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Homogenization and mechanical behavior under impact of auxetic metamaterials

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Quelle belle lumière

Abstract

Metamaterials constitute a new class of materials with augmented properties compared to what one can usually find in nature. The properties of this class of materials rely principally on the microstructure that one designs rather than on the structural material itself from which the microstructure is made. In recent years, the progress in the techniques of additive manufacturing made possible the realization of very precise microstructures. The fabrication and testing of metamaterials have therefore become more accessible.

In this work, the so-called *auxetic* metamaterials are studied. This category of metamaterials has the property to shrink in all directions under the application of compression in one direction. This tendency to densify confers an augmented capacity for impact absorption to such structures.

The interest in using auxetic structure in the design of protective masks is investigated. Several tests of impact are realized on sandwich structures containing a layer of auxetic material. The analyses will lead to evidence of a global improved response in terms of transmitted force through the portion of the protective device.

Furthermore, the possibility to evaluate the homogenized effective properties of such materials will be investigated. The equivalent stiffness matrices will be computed and the possibility to reduce the time of computation through the substitution of the real auxetic structure by the homogenized media is studied. The homogenization method accounts for the elastic behavior only, it is not able to predict the complex non-linear phenomena, such as local instability or plasticity which occur under impact. However, the results of the experiments show a reasonable agreement between the two analyses.

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Chapter 1

Introduction

The term *Metamaterial* etymologically comes from Greek. It results from the concatenation of $\mu\epsilon\tau\alpha$ which in Greek means beyond and *materials*. This word describes the capacity of this class of materials to have properties beyond what one can usually find in nature. Even though the term is becoming more and more known, its formalization is quite recent and it is usually accredited to Rodger M. Walser, University of Texas at Austin, around 1999 [36]. This class of materials can be defined as:

"A metamaterial is macroscopic media made of periodic or non-periodic microstructures, whose cellular architecture has been designed to achieve certain behaviors difficult to be found in nature." [14]

The concept of metamaterials first appeared in the field of electromagnetism. Before 1999, they were known under the term LHM (left-handed material) or NIM (negative index materials) due to their negative permittivity and permeability. The first studies on the subject started around 1951 with the work of Malyuzhinets [26] and made significant progress with the work of Veselago in 1968 [35]. This latest suggested a standard method to analyze the electromagnetic behavior of NIM. These works did not however encourage the scientific community toward further investigations [14]. The lack of experimental validations, as well as the quasi-absence of these materials in nature, were the main causes. In 1996, Pendry et al. [28] were able to create an electric plasma made of wire medium that achieved a negative permittivity. Three years later a new material with negative permeability was synthesized.

Within the same decades, works were conducted in other fields of physic such as thermodynamic and mechanics of solids [8,9,29,33]. The idea of designing the microstructure was used to obtain materials characterized by negative physical coefficients such as the thermal expansion or the Poisson coefficient. This last class of materials, having a negative effective Poisson coefficient, are often referred to as *auxetic* materials. Other classes of materials such as phononic crystal and locally resonant materials also gained popularity at the same time.

The term *auxetic* was introduced by Professor Ken Evans (University of Exeter) in a paper about polymers [16].

"A material with a negative Poisson's ratio is one that expands in all directions when pulled in only one, thus increasing its volume. Such materials may be described as auxetic materials (from the Greek auxetos: that may be increased) to avoid the cumbersome phrase: materials with negative Poisson's ratios".

A large variety of micro-architectures are present in the literature nowadays. They are often sorted according to their specific mechanism of deformation. There are three main categories [25].

1. Re-entrant geometry:

The mechanism of deformation is characterized by the re-alignment of the cell ribs. The auxetic foams are part of this first class. They were the first auxetic structures to be synthesized. Other typical microstructures of this class can be observed in Figure 1.1. Among these: the re-entrant honeycomb (Figure 1.1 a), the re-entrant triangles (Figure 1.1 b), the star-system (Figure 1.1 c) are the most common. At this point, it is of interest to precise that auxetic foams are nowadays the most common auxetic structure to be used. Usually, foams are made auxetic through mechanical and thermal treatment but one does not have real control over the microstructure. The recent developments of additive manufacturing allowed the use of creative microstructures. Moreover, this technique allows very precise design and control on the parameters such as the Young modulus and the Poisson's ratio [15].

2. Chiral structures:

Within this class, the mechanism is characterized by both the rotation of the central ring and the bending and stretching of the ligaments. The more studied chiral structures are the tetra-chiral (d) and the hexa-chiral (e).

3. Rotating rigid structures:

The last class mechanism is based on the relative rotation between the rigid elements. Among those structures, the one made of a rectangle (f) is the most common.

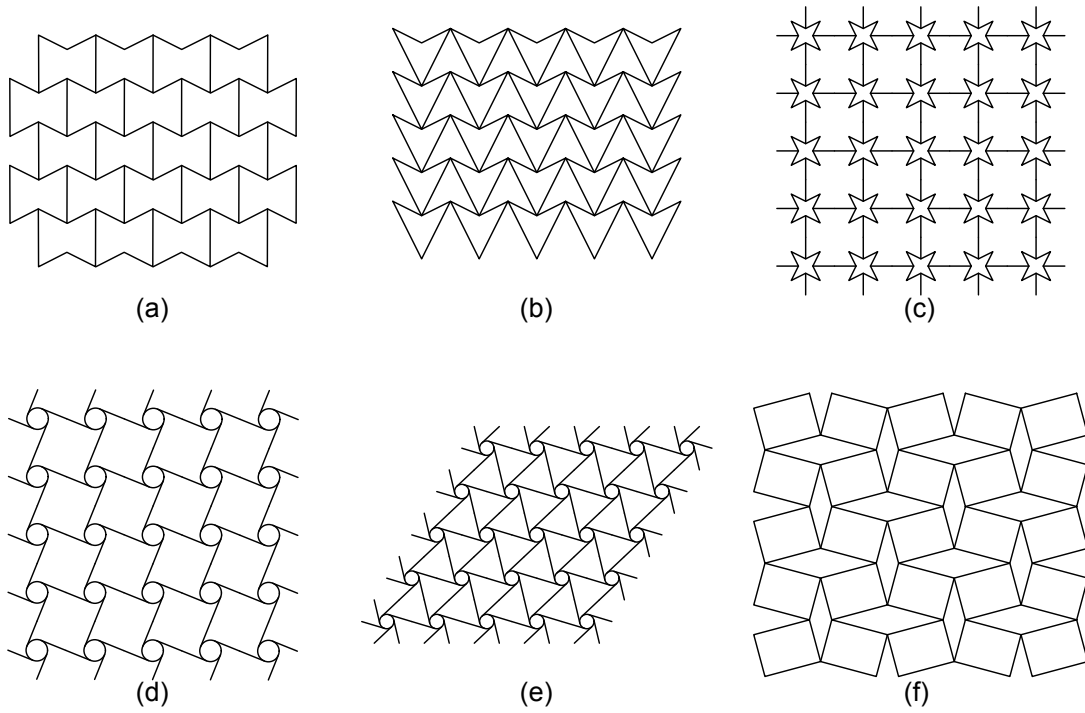


Figure 1.1: Most popular auxetic micro-structures.

For the sake of completeness, other microstructures have also been proven as auxetic. Among them: crumpled sheets, perforated sheets, or origami-like structures among which the Miura-ori has been the more studied [22, 25, 34]. Most of these structures can be also generalized in 3D.

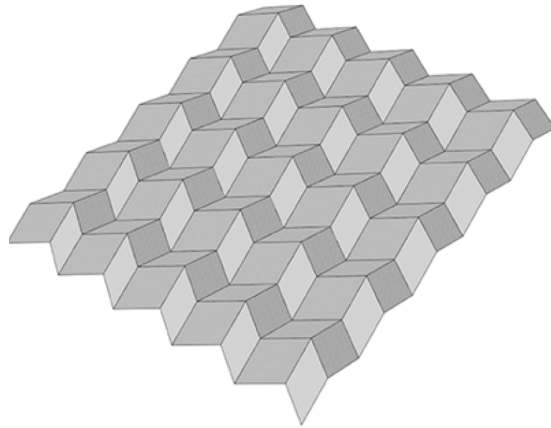


Figure 1.2: Origami inspired auxetic structure: Miura-ori [22].

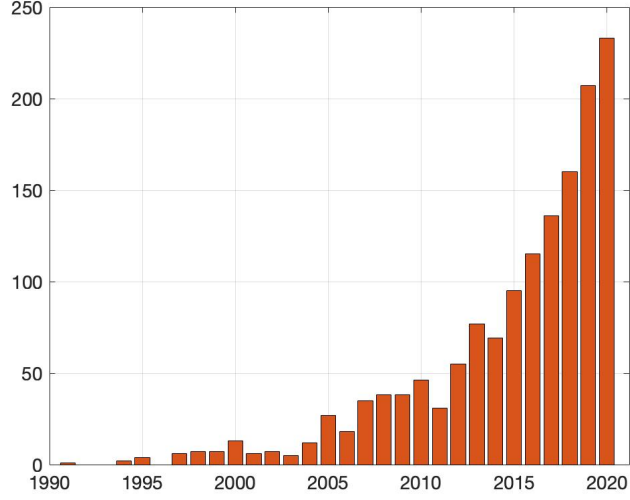


Figure 1.3: Annual publications on "auxetic materials". Data from Scopus search.

The number of works on auxetic meta-materials, since the first appearance in 1991, never stopped growing in number as it can be seen in [Figure 1.3](#). This continuously increasing interest can be explained by the multitude of applications for which auxetic materials behave in a more efficient way than classical materials. Among those properties, the most pertinent ones will be presented hereafter [\[31\]](#).

In order to understand better the behavior, the fundamental case of linear isotropic elasticity is considered. For this case, two independent constants are usually used to describe the behavior of a material. The Young's Modulus E which relates the axial strain to the axial stress and the Poisson's ratio ν that links the axial strain to the transversal strain. Other two constants can alternatively be used. The bulk modulus K which relates the hydrostatic pressure to the change of volume and the shear modulus G which links the shear strain and the shear stress. These latter ones are linked to E and ν by the following formulas:

$$K = \frac{E}{3(1 - 2\nu)} \quad G = \frac{E}{2(1 + \nu)} \quad (1.1)$$

From these relations, a certain number of properties can be derived considering the specific case where the Poisson's ratio becomes negative.

- Superior shear stiffness:

In [Equation 1.1](#), it can be seen that as ν approaches $1/2$, the bulk modulus tends to extreme values. On the opposite, when ν tends to -1 , the shear modulus becomes very large. The high shear stiffness can be beneficial in many structural elements such as plates, shells, or beams.

- Resistance to indentation:

Considering the case of local indentation, conventional materials would tend to spread the matter away from the indentation. On the contrary, auxetic materials in the situation of local compression tend to concentrate the matter in the indentation zone. This phenomenon is also known as densification (Figure 1.4) and will be of great interest when facing the problem of impact.

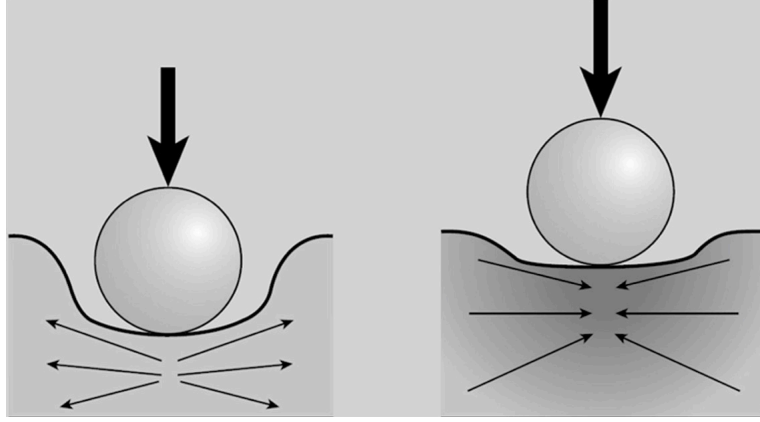


Figure 1.4: Indentation of a conventional material (on the left) and an auxetic material (on the right) from [1].

- Synclastic properties:

The plates made of auxetic material are reported to have synclastic properties [17]. When an auxetic material is used, a rectangular plate subject to bending deforms assuming the shape of a dome instead of a saddle which is the case for conventional materials, as shown in Figure 1.5.

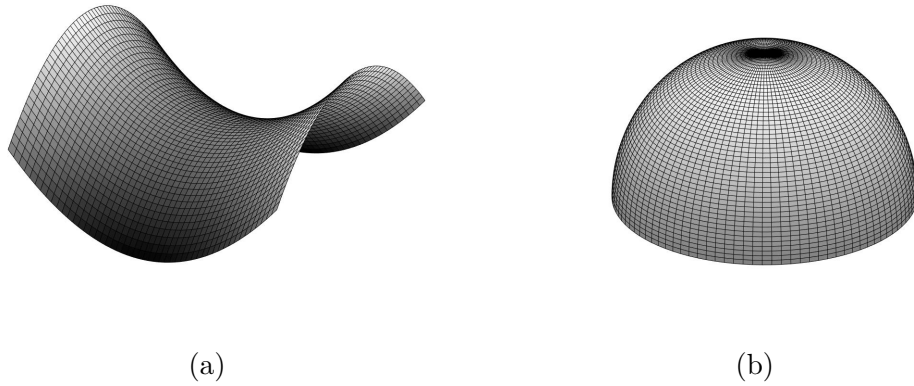


Figure 1.5: Representation of (a) anti-synclastic behavior and (b) synclastic behavior.

"This property is reported to be very useful based on the fact that it provides a way to fabricate a dome-shaped structure with using damaging techniques nor additional machining [31]."

- Augmented dynamic and static energy absorption:
The auxetic structures are often reported to have a higher energy absorption. In [39] studies were done with ballistic tests. The test was performed on sandwich panels using both auxetic foams and aluminum foam. The property of condensation at the impact zone makes it better [12].
- Variable permeability:
Due to the specific property of pores dilatation, the auxetic structures are good candidates for smart filters from macro to nanoscale. Work such as [3] reported the effectiveness and interest of such property.

These properties make auxetic materials very pertinent for a large range of applications. In the aspect of medical application, auxetic materials are also reported to be very promising. They could be used for stents, smart bandages, skin grafts, or blood vessels. In the work of Ali et al. [4], an auxetic stent was designed and manufactured. The use of this film was to expand the esophagus and therefore releases the pain of people suffering esophageal cancer. Later on, Gatt et al. [20] presented a geometry based on a two levels hierarchical rotating geometry. This structure was reported to be very suitable for skin grafts due to the relieving of the pressure on the swelling area.

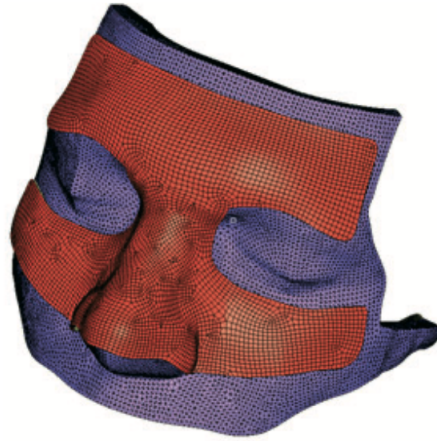


Figure 1.6: Finite element model of a human face and of the protective design in red. [13]

Huge potential for protective devices is also conferred to auxetic meta-materials. Both energy absorption and comfort could be improved with auxetic structures. Many works have been presented on auxetic devices like cushions [38], face masks [19] (Figure 1.6) but also more generally on the diminution of the contact stress concentration [23].

