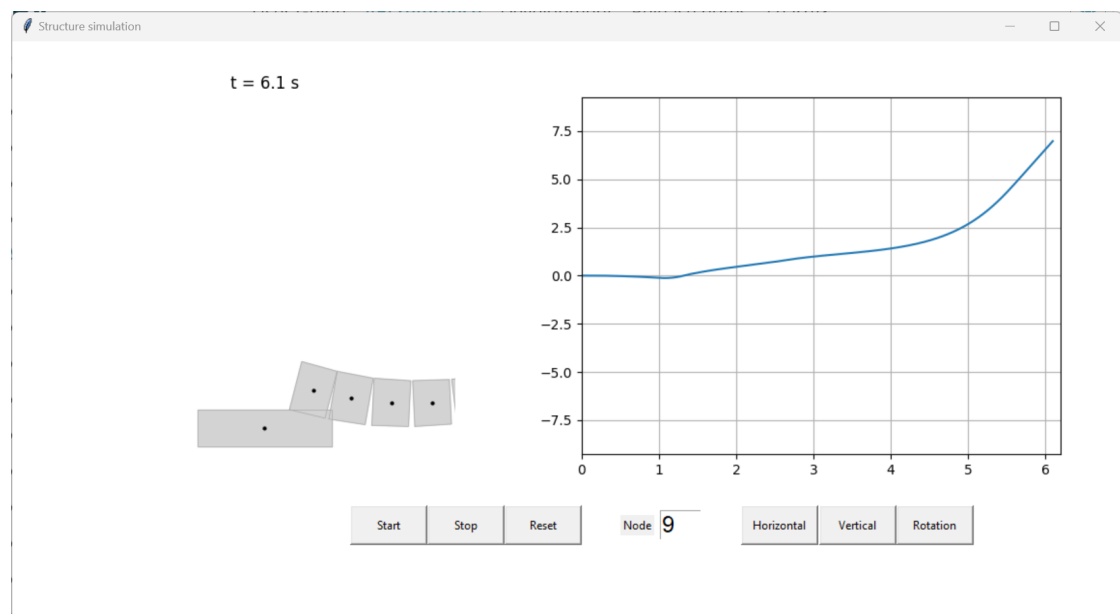


Figure B.6: Horizontal displacement at the top of the column during collapse.

UNIVERSITÉ CATHOLIQUE DE LOUVAIN
École polytechnique de Louvain

Rue Archimède, 1 bte L6.11.01, 1348 Louvain-la-Neuve, Belgique | www.uclouvain.be/epl