Finally, in the work of Sanami et al. [32], the possibility of manufacturing helmets with a new honeycomb structure was studied. This study was carried out together with an indentation test of auxetic and non-auxetic foams for evaluating the applications of protective pads and running shoes. In Table 1.1, other applications among the most promising ones are summarized.

Field	Application and rationale
Aerospace	Vanes for engine, thermal protection, aircraft nose-cones, wing panel, vibration absorber.
Automotive	Bumper, cushion, thermal protection ,sounds and vibrations absorber parts, fastener.
Biomedical	Bandage, wound pressure pad, dental floss, artificial blood vessel, artificial skin, drug-release unit, ligament anchors , surgical implants.
Composite	Fibre reinforcement (because it reduces the cracking between matrix and fiber)
Military(Defence)	Helmet, bullet proof vest, knee pad, glove, protective gear (better impact property)
Sensor/actuators	Hydrophone, piezoelectric devices, various sensors
Textiles (Industry)	Fibres, functional fabric, color-change straps of fabrics, threads

Table 1.1: Summary of the potential applications for auxetic materials from [30].

The present work is inserted into this fruitful line of researches. The main objective is the study of auxetic materials for the structure of protective facial masks. This thesis follows the work of the previous thesis [18] and compares different two-dimensional and three-dimensional geometries also by making use of the theory of homogenization. The capacity of three-dimensional auxetic structures to absorb impacts with respect to classical non-auxetic materials is studied on a small portion of a potential protective design. The impact is simulated between a rigid sphere and a sandwich structure representing a portion of the mask.

In parallel, a new 3D auxetic structure is defined and described geometrically. This new geometry will be compared with one of the most classical auxetic structures, namely: the three-dimensional re-entrant honeycomb, in terms of performances under impact tests.

Throughout the thesis, the theory of homogenization is used for the analysis of two-dimensional and three-dimensional auxetic materials. Ultimately, the main goal will be to test the possibility for the use of the mathematical tool within a large scale model. More precisely, the tests of impact will be performed using an equivalent homogeneous material in order to reduce the computational time.

The thesis is organized as follows. After the present introduction, the second chapter covers the main aspects of two and three-dimensional periodic structures. The different types of lattices are presented in 2D and 3D cases. The specific terminology that will be used to describe the periodic auxetic cellular architecture is defined.

In the third chapter, the theory of homogenization is presented for the case of linear elastic deformable bodies. The formulation is developed for the general case of elastic dynamic problem and finally, the expression for the homogenized stiffness coefficients is derived.

In the fourth chapter, the main two-dimensional auxetic structures are studied, namely: the re-entrant honeycomb, the re-entrant triangle, the tetra-chiral, and the hexa-chiral geometry. The goal is to define the elastic parameters for each structure. The objective is achieved on one hand using the classic test on the Representative Volume Element (R.V.E.) and then using the previously mentioned theory of homogenization.

In the fifth chapter, a parametric study is done on the first two geometries (re-entrant honeycomb and re-entrant triangle). The influence of each geometrical parameter is assessed and a comparisons between the two cells are drawn.

The sixth chapter is focused on three-dimensional problems. The new three-dimensional structure is presented and studied. The procedure of homogenization is applied successively on the new geometry and on the well-known 3D re-entrant honeycomb.

In the seventh chapter, the final simulation of impact is developed. The model is described and the main assumptions are presented. The analyses are performed with the two geometries studied in the previous chapter (re-entrant triangle and re-entrant honeycomb). Then, the possibility to use the theory of homogenization for the simulation of impact is addressed. Both the global and local responses are analyzed. The advantages and limitations of the theory of homogenization are discussed.

Finally, the last chapter summarizes the main results obtained throughout the work and the ulterior developments are presented.

Notation: Throughout this whole work, the vectors will be underlined once and the tensors twice. The dimensions of the tensor will always be specified.

Chapter 2

Periodic structures

In this work, the micro-architectures studied are built in the form of periodic structures. In the third chapter concerning the theory of homogenization, the notions of periodic media and unit cell are also needed for the formulation of the mathematical problem. For these reasons, the following chapter presents the theoretical bases of periodic media in 2D and 3D.

A lattice can be defined as an infinite repetition of identical parts, see [Figure 2.1\(a\)](#). These identical parts are often referred to a *basis* or *reference system*. Formally, this part can be anything: atoms, molecules or, just any object. Within the lattice, the basis can be localized by a point. The lattice can therefore also be described as the mathematical set of points to which the basis is attached.

A three-dimensional Bravais lattice can be defined as a translational lattice characterized by three linearly independent vectors: $\underline{a}_1$, $\underline{a}_2$, and $\underline{a}_3$. The latest ones are called *the translation vectors*. These vectors define the position of any point within the Bravais lattice as:

$$\underline{r} = u_1 \underline{a}_1 + u_2 \underline{a}_2 + u_3 \underline{a}_3 \quad u_1, u_2, u_3 \in \mathbb{Z} \quad (2.1)$$

Another property of a periodic structure is that the arrangement remains unchanged when looking from a point $\underline{r}$ or another point $\underline{r}'$ defined as:

$$\underline{r}' = \underline{r} + v_1 \underline{a}_1 + v_2 \underline{a}_2 + v_3 \underline{a}_3 \quad v_1, v_2, v_3 \in \mathbb{Z} \quad (2.2)$$

Considering a periodic media, any portion of this media that translated can reproduce the whole lattice is called a *unit cell*, see [Figure 2.1\(b\)](#). It is important to notice that the choice of this cell is arbitrary and therefore not unique. It might seem at this point without interest but this specific non-uniqueness will be discussed later in this work. Among all the choices of unit cells, the so-called *primitive cell* is of particular interest. This unit cell has the particularity to be the unit cell with minimum volume. It can therefore be seen as the more economic way to represent the lattice. The translation vectors building the primitive cell are often called the primitive vectors. In [Figure 2.1](#),

an example of these components for the 2D case are represented in sub-figures (b) and (c). The same comment as for the unit cell holds for the primitive cell in the sense that it is not unique. Nevertheless, the volume is uniquely defined for a given lattice. The latter is defined through the translation vectors:

$$V_c = \underline{a}_1 \cdot \underline{a}_2 \times \underline{a}_3 \quad (2.3)$$

This concept of primitive cell will be of most importance when developing the theory of homogenization.

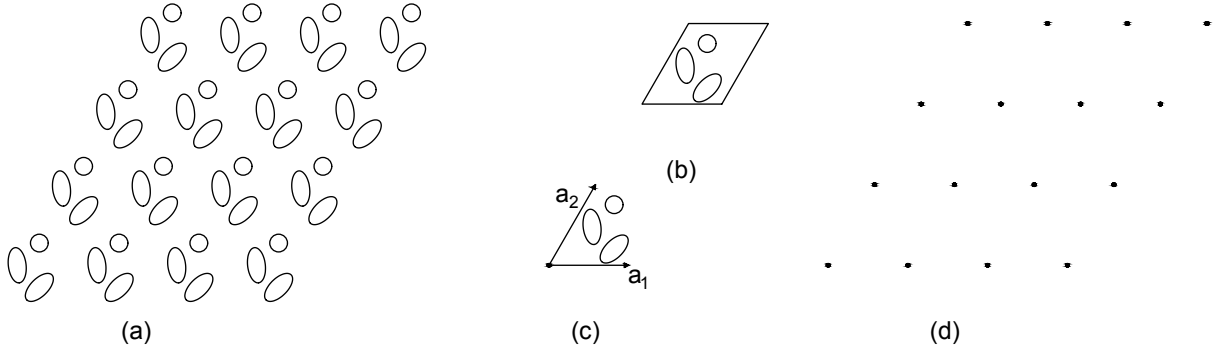


Figure 2.1: (a) Periodic structure, (b) choice of the cell, (c) point location of the basis and primitive vectors, (d) Bravais lattice of the periodic structure.

Other types of primitive cells can be considered without needing to define the translation vectors. One of the most known is the so-called Wigner-Seitz primitive cell. To build this cell, the procedure consists of linking all the points of the lattice to all nearby points, then lines/planes are drawn from the middle of each previously drawn line. The intersection of these lines/planes defines the Wigner-Seitz cell. The construction can be seen in [Figure 2.2](#) for a 2D lattice.

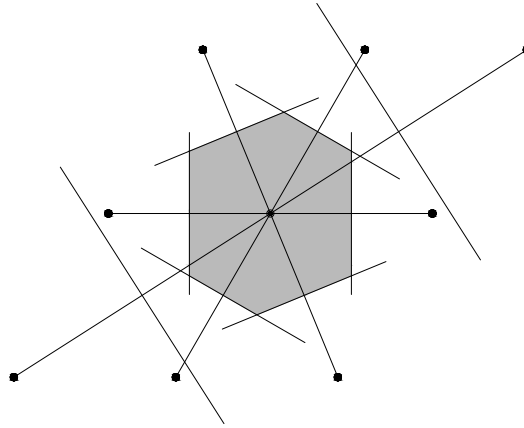


Figure 2.2: Construction of the Wigner-Seitz primitive cell.

2.1 Fundamental lattice type

As it has been seen, lattices can be mapped into themselves by translation considering the translation vectors. Other types of transformation leave also the lattice unchanged. The different lattices are often classified by the types of rotation that carry them into themselves. The possible rotations are the one-,two-,three-,four- and sixfold rotation corresponding respectively to rotations of 2π , $2\pi/2$, $2\pi/3$, $2\pi/4$ and $2\pi/6$. Other rotations such that five- or sevenfold are not possible when considering an infinite periodic media. It is in fact impossible to consider these point symmetry and to fill the space. Hereafter, the description and classification of the unit cell in 2D and then in 3D are presented also considering the rotation symmetries that they possess.

2.1.1 Two-dimensional Bravais lattices

In the two-dimensional case, the lattices can be sorted into five types. The general case is the oblique lattice which is the one shown in [Figure 2.1](#) and [Figure 2.3\(d\)](#). This lattice is invariant under rotation of π and 2π . By imposing proper conditions on the vectors $\underline{a}_1$ and $\underline{a}_2$ and the angle between them, it is possible to achieve invariance under the other mentioned rotations. Each type is represented hereafter in [Figure 2.3](#).

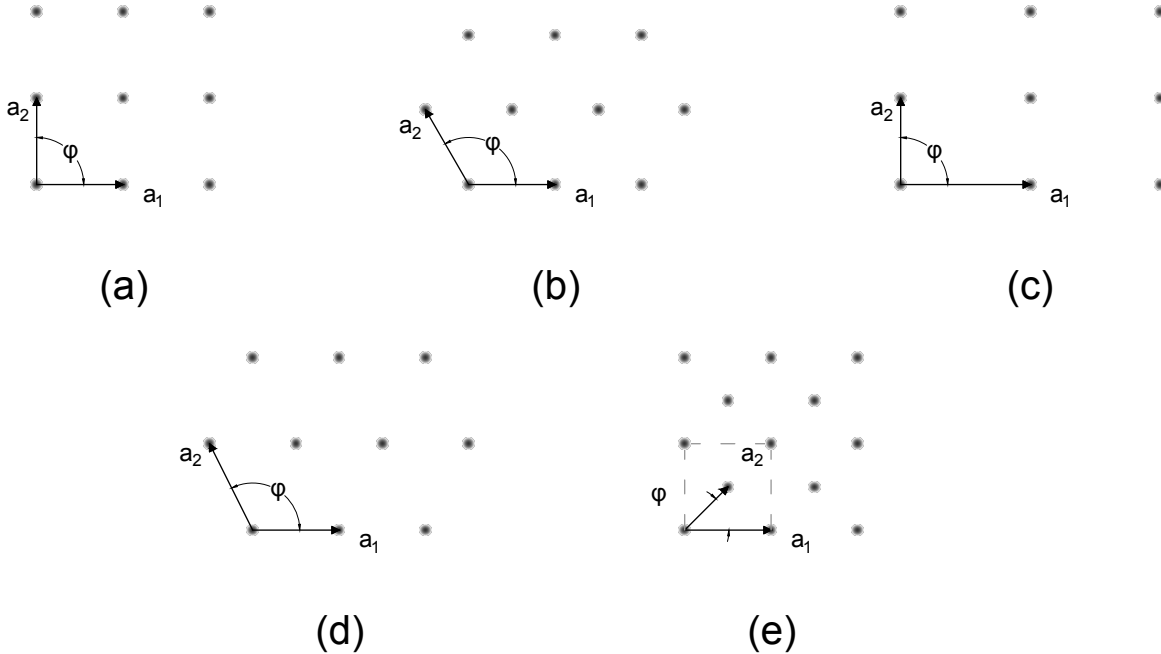


Figure 2.3: Two-dimensional Bravais lattice: (a) square, (b) hexagonal, (c) rectangular, (d) oblique, (e) centered rectangular.

The squared lattice is characterized by two vectors of equal length and an angle φ equal to 90° (Figure 2.3 a). The hexagonal one has $|\underline{a}_1| = |\underline{a}_2|$ and $\varphi = 120^\circ$ (Figure 2.3 b). The rectangular one is defined by $|\underline{a}_1| \neq |\underline{a}_2|$ and $\varphi = 90^\circ$ (Figure 2.3 c). The oblique is the most generic one and has no specification on the norm of the vectors nor the angle that they make (Figure 2.3 d). The last one, the centered rectangular, is only specified by the relation $(2\underline{a}_2 - \underline{a}_1) \cdot \underline{a}_1 = 0$ (Figure 2.3 e).

2.1.2 Three-dimensional Bravais lattices

In three dimensions, the problem becomes a little bit more cumbersome, and representations are more difficult. As it has been done for the 2D, the lattice in 3D can be characterized by the angles (α, β, γ) between the primitive vectors and their length $(|\underline{a}_1|, |\underline{a}_2|, |\underline{a}_3|)$. There exist fourteen lattice types and they can be regrouped into seven types of cells, which are reported in Table 2.1

Systems	Number of lattices	Restrictions on conventional cell axes and angles
Triclinic	1	$ \underline{a}_1 \neq \underline{a}_2 \neq \underline{a}_3 $ $\alpha \neq \beta \neq \gamma$
Monoclinic	2	$ \underline{a}_1 \neq \underline{a}_2 \neq \underline{a}_3 $ $\alpha = \gamma = 90^\circ \neq \beta$
Orthorhombic	4	$ \underline{a}_1 \neq \underline{a}_2 \neq \underline{a}_3 $ $\alpha = \beta = \gamma = 90^\circ$
Tetragonal	2	$ \underline{a}_1 = \underline{a}_2 \neq \underline{a}_3 $ $\alpha = \beta = \gamma = 90^\circ$
Cubic	3	$ \underline{a}_1 = \underline{a}_2 = \underline{a}_3 $ $\alpha = \beta = \gamma = 90^\circ$
Trigonal	1	$ \underline{a}_1 = \underline{a}_2 = \underline{a}_3 $ $\alpha = \beta = \gamma \leq 120^\circ \neq 90^\circ$
Hexagonal	1	$ \underline{a}_1 = \underline{a}_2 \neq \underline{a}_3 $ $\alpha = \beta = 90^\circ$ $\gamma = 120^\circ$

Table 2.1: The seven systems for 3D Bravais-lattices [24].

The reason for which some systems, such as monoclinic, have more than one number of lattices is that within the monoclinic cell more than one arrangements are possible. Still speaking of the monoclinic lattice system, it can be either monoclinic or base-centered monoclinic with one element on the top and bottom face.

Chapter 3

Two-scale asymptotic homogenization

When facing the problem of computing the elastic parameters of a heterogeneous media, two procedures are often presented in the literature. The first one is based on the study of the Representative Volume Element (R.V.E). The latest can be defined as: the smallest portion of the media which is representative of the behavior of the whole media. The method is based on averaging the response over the R.V.E. The elastic parameters are then computed as the ratio between the averaged values. These coefficients are considered representative of the macroscopic behavior. This first method has the advantages of being easy to implement when the R.V.E is clearly defined and has an easy physical and intuitive interpretation.

On the other hand, the theory of mathematical homogenization also seeks to describe the macroscopic behavior of a media which is "microscopically heterogeneous". Instead of working as in the first method on a problem with a specific size (the RVE) and averaging the response to obtain the parameters, the homogenization is based on an asymptotic development of the unknown field and on the separation of scales between the heterogeneities and the considered problem. More explicitly, the procedure of homogenization allows determining the equivalent homogeneous properties of a media. The homogenization pursues to solve a series of problems on a *unit cell*. For the linear elastic case, the equivalent homogeneous tensors can be written in closed form as a function of the solutions for the series of problems solved on the unit cell. The method of homogenization is a mathematically well-posed problem [10]. Among the principal advantages of this method, one can cite: the uniqueness of the homogenized properties and the proof of the convergence. Even though the method of two-scale asymptotic homogenization is a relatively widespread mathematical tool, it seems to be perfectly adapted for the study of periodic media such as those studied in this work. Several recent works used this tool to describe metamaterials, see e.g. [11, 21].

In this chapter, an introduction to the theory of first-order homogenization will be presented for periodic media. This introduction will follow the same developments as the ones found in [10].

The problem can also be formulated as: the procedure of homogenization seeks for the macroscopic properties of the composites media taking into account the properties of the microscopic structure. The theory, only developed in the linear elastic regime, relies on two principles. The first one is that for a large class of problems, the values of interest are not the pointwise ones but rather the global ones, and secondly that the size of the periodic heterogeneities is often very small compared to the size of the media. For the problems that will be studied later, we assume that these hypotheses will always be met. One should remark, however, that in the example concerning a portion of the mask the hypotheses are not fully satisfied and less accurate results are expected. This point will be further discussed in chapter seven. The homogenization theory is very general and can be extended to any kind of differential problem such as heat conduction, conductivity, or in our case, deformable body. It is also of interest to notice that, very few assumptions on the geometry are requested. A representation of the periodic domain can be found in Figure 3.1 where also the unit cell (b) and the re-scaled unit cell (c) are shown.

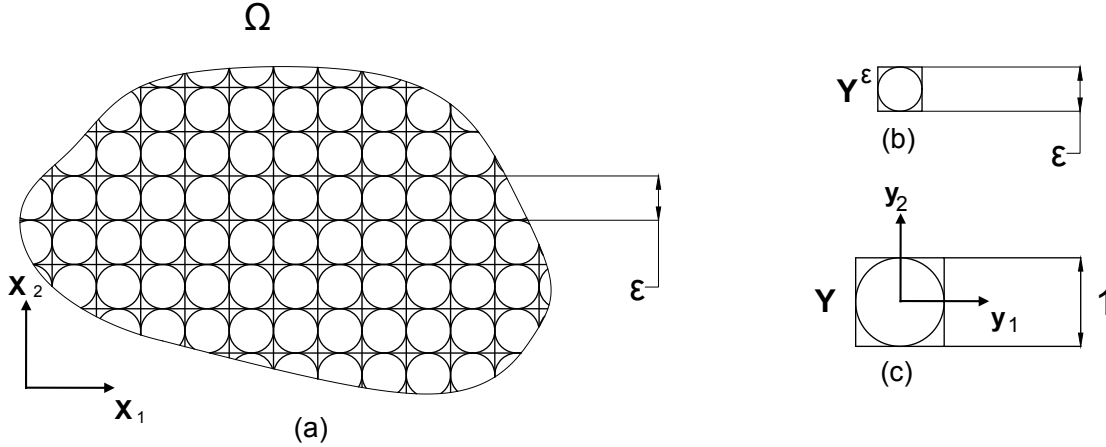


Figure 3.1: (a) Periodic material; (b) unit cell; (c) re-scaled unit-cell.

One should pay attention in the following chapter that the Einstein notation will not be used, in the sense that summation will always be indicated specifically.

3.1 Formulation of the problem

Let us consider a linear elastic body Ω with periodic heterogeneities, as represented in Figure 3.1 for a 2D solid. The periodicity is described by a Bravais lattice with unit cell

Y^ϵ . Under the hypothesis of small strains and displacements, the governing equation of the problem are:

$$\begin{cases} \rho^\epsilon(\underline{x})\ddot{\underline{u}}^\epsilon(t, \underline{x}) - \operatorname{div}\left[\underline{\underline{\sigma}}(\underline{u}^\epsilon(t, \underline{x}))\right] = \underline{F}^\epsilon(t, \underline{x}) & \text{in } \Omega \\ \underline{\underline{\sigma}}(\underline{u}^\epsilon(t, \underline{x})) = \underline{\underline{A}}^\epsilon(\underline{x}) : \underline{\underline{\varepsilon}}(\underline{u}^\epsilon(t, \underline{x})) & \text{in } \Omega \end{cases} \quad (3.1)$$

The first equation is the equilibrium equation while the second one is the constitutive law. Together with these equations, the boundary conditions should be defined. The terms appearing in the expressions are defined hereafter:

- t is the time
- ρ^ϵ is the mass density of the periodic media;
- $\underline{u}^\epsilon$ is the displacement field.
- $\underline{\underline{\sigma}}$ is the second-order Cauchy's stress tensor;
- $\underline{\underline{A}}^\epsilon$ is the fourth-order stiffness tensor of the periodic media.
- $\underline{\underline{\varepsilon}}$ is the second-order strain tensor, i.e. the symmetric part of the gradient of displacement;
- $\underline{F}^\epsilon$ are the body forces.

Denoting by $\underline{x}$ the macroscopic coordinate, let us introduce a microscopic coordinate $\underline{y} = \underline{x}/\epsilon$ where ϵ is the cell dimension. This second variable is related to the re-scaled cell $Y = Y^\epsilon$.

3.2 Asymptotic expansion

The starting point for the derivation of the homogenization theory is the two-scale asymptotic expansion. Introducing the *fast variable* (or microscopic) $\underline{y} = \underline{x}/\epsilon$ and the *slow variable* (or macroscopic) $\underline{x}$, the unknown displacement field $\underline{u}^\epsilon(t, \underline{x})$ can be expanded in the asymptotic series:

$$\underline{u}^\epsilon(t, \underline{x}) = \underline{u}^0\left(t, \underline{x}, \frac{\underline{x}}{\epsilon}\right) + \epsilon \underline{u}^1\left(t, \underline{x}, \frac{\underline{x}}{\epsilon}\right) + \epsilon^2 \underline{u}^2\left(t, \underline{x}, \frac{\underline{x}}{\epsilon}\right) + \dots \quad (3.2)$$

where the functions $\underline{u}^i(t, \underline{x}, \underline{y})$ are periodic in $\underline{y}$. The stress tensor can be expanded in the same form starting from order -1 since it involves partial derivatives of the displacement, namely:

$$\underline{\underline{\sigma}}^\epsilon(t, \underline{x}) = \epsilon^{-1} \underline{\underline{\sigma}}^{-1}\left(t, \underline{x}, \frac{\underline{x}}{\epsilon}\right) + \underline{\underline{\sigma}}^0\left(t, \underline{x}, \frac{\underline{x}}{\epsilon}\right) + \epsilon \underline{\underline{\sigma}}^1\left(t, \underline{x}, \frac{\underline{x}}{\epsilon}\right) + \dots \quad (3.3)$$

where the tensors $\underline{\underline{\sigma}}^i(t, \underline{x}, \underline{y})$ are periodic with respect to the fast variable. From [Equation 3.2](#), it is possible to express the small strain tensor:

$$\underline{\underline{\epsilon}}^\epsilon(\underline{u}) = \epsilon^{-1} \underline{\underline{\epsilon}}_y(\underline{u}^0) + \epsilon^0 \left(\underline{\underline{\epsilon}}_x(\underline{u}^0) + \underline{\underline{\epsilon}}_y(\underline{u}^1) \right) + \epsilon^1 \left(\underline{\underline{\epsilon}}_x(\underline{u}^1) + \underline{\underline{\epsilon}}_y(\underline{u}^2) \right) + \dots \quad (3.4)$$

Where the terms $\underline{\underline{\epsilon}}_x$ and $\underline{\underline{\epsilon}}_y$ denotes the symmetric parts of the gradient of the displacement evaluated with respect to the variables $\underline{x}$ and $\underline{y}$ respectively. Substituting [Equation 3.4](#) in the constitutive law ([Equation 3.1](#)), and comparing with the [Equation 3.3](#), one can retrieve:

$$\underline{\underline{\sigma}}^\epsilon(t, \underline{x}) = \epsilon^{-1} \underline{\underline{A}} : \underline{\underline{\epsilon}}_y(\underline{u}^0) + \epsilon^0 \underline{\underline{A}} : \left(\underline{\underline{\epsilon}}_x(\underline{u}^0) + \underline{\underline{\epsilon}}_y(\underline{u}^1) \right) + \dots \quad (3.5)$$

Comparing term by term the expression [Equation 3.3](#) and the [Equation 3.5](#), and finally comparing with the expansion, the following expressions can be retrieved:

$$\underline{\underline{\sigma}}^{-1} = \epsilon^{-1} \underline{\underline{A}} : \underline{\underline{\epsilon}}_y(\underline{u}^0) \quad (3.6)$$

and for $i \geq 0$:

$$\underline{\underline{\sigma}}^i = \underline{\underline{A}} : \left[\underline{\underline{\epsilon}}_x(\underline{u}^i) + \underline{\underline{\epsilon}}_y(\underline{u}^{i+1}) \right] \quad (3.7)$$

The acceleration can also be written following the same expansion procedure:

$$\underline{\underline{u}}^\epsilon(t, \underline{x}) = \epsilon^0 \underline{\underline{u}}^0 \left(t, \underline{x}, \frac{\underline{x}}{\epsilon} \right) + \epsilon^1 \underline{\underline{u}}^1 \left(t, \underline{x}, \frac{\underline{x}}{\epsilon} \right) + \dots \quad (3.8)$$

The divergence of the stress tensor is developed and expressed as a sum of increasing power of ϵ and replaced in [Equation 3.1](#))

$$\begin{aligned} \underline{\underline{F}} = & -\epsilon^{-2} \text{div}_y(\underline{\underline{\sigma}}^{-1}) \\ & -\epsilon^{-1} \left[\text{div}_y(\underline{\underline{\sigma}}^0) + \text{div}_x(\underline{\underline{\sigma}}^{-1}) \right] \\ & +\epsilon^0 \left[\rho \underline{\underline{u}}^0 - \text{div}_y(\underline{\underline{\sigma}}^1) - \text{div}_x(\underline{\underline{\sigma}}^0) \right] \\ & + \dots \end{aligned} \quad (3.9)$$

where div_x and div_y are the divergence operator with respect to $\underline{x}$ and $\underline{y}$. Since [Equation 3.9](#) should be satisfied for any values of ϵ , one obtains a sequence of equations for each order of ϵ . At order -2 one has:

$$\text{div}_y(\underline{\underline{\sigma}}^{-1}) = 0 \quad (3.10)$$

Taking [Equation 3.10](#) and the equation [Equation 3.6](#). The following PDE is obtained:

$$\begin{cases} \text{div}_y \left[\underline{\underline{A}} : \underline{\underline{\epsilon}}_y(\underline{u}^0) \right] = 0 & \text{in } \Omega \times Y \\ \underline{u}_0(t, \underline{x}, \underline{y}) \text{ periodic w.r.t } \underline{y} & \text{on } \Omega \times \partial Y \end{cases} \quad (3.11)$$

At this point, it is possible to prove that this problem is well-posed in the sense that it admits a unique solution up to a constant. It can also be noticed that t and $\underline{x}$ are parameters in the problem. Taking into account these two facts, this implies that $\underline{u}_0$ is independent of $\underline{y}$:

$$\underline{u}^0(t, \underline{x}, \underline{y}) = \underline{u}^0(t, \underline{x}) \quad (3.12)$$

3.3 Cell problems

The second problem is the one of order -1 . Looking at the second line of the [Equation 3.9](#):

$$\operatorname{div}_y(\underline{\sigma}^0) + \operatorname{div}_x(\underline{\sigma}^{-1}) = 0 \quad (3.13)$$

Considering [Equation 3.6](#), [Equation 3.7](#) and [Equation 3.13](#), one can retrieve:

$$\begin{cases} \operatorname{div}_y [\underline{A}(\underline{y}) : \underline{\varepsilon}_y(\underline{u}^1)] = -\operatorname{div}_y [\underline{A}(\underline{y}) : \underline{\varepsilon}_x(\underline{u}^0)] & \text{in } \Omega \times Y \\ \underline{u}^1(t, \underline{x}, \underline{y}) \text{ periodic w.r.t } \underline{y} & \text{on } \Omega \times \partial Y \end{cases} \quad (3.14)$$

Since [Equation 3.14](#) is linear, and $\underline{u}^0$ depends only on $\underline{x}$, each components of the strain tensor $\underline{\varepsilon}_x(\underline{u}^0)$ can be treated as an inelastic eigenstrain applied separately to the others. The function $\underline{u}^1$ can be therefore expresses as:

$$\underline{u}^1(t, \underline{x}, \underline{y}) = \underline{u}^1(t, \underline{x}) + \sum_{i,j=1}^3 \varepsilon_{ij}(\underline{u}_0(t, \underline{x})) \underline{\chi}^{ij}(\underline{y}) \quad (3.15)$$

where the functions $\underline{\chi}^{ij}(\underline{y})$ are the solutions of the so called cell problem:

$$\begin{cases} \operatorname{div}_y [\underline{A} : (\underline{\varepsilon}_y(\underline{\chi}^{ij}) + \underline{e}_i \odot \underline{e}_j)] = \underline{0} & \text{in } Y \\ \underline{\chi}^{ij}(\underline{y}) \text{ periodic w.r.t } \underline{y} & \text{on } \partial Y \\ \underline{n} \cdot \underline{A} : (\underline{\varepsilon}_y(\underline{\chi}^{ij}) + \underline{e}_i \odot \underline{e}_j) \text{ antiperiodic} & \text{on } \partial Y \end{cases} \quad (3.16)$$

for $i, j = 1, 2, 3$

- The term $\underline{e}_i \odot \underline{e}_j$ represent an initial strain state within the cell
- Due to the symmetry of the problem, $\underline{\chi}^{ij} = \underline{\chi}^{ji}$. There are therefore 6 different cell problems.

For further development, it is useful to express the cell problems (Equation 3.16) in their weak forms. For any kinematically admissible displacement fields $\underline{\phi}$ one has that:

$$\int_Y \underline{\phi} \cdot \operatorname{div}_y \left[\underline{A} : (\underline{\varepsilon}_y(\underline{\chi}^{ij}) + \underline{e}_i \odot \underline{e}_j) \right] dY = 0 \quad (3.17)$$

After intergrating by parts, one can retrieve the weak form of the cell problem: find $\underline{\chi}^{ij}$ such that:

$$\int_Y \underline{\varepsilon}_y(\underline{\phi}) : \underline{A} : (\underline{\varepsilon}_y(\underline{\chi}^{ij}) + \underline{e}_i \odot \underline{e}_j) dY = 0 \quad \text{for any } \underline{\phi} \text{ kinematically admissible} \quad (3.18)$$

3.4 Homogenized equation of motion

The third problem, at order 0 reads:

$$\rho \ddot{\underline{u}}^0 - \operatorname{div}_y(\underline{\sigma}^1) - \operatorname{div}_x(\underline{\sigma}^0) = \underline{F} \quad (3.19)$$

The second term on the left-side of the equation can be integrated over Y . Due to the periodicity condition on $\underline{u}_1$ and $\underline{u}_2$, the following holds:

$$\int_Y \operatorname{div}_y(\underline{\sigma}^1) dY = \int_{\partial Y} \underline{\sigma}^1 \cdot \underline{n} dY = \int_{\partial Y} \underline{A} : [\underline{\varepsilon}_x(\underline{u}^1) + \underline{\varepsilon}_y(\underline{u}^2)] \cdot \underline{n} dY = \underline{0} \quad (3.20)$$

where $\underline{n}$ is the unit normal vector pointing outside the boundary. Therefore, after integration over the unit cell, Equation 3.19 can be rewritten as:

$$\frac{1}{|Y|} \int_Y \rho dY \ddot{\underline{u}}^0 - \operatorname{div}_x \left(\frac{1}{|Y|} \int_Y \underline{\sigma}^0 dY \right) = \frac{1}{|Y|} \int_Y \underline{F} dY \quad (3.21)$$

where $|Y|$ is the volume of the re-scaled unit cell. Then one can define the homogenized mass density:

$$\rho^0 = \frac{1}{|Y|} \int_Y \rho(\underline{y}) dY \quad (3.22)$$

the homogenized body forces:

$$\underline{F}^0 = \frac{1}{|Y|} \int_Y \underline{F} dY \quad (3.23)$$

and the homogenized stiffness tensor $\underline{A}^0$ such that:

$$\underline{A}^0 : \underline{\varepsilon}_x(\underline{u}^0) = \frac{1}{|Y|} \int_Y \underline{\sigma}^0 dY \quad (3.24)$$

Considering these definitions, Equation 3.21 can finally be written in the form of an homogenized differential equation:

$$\rho_0 \ddot{\underline{u}}^0 - \operatorname{div}_x \left(\underline{A}^0 : \underline{\varepsilon}_x(\underline{u}^0) \right) = \underline{F}^0 \quad \text{in } \Omega \quad (3.25)$$

According to Equation 3.7, Equation 3.24 can be rewritten as:

$$\underline{\underline{A}}^0 : \underline{\underline{\varepsilon}}_x(\underline{u}^0) = \frac{1}{|Y|} \int_Y \underline{\underline{A}}^0 : [\underline{\underline{\varepsilon}}_x(\underline{u}^0) + \underline{\underline{\varepsilon}}_y(\underline{u}^1)] dY \quad (3.26)$$

From Equation 3.15 one can evaluate:

$$\underline{\underline{\varepsilon}}_y(\underline{u}^1) = \sum_{i,j=1}^3 \varepsilon_{ij}(\underline{u}_0) \underline{\underline{\varepsilon}}_y(\underline{\chi}^{ij}) \quad (3.27)$$

Thus, after substituting Equation 3.27 in Equation 3.26, one retrieve that:

$$\underline{\underline{A}}^0 : \underline{\underline{\varepsilon}}_x(\underline{u}^0) = \frac{1}{|Y|} \int_Y \underline{\underline{A}} : \sum_{i,j=1}^3 \varepsilon_{ij}(\underline{u}^0) [\underline{e}_i \odot \underline{e}_j + \underline{\underline{\varepsilon}}_y(\underline{\chi}^{ij})] dY \quad (3.28)$$

which implies that the components of the homogenized stiffness tensor can be evaluated as:

$$A_{ijkl}^0 = \frac{1}{|Y|} \int_Y (\underline{e}_k \odot \underline{e}_l) : \underline{\underline{A}} : (\underline{\underline{\varepsilon}}_y(\underline{\chi}^{ij}) + \underline{e}_i \odot \underline{e}_j) dY \quad (3.29)$$

From the weak formulation (Equation 3.18) of the cell problem, one can select $\underline{\phi} = \underline{\chi}^{kl}$ and obtain that:

$$\frac{1}{|Y|} \int_Y \underline{\underline{\varepsilon}}_y(\underline{\chi}^{kl}) : \underline{\underline{A}} : (\underline{\underline{\varepsilon}}_y(\underline{\chi}^{ij}) + \underline{e}_i \odot \underline{e}_j) dY = 0 \quad (3.30)$$

Summing up Equation 3.29 and Equation 3.30, one retrieve the final form of the homogenized stiffness tensor components:

$$A_{ijkl}^0 = \frac{1}{|Y|} \int_Y (\underline{\underline{\varepsilon}}_y(\underline{\chi}^{ij}) + \underline{e}_i \odot \underline{e}_j) : \underline{\underline{A}} : (\underline{\underline{\varepsilon}}_y(\underline{\chi}^{kl}) + \underline{e}_k \odot \underline{e}_l) dY \quad (3.31)$$

From the last equation, it can be seen that the homogenized stiffness tensor posses:

- minor symmetries : $A_{ijkl}^0 = A_{jikl}^0 = A_{ijlk}^0 = A_{jilk}^0$
- major symmetry : $A_{ijkl}^0 = A_{klij}^0$

It can also be proven that homogenized stiffness tensor is positive defined.

Chapter 4

Numerical analysis of auxetic materials with two dimensional periodicity

This chapter will study more in detail the case of 2D geometries in the plane-strain regime. The main objectives are to describe the different cells of interest and use the theories presented in the first part of this work to characterize them. More precisely, two re-entrant geometries (honeycomb and triangle) and two chiral geometries (tetra-chiral and hexa-chiral) are investigated. Their effective elastic parameters are computed and their ability to produce auxetic behavior is discussed. The analysis for each geometry will be divided into three phases.

The first phase is performed on the Representative Volume Element with the commercial finite element program *ABAQUS*. Periodic boundary conditions are imposed on the RVE and then, an imposed displacement is applied successively along the vertical and horizontal direction. For the periodic boundary conditions, the continuity of displacement is imposed on the faces of the boundary for the unit cells studied. This first step of the analysis provides results to be compared with those obtained through homogenization.

Secondly, the theory of homogenization is employed and particularized to the 2D case. The three cell problems pertinent for the 2D case will successively be solved and the homogenized elastic coefficient will be retrieved. The tests for the homogenization procedure will be executed on *COMSOL Multiphysics*. The main reason is that this program is more efficient when there is the need to perform several tests with one or more parameters varying between each test. Then, the homogenized effective properties obtained from the two procedures are compared in order to attest the equivalence of the two approaches.

Finally, two experiments will be performed using on one side a structure made of several cells and on the other hand a homogenized media. This last phase will allow to appreciate the accuracy of the homogenization procedure. For the second experiment, the influence of the number of cells is analyzed to better understand the importance of the assumptions needed for the homogenization, i.e. that the size of the unit cell should be sufficiently small if compared with those of the periodic media.

After the analysis of the characterization of each structured material for a particular choice of the geometry, a parametric study will also be completed. These tests will have a coupled goal. First, it will allow to understand the mechanisms of deformation of each geometry and the influence of the different geometric parameters on the elastic response. The second goal will be to underline the pertinence of the homogenization procedure to produce this type of characterization.

At the end of the chapter, a more complete and general parametric comparison between the two re-entrant cells will be performed. This last study will allow to compare the advantages of one geometry over the other in terms of its homogenized elastic parameters.

4.1 Re-entrant honeycomb structure

The first structure to be analyzed is the re-entrant honeycomb. This geometry is one among the most known and studied among the re-entrant geometries. It was first presented in 1985 [5]. Since that time, it has been more and more studied and is still a subject of interest as it can be seen in the recent publications [27,37,40]. The geometry will successively be presented and then analyzed following the procedure described above.

4.1.1 Geometry of the cell

Figure 4.1 represents in black a unique element from which the periodic mesh will be generated and in red the contour of the chosen unit cell. As explained before, this particular choice of unit cell is arbitrary and has been chosen to facilitate the different tests to be performed on the cell.

The main parametric dimensions are also indicated. From the geometry, it can be seen that a certain choice has been made for the number of parameters. This decision is arbitrary and one could achieve the same objective by choosing different parameters. Nevertheless, for the definition of the geometry, the goal was to have a clear geometry while keeping a reasonable number of parameters.

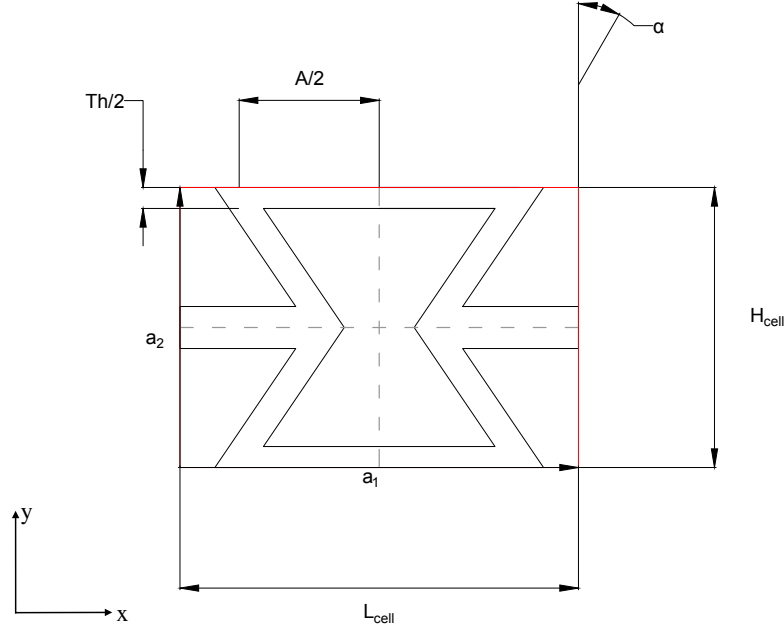


Figure 4.1: Re-entrant honeycomb unit-cell.

Symbol	Description	Formula	Value
A	Width of the midline	/	1 mm
Th	Thickness of the segments	/	0.15 mm
α	Angle of the inclined part	/	30°
H_{cell}	Height of the cell	/	1 mm
L_{cell}	Length of the cell	$A - (H/2) \cdot \tan \alpha$	1.4226 mm

Table 4.1: Main dimensions for the re-entrant honeycomb lattice.

Looking at [Table 4.1](#), it can be noticed (also referring to [Figure 2.1](#)) that the Bravais lattice can be of two types. The unit cell can be either *square* or *rectangular* depending on the specific choice for the parameters. Following the notations presented in Chapter 2, the vectors $\underline{a}_1$ and $\underline{a}_2$ have the same length and orientation as, respectively, L_{cell} and H_{cell} . For the simulations within this chapter, we assume the validity of small strains and displacement hypothesis. We consider the periodic media made of a rigid ethylene-vinyl acetate (EVA), which can be modeled as a linear-elastic-isotropic material. The properties of the material under these hypotheses are listed in [Table 4.2](#).

Parameter	Symbol	Value
Young's modulus	E	480 MPa
Poisson's ratio	ν	0.48
Mass density	ρ^{bulk}	940 kg/m ³

Table 4.2: Mechanical properties of rigid EVA.

4.1.2 Test on the RVE

For the tests on the RVE two different settings will be tested. The first one will be made of one unique cell while the second will be built with four cells. This approach will allow to verify the results by checking that the test performed on one cell is effectively representative of the behavior of multiple cells.

Imposed displacement

The test is performed by imposing a uniform displacement. The test is performed twice as the loading is imposed first along the longitudinal direction (X) and then along the vertical direction (Y). For the geometry, the displacement is imposed symmetrically perpendicularly to the edges.

For the test along the longitudinal direction (Figure 4.2 a), a uniform displacement of 0.05 millimeters is imposed on the right side boundary of the cell along the X direction while a uniform displacement of -0.05 millimeters is imposed on the left side of the cell. In order to avoid rigid body motions, a zero vertical displacement is imposed on the horizontal symmetry axis. For the test along the vertical direction, a uniform displacement of 0.05 millimeters is imposed along the Y direction on the top boundary and a uniform displacement of -0.05 millimeters is forced on the bottom edge of the cell. At last, a zero horizontal displacement is added on the vertical symmetry line to avoid rigid body motions.

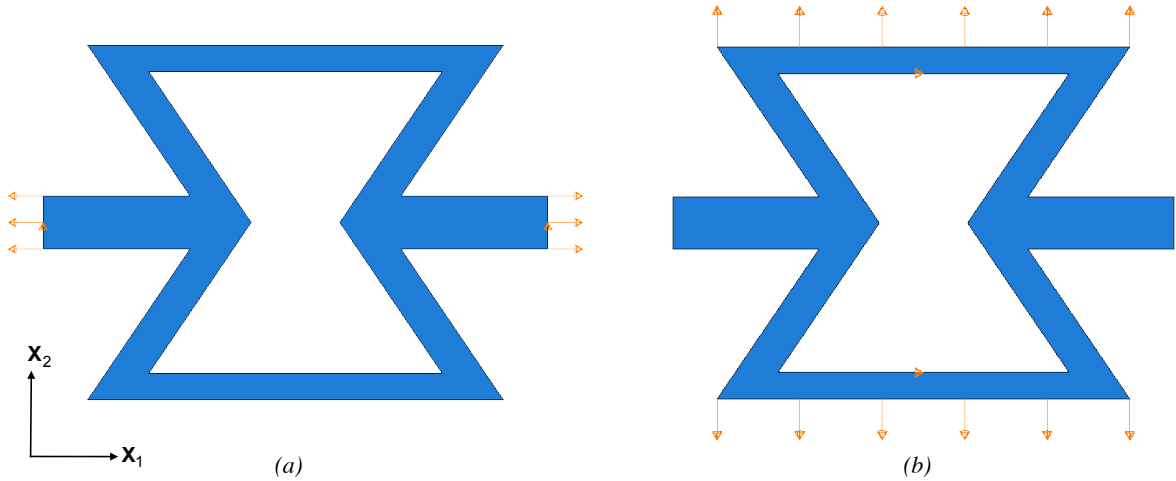


Figure 4.2: Imposed displacement (a) along the longitudinal direction, (b) along the vertical direction.

Periodicity conditions

In order to reproduce the behavior of a periodic media, periodic boundary conditions are imposed on the edges of the cell. For the test with imposed displacement along the longitudinal direction, the horizontal displacement of each node from each side of the axis of symmetry was imposed to be equal. This was imposed using the *Equation* feature in the *Constraints* section of *ABAQUS*. For each node on each side of the horizontal axis of symmetry, the difference of displacement along the horizontal displacement was set equal to zero. Then, the top and bottom edges were modeled as rigid. This last condition was also imposed with the *Equation* feature. More specifically, the sum of the vertical displacement of two successive nodes on the boundary was forced to be equal to zero. For the left and right boundaries, the same types of conditions were imposed. The pair of nodes from one part and the other of the vertical symmetry axis were imposed to have the same vertical displacement. It is to say, that the difference of the vertical displacement for two symmetric nodes was imposed equal to zero.

For the test along the vertical direction, the interactions were imposed with the same procedure as it was done for the longitudinal loading. The horizontal displacement was forced to be of the same amplitude and opposite sign on the left and right edges. The vertical displacement was set to be equal for symmetric nodes and finally, rigidity was also imposed for these edges. For the top and bottom faces, only the horizontal displacement was constrained to be equal for symmetric nodes.

Results of the tests

For the load case along the longitudinal direction (X), the elastic parameters were computed as follow. The equivalent Poisson's ratio is computed as the opposite of the ratio between mean transversal and longitudinal strains ε_t and ε_l . These latter ones read:

$$\varepsilon_l = \varepsilon_x = \frac{U_{1,right} - U_{1,left}}{L_{cell}} = \frac{0.05 - (-0.05)}{1.4226} = 0.0351 \quad (4.1)$$

$$\varepsilon_t = \varepsilon_y = \frac{U_{2,top} - U_{2,bottom}}{H_{cell}} = \frac{0.02415 - (-0.02415)}{1} = 0.04831 \quad (4.2)$$

Hence the resulting equivalent Poisson's ratio is:

$$\nu_{xy}^* = \frac{-\varepsilon_t}{\varepsilon_l} = -0.6872 \quad (4.3)$$

Similarly, the equivalent Young's modulus is obtained as the ratio between the stress and the strain acting in the longitudinal direction. The strain is computed as presented above and the stress is computed as the ratio between the total force acting on one of the vertical edges and the height of the unit cell (H_{cell}). The total force was computed as the

sum of the forces acting along the longitudinal direction for each node of the right-side vertical edge.

$$F_{x,total} = 0.7617 \text{ N/m} \quad (4.4)$$

$$E_x^* = \frac{\sigma_x}{\varepsilon_x} = 10.8365 \text{ MPa} \quad (4.5)$$

For the loading along vertical direction (Y) the same procedure is followed:

$$\varepsilon_t = \varepsilon_x = \frac{U_{1,right} - U_{1,left}}{L_{cell}} = \frac{0.0655 - (-0.0655)}{1.4226} = 0.0915 \quad (4.6)$$

$$\varepsilon_l = \varepsilon_y = \frac{U_{2,top} - U_{2,bot}}{H_{cell}} = \frac{0.05 - (-0.05)}{1} = 0.1 \quad (4.7)$$

$$\nu_{yx}^* = \frac{-\varepsilon_t}{\varepsilon_l} = -0.9155 \quad (4.8)$$

The Young's modulus was computed similarly to the one for the longitudinal imposed displacement. More specifically, the strain is the vertical one instead and the strain is computed as the ratio between the total vertical force and the length of the unit cell (L_{cell}):

$$F_{y,total} = 2.0626 \text{ N/m} \quad (4.9)$$

$$E_y = \frac{\sigma_y}{\varepsilon_y} = 14.4987 \text{ MPa} \quad (4.10)$$

Figure 4.3 shows the deformed shape of the unit cell for the test along the longitudinal direction.

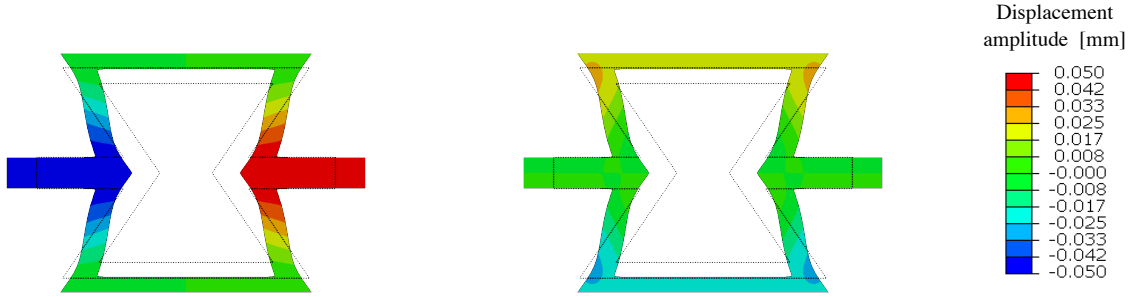


Figure 4.3: Deformed configuration of the cell for the test along the X direction. In black the undeformed mesh is represented. The left contour represents the amplitude of the longitudinal displacement and the right one, the amplitude of the vertical displacement.

It can be seen from the deformed shape that the re-entrant honeycomb belongs, as described in the introduction, to the re-entrant geometry. The auxetic behavior of the cell is produced by the re-alignment of the ribs. Finally, the two elastic matrices for the plane strain problem can be obtained.

The compliance matrix reads:

$$\underline{\underline{D}} = \begin{bmatrix} 0.0923 & 0.0626 & 0 \\ 0.0626 & 0.0690 & 0 \\ 0 & 0 & \frac{1}{G_{xy}^*} \end{bmatrix} \text{MPa}^{-1} \quad (4.11)$$

The stiffness matrix reads :

$$\underline{\underline{C}} = \begin{bmatrix} 28.1640 & -25.5516 & 0 \\ -25.5516 & 37.6744 & 0 \\ 0 & 0 & G_{xy}^* \end{bmatrix} \text{MPa} \quad (4.12)$$

Note that the shear modulus G_{xy}^* could have been computed by a third analysis imposing a horizontal displacement on the top edge and computing the ratio between the shear strain and the reaction stress acting against the imposed displacement.

The same experiment was considered on an RVE made of 4 individual cells to check the results on the unique cell. The procedure for the test on four cells can directly be deduced from the procedure for a unique cell. [Figure 4.4](#) shows the deformed shape of the four-unit cells for the testing along the longitudinal direction.

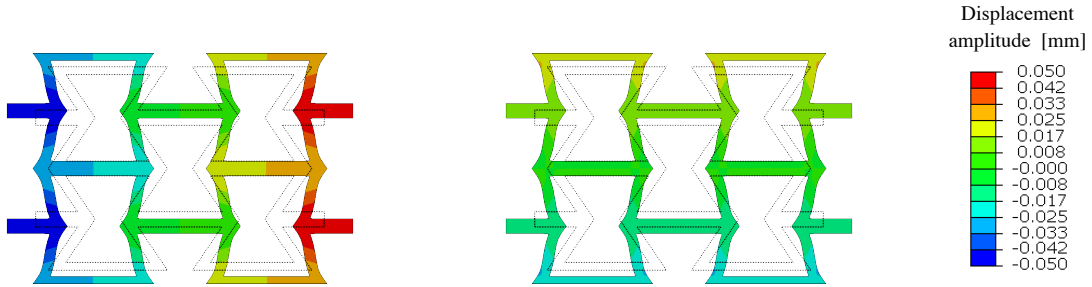


Figure 4.4: Deformation mechanism of the four cells for the test along the X direction. In black the undeformed geometry is represented. The left contour represents the amplitude of the longitudinal displacement and the right one, the amplitude of the vertical displacement.

The compliance matrix reads:

$$\underline{\underline{D}} = \begin{bmatrix} 0.0926 & 0.0637 & 0 \\ 0.0637 & 0.0691 & 0 \\ 0 & 0 & \frac{1}{G_{xy}^*} \end{bmatrix} \text{MPa}^{-1} \quad (4.13)$$

The stiffness matrix:

$$\underline{\underline{C}} = \begin{bmatrix} 29.50208 & -27.1960 & 0 \\ -27.1960 & 39.5361 & 0 \\ 0 & 0 & G_{xy}^* \end{bmatrix} \text{ MPa} \quad (4.14)$$

At this point, it can be seen that the two experiments give rise to similar estimates of the equivalent parameters even though a difference of 0.33 % is obtained on E_x^* and of 0.15 % on E_y^* . As explained, in the introduction, these results justify the capacity of the RVE made of one cell to represent the behavior of the periodic media. Moreover, these results will serve as the point of comparison for the next step, namely, homogenization.

4.1.3 Homogenization

The homogenization was performed on the same unit cell that was used for the first experiment (Figure 4.1). More precisely, in the case of two-dimensional geometries, only three cell problems are necessary to retrieve the homogenized two-dimensional stiffness matrix. The procedure for solving the cell problems is explained hereafter.

Boundary conditions

The boundary for this choice of the unit cell is made of four edges: two on the top and the bottom of the cell, and two on the right and left side of the cell. Because the displacement field of the cell problem should be periodic on the boundary (3.16), the boundary condition *Periodic displacement* was imposed in *COMSOL Multiphysics* respectively for the two pairs of opposite edges. To avoid any rigid body motion, one point on the geometry was fixed with all the components of displacement equal to zero. The choice of the point is arbitrary and does not affect the procedure since the homogenized stiffness depends only on the strains of the cell problems. The boundary conditions remain the same for the three different cell problems.

Loading

For each of the three cell problems, a uniform strain state is imposed through the condition *Initial strain state*. The three problems are characterized by a different imposed eigenstrain.

$$\underline{\underline{\varepsilon}}_0 = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_y \end{bmatrix} \quad (4.15)$$

For the first case, all the entries of the initial strain matrix (4.15) will be set equal to zero except for the strain along the X direction (ε_x). For this first case, the strain ε_x will be set equal to -0.1 . For the second case, the only the strain along the Y direction

will be different from zero and equal to $\varepsilon_y = -0.1$. The third case will have a loading with all the elements of the strain matrix equal to zero except for the two strains ε_{xy} and ε_{yx} which will be set equal to -0.05 . Table 4.3 gathers the values of eigenstrains for the three cell problems which allows to compute the solution of the cell problems $\underline{\chi}^{ij}$

Cell problem	ε_{xx}	ε_{yy}	$\varepsilon_{xy} = \varepsilon_{yx}$
$\underline{\chi}^{11}$	-0.1	0	0
$\underline{\chi}^{22}$	0	-0.1	0
$\underline{\chi}^{12}$	0	0	-0.05

Table 4.3: Imposed eigenstrains for each cell problem.

The representation of the solution for three cell problems described can be find in Figure 4.5, 4.6 and 4.7.

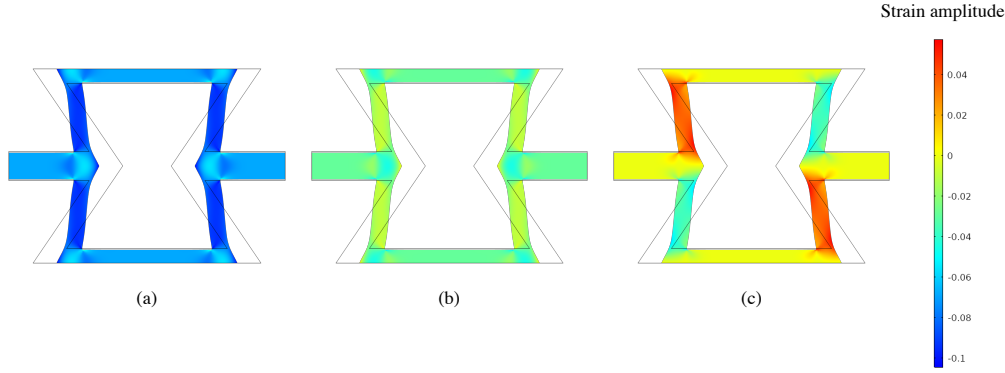


Figure 4.5: First cell problem: contour plots on the deformed shape of (a) ε_x , (b) ε_y and (c) ε_{xy} .

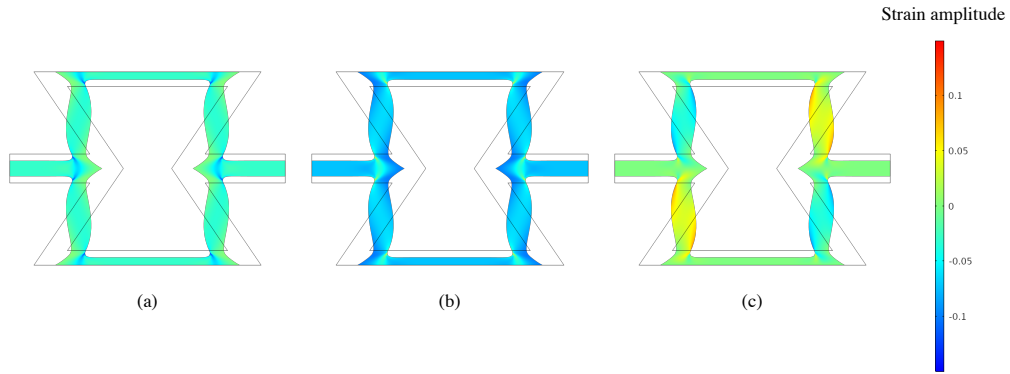


Figure 4.6: Second cell problem: contour plots on the deformed shape of (a) ε_x , (b) ε_y and (c) ε_{xy} .

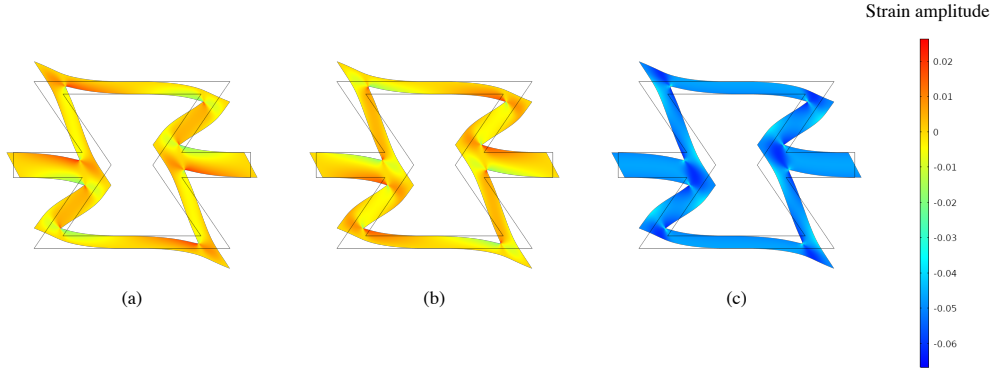


Figure 4.7: Third cell problem: contour plots on the deformed shape of (a) ε_x , (b) ε_y and (c) ε_{xy} .

From the solution of these three cell problems, the equivalent elastic matrix can be retrieved using the equations (3.31) and applying a proper scaling of the imposed strains

$$\underline{\underline{C}} = \begin{bmatrix} 29.313 & -27.183 & 0 \\ -27.183 & 38.508 & 0 \\ 0 & 0 & 112.54 \end{bmatrix} \text{ MPa} \quad (4.16)$$

It can be seen that the above stiffness matrix describes an orthotropic material. To compare this matrix with the stiffness matrix obtained by the previous analysis on the RVE, we compute the relative difference on the single terms defined as:

$$E(i, j) = 100 \cdot \frac{|C_{RVE}(i, j) - C_{Homo}(i, j)|}{C_{RVE}(i, j)} \quad (4.17)$$

Considering $\underline{\underline{C}}_{RVE}$ as the stiffness matrix obtained with the test on 4 cells and $\underline{\underline{C}}_{Homo}$ the one presented above, the following result is obtained:

$$[\underline{\underline{E}}] = \begin{bmatrix} 0.6409 & 0.0478 \\ 0.0478 & 2.6004 \end{bmatrix} \% \quad (4.18)$$

It can be seen that except for the term on the second diagonal term, the relative difference is always smaller than 1%. The two formulations for the characterization give rise to the same elastic parameters.

4.1.4 Validation of the homogenization

In this section, the analyses for the validation of the homogenization procedure are presented. The objective is mainly to assess the accuracy of the homogenized parameters to reproduce the behavior of a real periodic media as a function of the number of cells.

Two pairs of analyses are performed for the assessment of the homogenization. The first one will be loaded with a uniform displacement of 0.1 mm on the right side and rollers will be placed on the left side of the real mesh (Figure 4.8 (a)). In addition, a point will be fixed to avoid rigid body motion. For the first part of the second pair of analyses, the real mesh will be loaded with a surface load along the longitudinal direction with an amplitude of 1000 N/m². As for the boundary conditions, the choice was made to clamp the bottom edge of the mesh forcing both displacements and rotation equal to zero (Figure 4.8 (c)). On this second analysis, the number of cells will also vary in order to understand the influence of this parameter on the accuracy of the results.

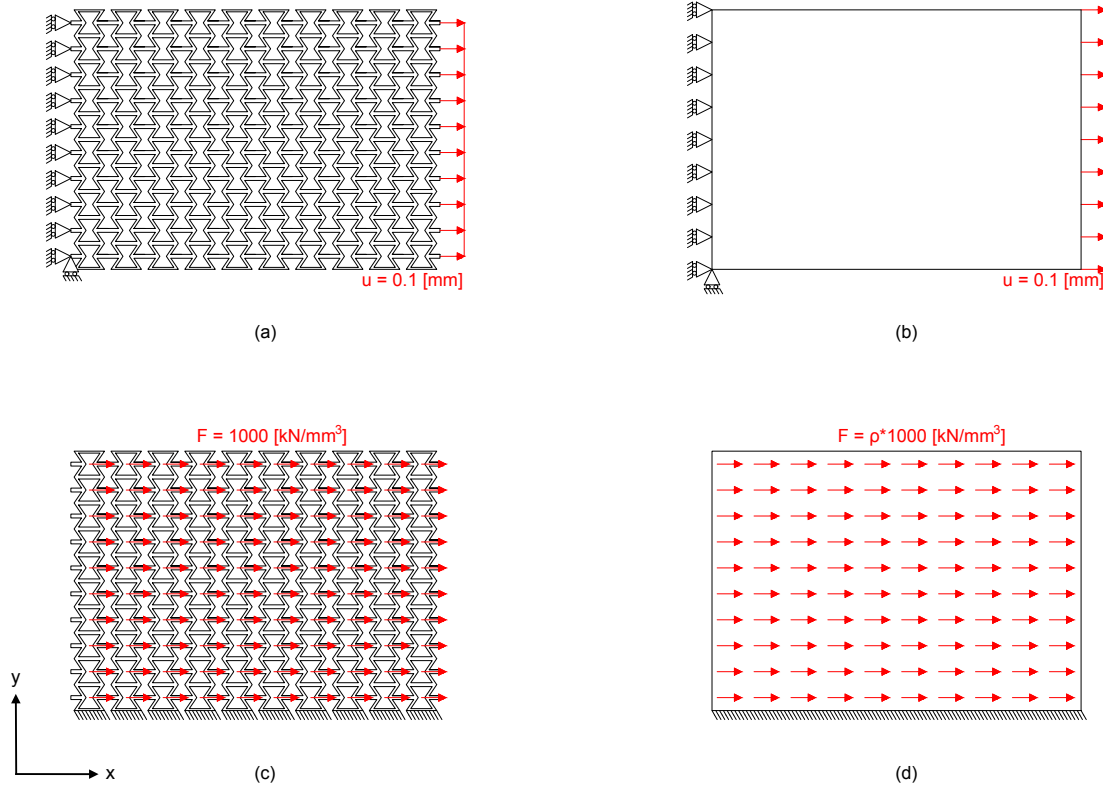


Figure 4.8: Sketch of the two analyses on the re-entrant honeycomb geometry: (a) real geometry and its boundary conditions of the first part of the first analysis, (b) homogeneous media and its boundary conditions of the second part of first analysis, (c) real geometry and its boundary conditions of the first part of second analysis, (d) homogeneous media and its boundary conditions of the second part of the second analysis.

The second part of the analyses of each pairs is performed on the homogeneous media. A rectangle with the same global dimensions as the real mesh will be created. This means that if the real mesh is made out of an array of N unit cells by M unit cells the homogeneous media will have as length $N \cdot L_{cell}$ and as height $M \cdot H_{cell}$.

The material for the homogeneous media was modeled as *linear elastic orthotropic*. For the first pair of analysis, the boundary conditions remain exactly the same. (Figure 4.8 (b)). For the second pair of analysis, the homogenized load was computed using equation (3.23) for the homogenized force vector. The bottom edge was also clamped (all the degrees of freedom are set equal to zero) as to reproduce the same boundary condition as for the real mesh (Figure 4.8 (d)).

For the first analysis, the total reaction force on the right edge is monitored for the two parts of the analysis and will allow assessing the accuracy of the equivalent homogenized coefficients. In Figure 4.9 the deformed shapes of the two parts of the analysis are represented. On the left is the result of the part presented in Figure 4.8 (a) while on the right, is the result of the part presented in Figure 4.8 (b).

Number of cells in the real media	F in the real media	F [N] in the homogeneous mesh	Relative error
10×10	675.779 N	678.568 N	0.413 %

Table 4.4: Comparison of the results for the first analysis.

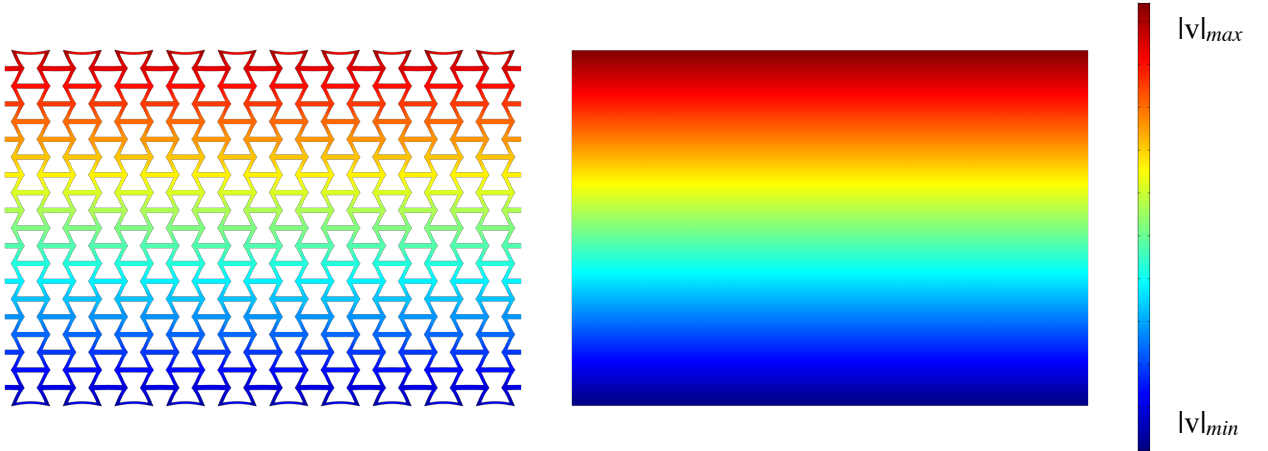


Figure 4.9: Deformed shapes and contour plots of the vertical displacement amplitude for the two parts of the first analysis with the same scale and 10×10 cells.

Looking both at Table 4.4 and Figure 4.9, it can be seen that the homogeneous media is able to reproduce both the vertical displacement field and the total reaction force with a relative error smaller than 0.5% in term of total reaction force. This first analysis also demonstrates the auxetic behavior as the action of pulling along the horizontal direction produces an expansion along the vertical direction.

The comparison between the two steps of the second analysis will be performed in terms of the displacements. The displacement of the node of the top right cell at the extremity of the branch will be monitored. Within the homogeneous media, a node with the same geometric position as the one in the real mesh will be created and its displacement will as well be monitored. The relative error will then be computed as the ratio of the difference of displacement between the two experiments over the displacement in the real mesh and the displacement of the real mesh. On the left is the result of the analysis presented in Figure 4.8 (c) while on the right, is the result of the analysis presented in Figure 4.8 (d).

The displacements and the error can be found in the Table 4.5. The experiment was successively reproduced with different numbers of cells. The reason is that, as explained in the chapter about homogenization, the theory relies on a sufficient scale separation between the micro and macro-scale of the problem. The study of the tests as a function of the number of cells allows therefore to better understand the minimal number of cells for which the relative error becomes acceptable.

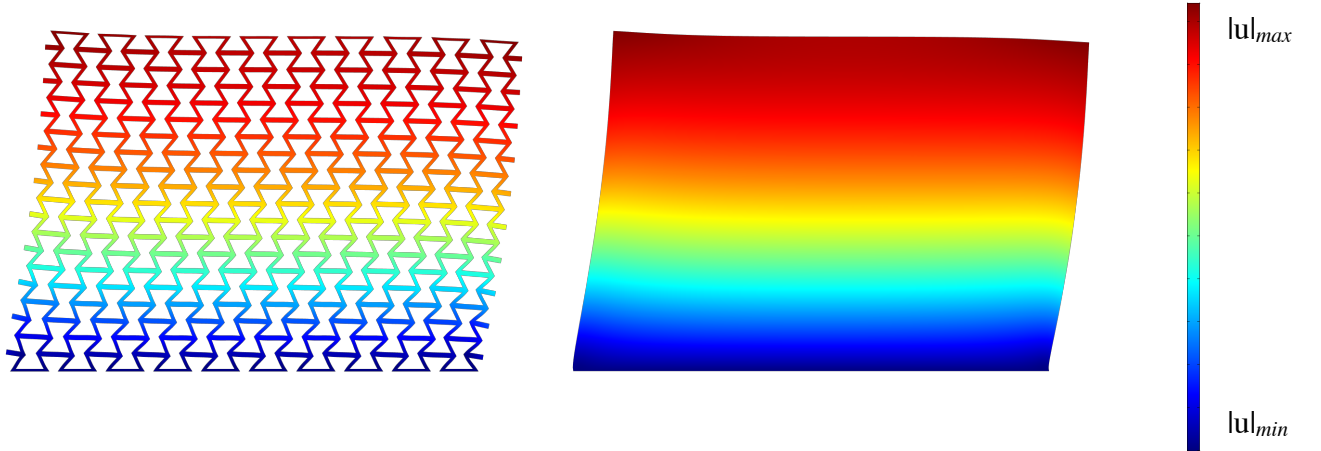


Figure 4.10: Deformed shapes and contour plots of the horizontal displacement amplitude for the two parts of the second analysis with the same scale and 10x10 cells.

For the geometry of re-entrant honeycomb, the relative error even for a small amount of cell is around 6% and decrease monotonously up to 1.3% (Table 4.5). For this specified geometry, the results within the homogeneous media tend to be more compliant in the sense that the displacement is always smaller for the analyses on the real geometry.

Number of cells in the real mesh	u [mm] in the real mesh	u [mm] in the homogeneous mesh	Relative error [%]
2×2	7.0798·e − 04	7.5331·e − 04	6.4026
3×3	1.6836·e − 03	1.7681·e − 03	5.0176
4×4	3.0726·e − 03	3.1991·e − 03	4.1177
5×5	4.8809·e − 03	5.0477·e − 03	3.4189
6×6	7.1095·e − 03	7.3147·e − 03	2.8868
7×7	9.7591·e − 03	1.0001·e − 02	2.4742
8×8	1.2830·e − 02	1.3105·e − 02	2.1475
9×9	1.6322·e − 02	1.6630·e − 02	1.8840
10×10	2.0236·e − 02	2.0573·e − 02	1.6677
11×11	2.4571·e − 02	2.4936·e − 02	1.4874
12×12	2.9328·e − 02	2.9719·e − 02	1.3349

Table 4.5: Comparison of the results with different number of cells for the second analysis.

Finally, in [Figure 4.11](#) the logarithm of the relative error on the displacement was represented as a function of the logarithm of the number of cells in the real mesh. It can be seen that the relative error seems to be inversely proportional to the inverse of the square roots of the total number of cells. It is to say inversely proportional to the number of cells in one direction as the array was created with the same number of cells in both directions. One should pay attention that the logarithm of the relative error has been normalized so that the logarithm of the error for the array of 12×12 cells is equal to one.

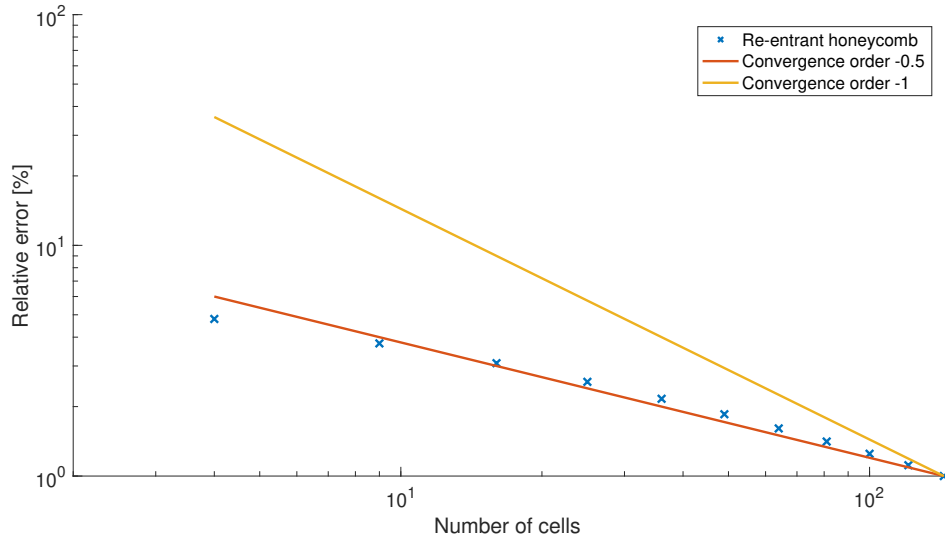


Figure 4.11: Bi-logarithmic plot of the relative error between the real and the homogenized displacement against the number of cells.

4.1.5 Parametric study of the cell

In this subsection, a parametric study is performed on the re-entrant honeycomb cell. For this part, the object is to quantify the elastic parameters of the cell as functions of the geometrical parameters. A certain number of parameters is chosen as variables and the procedure of homogenization is applied for each combination of parameters. The main goal is to give a first intuition of the influence that each parameter play on the elastic parameters.

The same geometry presented in [Figure 4.1](#) is used for the tests. Among all the parameters, the choice for this first parametric study has been made to take the angle α and the thickness Th as parameters. The study is done on a specific range of values for these two parameters. The range of values for the study can be found in the [Table 4.6](#).

The first parameter α is found to be a particularly interesting parameter for the study. The reason is that, as seen during the characterization of the cell, the re-entrant honeycomb belongs to the re-entrant geometries. As the re-alignment of the ribs is the mechanism producing the auxetic behavior, the re-entrant angle α is therefore directly linked to this feature.

Symbol	Description	Formula	Interval of values
A	Width of the midline	/	1 mm
Th	Thickness of the segments	/	0.1 - 0.31 mm
α	Angle of the inclined part	/	10-32°
H_{cell}	Height of the cell	/	1 mm
L_{cell}	Length of the cell	$A - (H/2) \cdot \tan(\alpha)$	0.35-1 mm

Table 4.6: Main dimensions and physics for the parametric problem.

The second parameter is directly linked to the density, as the thickness increases the material tends to occupy the whole area of the cell. One should therefore expect to retrieve the elastic parameters of the bulk material from which the auxetic structure is made. For this study, the relative density ρ is defined as:

$$\rho = \frac{\text{Area filled with the cell}}{L_{cell} \cdot H_{cell}} = \frac{\rho^0}{\rho^{bulk}} \quad (4.19)$$

ρ^0 being the equivalent mass density of the metamaterial and ρ^{bulk} the mass density of the constituent bulk material.

The parametric study is then performed. The choice is made to perform this test on *COMSOL Multiphysics*. Together with the sweep of parameters that defined the initial strain state for each cell problem, the two parameters for which the influence is studied are added to the sweep. The three pertinent cell problems for the 2D case are therefore

solved for each combination of the two parameters. The equivalent elastic parameters are then obtained as was done in the previous section dealing with homogenization. A contour plot of the four homogenized elastic parameters (E_x^0, E_y^0, ν_{xy}^0 and ν_{yx}^0) are then created. The vertical axis represents the re-entrant angle α and the horizontal one shows the relative density.

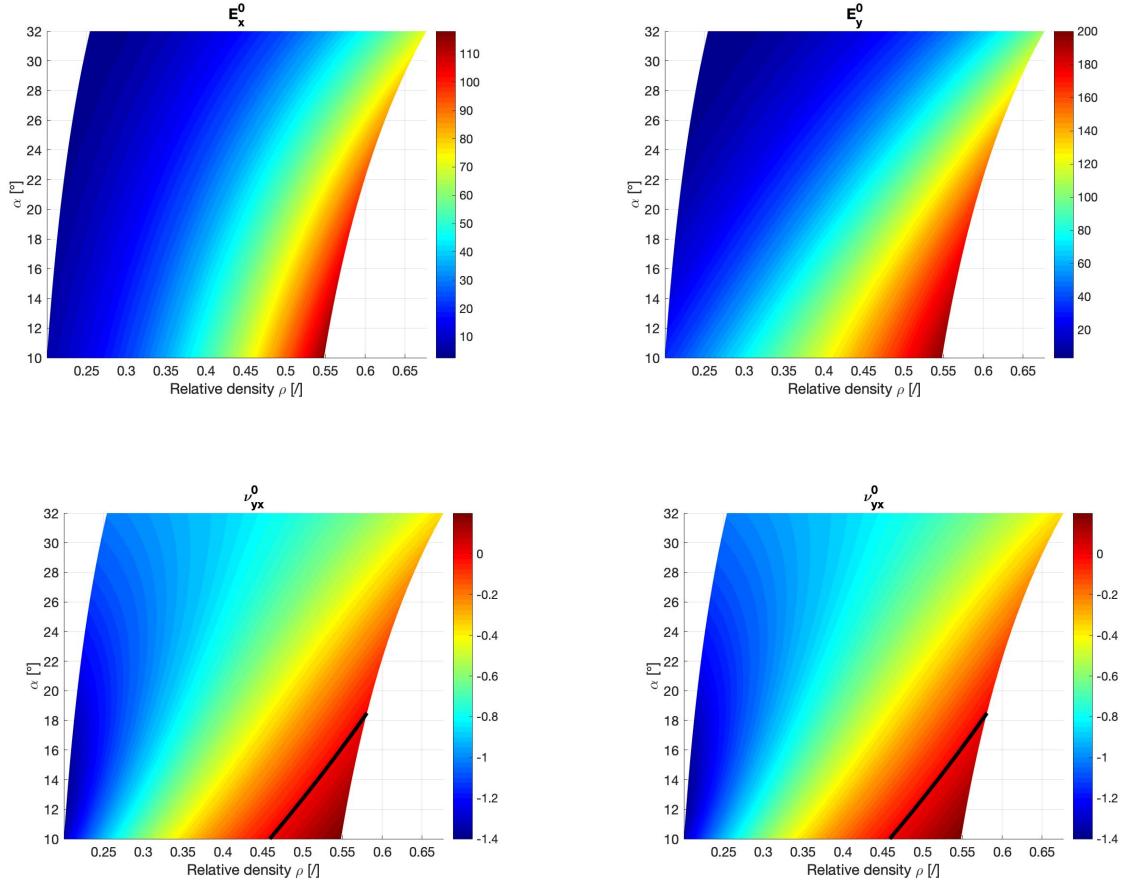


Figure 4.12: Contour plots of the equivalent Young's moduli (first row: E_x^0 left and E_y^0 right) and Poisson's ratio (second row: ν_{xy}^0 left and ν_{yx}^0 right) for the re-entrant honeycomb.

The first thing that can be noticed is that some zones of the graphs remain white. This is due to the fact that when the angle α and the thickness vary not all the cells can be filled and the relative is a non-linear function of these two parameters.

In terms of the elastic parameters, it can be seen that the two Young's moduli vary almost in the same way and that the value of E_x^0 is roughly half of E_y^0 . As expected, the Young's modulus tends to the value of the bulk material as the relative density tends

to 1. Concerning the Poisson's ratio, the metamaterial exhibits an auxetic behavior for a large variety of geometries. Only combinations with high relative density and small angle α have positive Poisson's ratios.

4.2 Re-entrant triangle structure

The second studied geometry is the so-called re-entrant triangle. The geometry, as the name indicates, is made of two arrowheads one on top of the other. Less reviewed than the previous geometry, it is nevertheless pertinent to study it. The reason is that the new 3D geometry that will be presented in the last part of this work will be inspired by this 2D geometry. The procedure to analyze this second cell is exactly the same as the one presented for the re-entrant honeycomb.

4.2.1 Geometry of the cell

In [Figure 4.13](#), the complete geometry is represented, it is to say the two arrowheads. Together with this, the choice of parameters for the description of the cell is also indicated. As for the re-entrant honeycomb, the choice was made to have at the same time an efficient representation of the cell while having a certain number of parameters to define the cell.

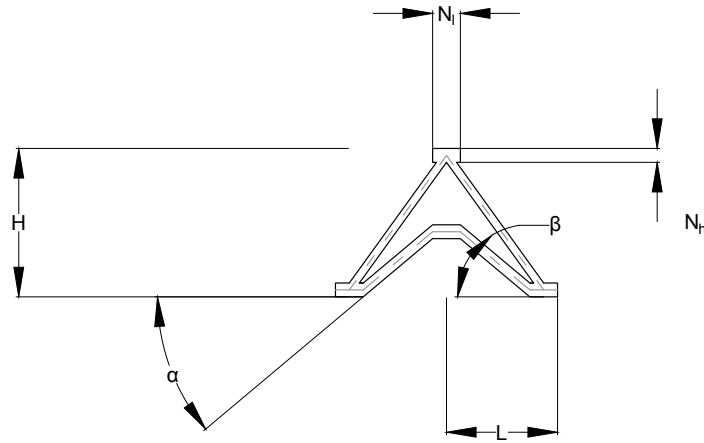


Figure 4.13: Single re-entrant triangle geometry.

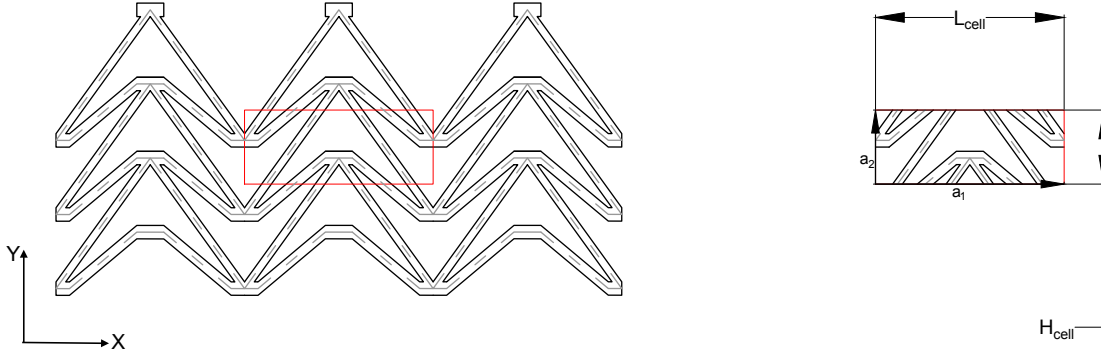


Figure 4.14: Single re-entrant triangle lattice and its unit-cell.

In [Figure 4.14](#), the chosen unit cell is drawn. For the re-entrant triangle, one should pay attention to the specific choice of unit cell. In the case of the re-entrant honeycomb, the choice for the unit cell was arbitrary but easier to make as the unit cell was coincident with the base geometry. Here it can be seen that the geometry presented in [Figure 4.13](#) does not represent a unit cell as the entire mesh can not be reproduce only by translation of the latest. The specific choice of unit cell plays an important role, especially when considering the test on the RVE as it will be explained later.

The geometrical quantities can be found in [Table 4.7](#). This specific choice of parameters will be chosen for the study on the RVE and for the unit cell used the homogenization. The specific choice is made to obtain the same relative density.

Symbol	Description	Formula	Value
L	Half width of the triangle	/	0.75 mm
H	Height of the triangle	/	1 mm
N_l	Length of the rectangular part	/	0.1875 mm
N_h	Thickness of the rectangular part	/	0.0938 mm
α	Angle of the inferior inclined part	/	40°
β	Angle of the external branches	$\text{atan}(\frac{H-N_h}{L-0.5 \cdot N_l})$	54.0903°
H_{cell}	Height of the cell	$H - N_h + -(L - 1.5N_l) \cdot \tan \alpha$	0.5129 mm
L_{cell}	Length of the cell	$2L - N_l$	1.3125 mm

Table 4.7: Main dimensions for the re-entrant triangle the lattice.

Here again, the unit cell can either be *square* or *rectangular* depending on the specific choice for the parameters. For the choice defined in [Table 4.7](#), the cell is rectangular.

4.2.2 Test on the RVE

Just as it has been done for the re-entrant honeycomb, the re-entrant triangle cell is characterized using the RVE. The procedure is the same and the results will be expressed in the same way. At this point, it should also be added that for the re-entrant triangle, the boundary conditions are not as evident to impose as it was for the case of re-entrant honeycomb. For the re-entrant honeycomb, the symmetries of the cell about both the vertical and horizontal axis gave the indication that no rotation was possible on the boundary of the cell. For the re-entrant triangle cell, the symmetry along the vertical axis was still present but not the one along the horizontal axis. There are no indications, a priori, on the rotation and the displacement of the top and bottom branches except the needed compatibility of the deformed shape inside the lattice. A more deep investigation and understanding of the deformation mechanism were needed to be able to impose the proper boundary conditions.

Imposed displacement and boundary conditions

For the test along the longitudinal direction (X), the longitudinal displacement is imposed uniformly on the left and right edge of the cell. The amplitude of the displacement is of -0.05 millimeters on the left edge and of 0.05 millimeters on the right edge. In order to avoid rigid body motions, the vertical displacement is fixed on two points localized on the same horizontal line. The value of this displacement is imposed so that the contour plots of the displacements turn out to be symmetric w.r.t. the vertical axis.

For the testing along the vertical direction, the displacement is imposed in one node in the middle of each branch. For the two branches on top, the amplitude of the vertical displacement is imposed equal to 0.05 millimeters on the two nodes localized respectively in the middle of the two branches. The displacement for the bottom branches is imposed in the same way but the amplitude was of -0.05 millimeters. To avoid rigid body motions the horizontal displacement in the vertical line of symmetry of the cell is set equal to zero.

For the test along the X direction the other imposed conditions are the following: The top and bottom branches must have the same horizontal displacement. This first interaction is imposed using the *Equation* feature on *Abaqus*. The equation is imposed such that the horizontal displacement of a node belonging to the top branches minus the displacement of the node having the same longitudinal coordinate on the bottom branches must be equal to zero. The second condition is imposed on the relative displacement namely: the relative displacement between two successive nodes on the top branches is the same. The last interaction is imposed such that the extremity of the branches remains straight, it is to say that no curvature is allowed. These three interactions ensure the compatibility of the deformed shape inside the lattice.

For the test along the vertical direction, the same interactions are imposed on the top and bottom branches. In addition, interaction of the left and right edges are added. The first one imposes that the two edges must remain straight and the second one that the same vertical displacement must be the same for two nodes on each side of the axis of symmetry.

Results of the tests

The results of the test along the X direction are shown in [Figure 4.15](#). The amplitude of the horizontal and vertical displacement are represented respectively on the right and left of the figure.

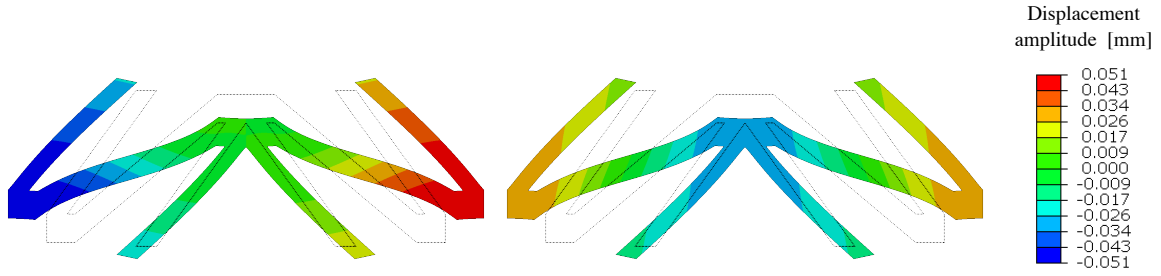


Figure 4.15: Deformed shapes of the triangular cell for the test along the X direction. In black the undeformed cell is represented. The left contour represents the amplitude of the longitudinal displacement and the right one, the amplitude of the vertical displacement.

Analyzing [Figure 4.15](#), it can be seen that the mechanism of deformation consists of the re-alignment of the ribs through the inflection of the branches. The auxetic property of the cell is therefore due to this specific mechanism of deformation. In terms of the stiffness matrix for the plane strain problem, the results are reported below and will be used as a comparison point for the homogenization procedure.

$$\underline{\underline{C}} = \begin{bmatrix} 6.1501 & -5.4215 & 0 \\ -5.4215 & 8.0416 & 0 \\ 0 & 0 & G_{xy}^* \end{bmatrix} \text{ MPa} \quad (4.20)$$

The analysis is also performed on four cells and the results are represented in [Figure 4.16](#). The procedure for the loading and the interaction was directly retrieved for the testing on the unit cell.

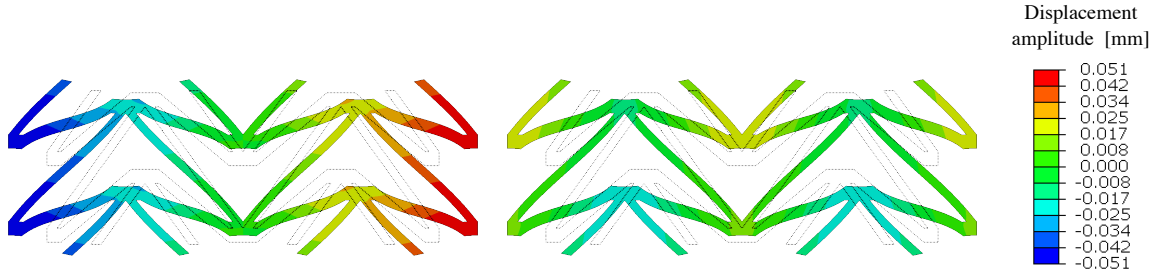


Figure 4.16: Deformed shapes of four triangular cells for the test along the X direction. In black the undeformed cell is represented. The left contour represents the amplitude of the longitudinal displacement and the right one, the amplitude of the vertical displacement.

The stiffness matrix computed through this analysis reads:

$$\underline{\underline{C}} = \begin{bmatrix} 6.0979 & -5.3421 & 0 \\ -5.3421 & 7.9281 & 0 \\ 0 & 0 & G_{xy}^* \end{bmatrix} \text{ MPa} \quad (4.21)$$

The same characterization is obtained for the testing on a unit cell and the one on four cells, with an error lower than 1.5%.

4.2.3 Homogenization

The homogenization procedure is performed on the unit cell shown in Figure 4.13. Periodic boundary conditions are imposed on the top and bottom branches as well as on right and left branches. One point was fixed to avoid rigid body motions. Finally, the three pertinent cell problems are solved with *COMSOL Multiphysics* with the *parametric sweep* as presented for the re-entrant honeycomb geometry.

The following stiffness matrix is obtained for the plane strain case.

$$\underline{\underline{C}} = \begin{bmatrix} 6.0671 & -5.3296 & 0 \\ -5.3296 & 7.9016 & 0 \\ 0 & 0 & 5.5600 \end{bmatrix} \text{ MPa} \quad (4.22)$$

Equation 4.17 is used and the following results is obtained:

$$[E] = \begin{bmatrix} 0.5081 & 0.2334 \\ 0.2334 & 0.3343 \end{bmatrix} \% \quad (4.23)$$

It can be seen that both experiments are in good agreement with a maximal relative error smaller than 0.6%. Nevertheless, it should be stressed that the procedure of homogenization was by far easier to apply in the sense that the interaction part for the test on the RVE was more challenging for this geometry.

4.2.4 Validation of the homogenization

As it was done for the re-entrant honeycomb, two different analyses are performed with a mesh made of unit cells and with the homogeneous media and the results are compared. The procedure, the loading, the boundary conditions, and the characterization of the material follow exactly the same path as the one presented for the re-entrant honeycomb geometry. (Figure 4.17).

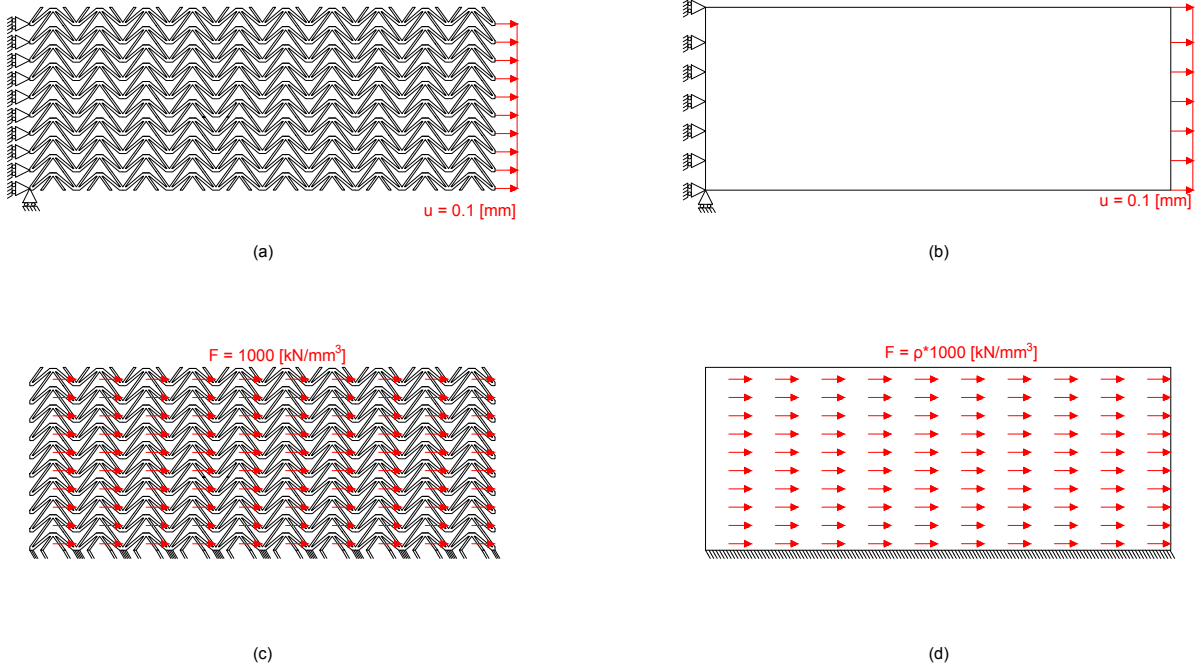


Figure 4.17: Sketch of the four analyses: (a) real geometry and its boundary conditions of the first analysis, (b) homogeneous media and its boundary conditions of the first analysis, (c) real geometry and its boundary conditions of the second analysis, (d) homogeneous media and its boundary conditions of the second analysis.

Considering the first analysis, the deformed shapes of the two parts of the analysis are represented in Figure 4.9. On the left is the result of the test presented in Figure 4.17 (a) while on the right, is the result of the test presented in Figure 4.17 (b).

Number of cells in the real media	F in the real mesh	F in the homogeneous mesh	Relative error
10×10	97.995 N	96.619 N	1.404 %

Table 4.8: Comparison of the results for the first analysis.

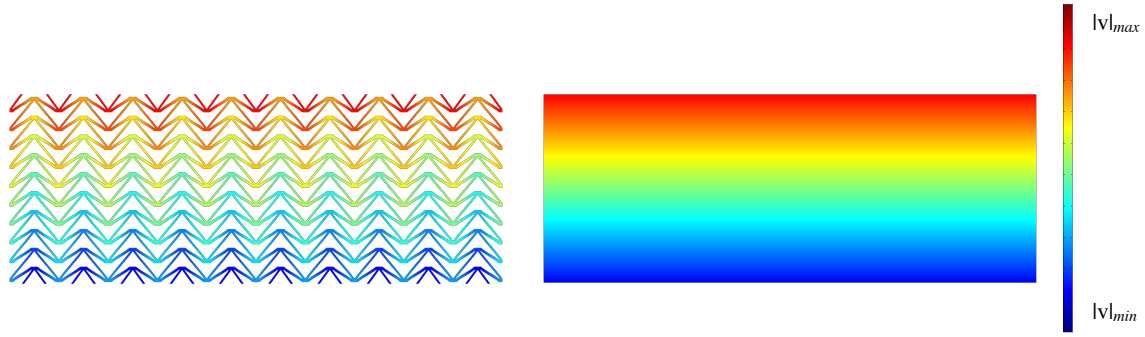


Figure 4.18: Deformed shapes and contour plots of the vertical displacement amplitude for the two parts of the first analysis with the same scale and 10×10 cells.

From the results of this analysis an important result of the procedure of homogenization is emphasized. The homogenization procedure produces a global macroscopic response and not a local one. In [Figure 6.8](#), the two contour plots of the vertical displacement do not seem equivalent to the homogenized one. In reality, if one looks carefully, the global displacements are the same. The homogeneous media does not keep track of the deformation mechanism occurring in the real mesh. On the contrary, at the level of the global response (total horizontal reaction force), the homogeneous media is able to predict the results with an error of less than 1.5%.

In [Figure 4.19](#), the results of the second analysis are represented. On the left is the result of the analysis presented in [Figure 4.17](#) (c) while on the right, is the result of the analysis presented in [Figure 4.17](#) (d).

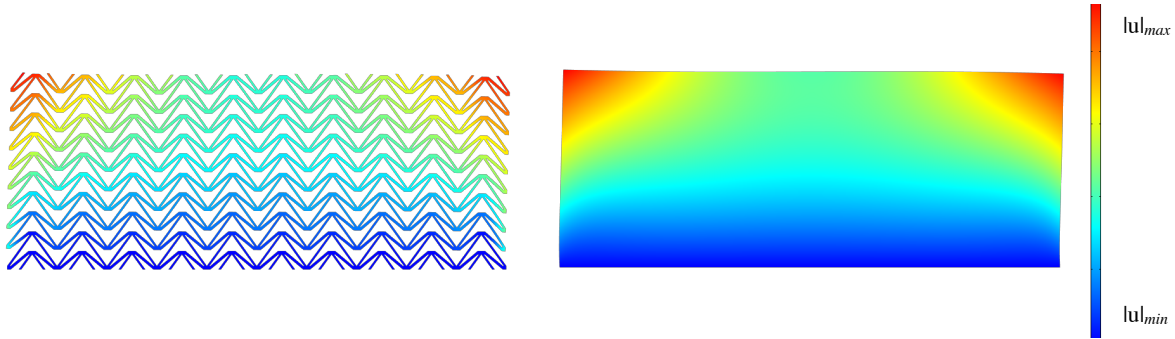


Figure 4.19: Deformed shapes and contour plots of the displacement amplitude for the two parts of the second analysis with the same scale and 10×10 cells.

In [Table 4.9](#) and [Figure 4.20](#), it can be seen that the initial error is greater than for the case of the re-entrant honeycomb. Nevertheless, the convergence of the relative error seems to be faster than the for the re-entrant honeycomb.

Number of cells in the real mesh	u [mm] in the real mesh	u [mm] in the homogeneous mesh	Relative error [%]
2×2	1.2471e-04	4.1627e-05	66.620
3×3	1.8379e-04	1.2545e-04	31.743
4×4	3.1989e-04	2.4921e-04	22.097
5×5	4.9484e-04	4.1334e-04	16.471
6×6	7.0189e-04	6.1809e-04	11.938
7×7	9.4592e-04	8.6361e-04	8.7017
8×8	1.2297e-03	1.1500e-03	6.4842
9×9	1.5546e-03	1.4772e-03	4.9785
10×10	1.9213e-03	1.8453e-03	3.9520
11×11	2.3300e-03	2.2544e-03	3.2423
12×12	2.7808e-03	2.7045e-03	2.7441

Table 4.9: Comparison of the results with different number of cells for the second analysis.

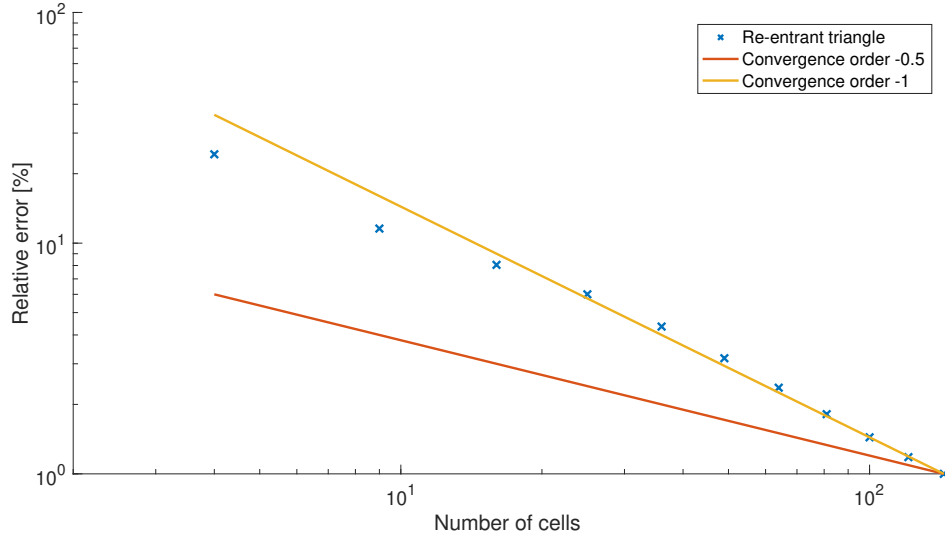


Figure 4.20: Bi-logarithmic plot of the relative error between the real and the homogenized displacement against the number of cells.

Contrary to the case of re-entrant honeycomb, the comparison shows that the homogeneous analysis gives results that are quite different from those obtained on the real geometry for few cells. The results become pertinent only when dealing with a large number of cells. For this geometry, the results for the homogeneous media tend to be more compliant in the sense that the displacements in the real mesh are always smaller. One of the hypotheses to explain these results could be the specific geometry of the boundary conditions. For the previous geometry, the clamped edge for the real mesh and the homogeneous one were almost the same. For this geometry, the bottom bound-

ary of the real mesh results in being very punctual which might cause inaccuracy when considering the loading and boundary conditions used for the experiments.

4.2.5 Parametric study of the cell

At last, the parametric study is performed on this geometry. The objective is to have a first comprehension of the influence of each geometrical parameter on the equivalent elastic parameters of the periodic media. Following the previously presented methodology, two parameters are chosen as variables to perform the study. The re-entrant angle α and the parameter N_l are chosen. It can directly be seen that this choice of parameters was made because they play the same role as the ones chosen for the parametric study of the other geometry. The re-entrant angle is here again the geometrical parameter strongly linked to the auxetic behavior of the cell. The second parameter (N_l) indirectly modifies the value of N_h as it is a fraction of it. Ultimately, the parameter N_h has a direct influence on the thickness of the branches. The parameter N_l is therefore directly linked to the relative density of the cell as defined in [Equation 4.19](#).

The geometry of the cell can therefore be re-written in terms of the two parameters. The half width of the triangle L and the height of the triangle H are kept constant and equal respectively to 0.75 millimeters and 1 millimeters. [Table 4.7](#) can be rewritten considering only the varying parameters and the range that they will take during the parametric study.

Symbol	Description	Formula	Interval of values
N_l	Length of the rectangular part	/	0.19 - 0.48 mm
N_h	Thickness of the rectangular part	$0.5 N_l$	0.095-0.24 mm
α	Angle of the inferior inclined part	/	10 - 28°
β	Angle of the external branches	$\text{atan}(\frac{1-0.5N_l}{0.75-0.5N_l})$	54.10 - 56.14°
H_{cell}	Height of the cell	$1 + (\frac{3}{2}\tan(\alpha) - 1)\frac{N_l}{2}$	0.66 - 0.82 mm
L_{cell}	Length of the cell	$1.5 - N_l$	1.02 - 1.31 mm

Table 4.10: Main dimensions for the parametric problem.

As explained above, the density of the cell is crucial to better understand the results of the parametric study. In order to explain the variation of the density more explicitly, the expression for the thickness of each branch can also be determined as a function of the two parameters α and N_l . Referring to the thickness of the outer branches as Th_o , and the one of the inner branches as Th_i , the following relations hold:

$$Th_o = N_h \cos(\beta) = \frac{N_l \cos(\beta)}{2} \quad Th_i = N_h \cos(\alpha) = \frac{N_l \cos(\alpha)}{2} \quad (4.24)$$

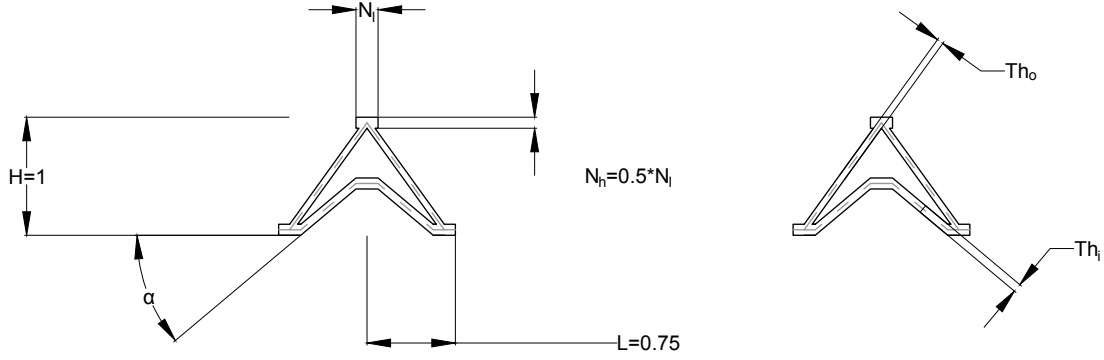
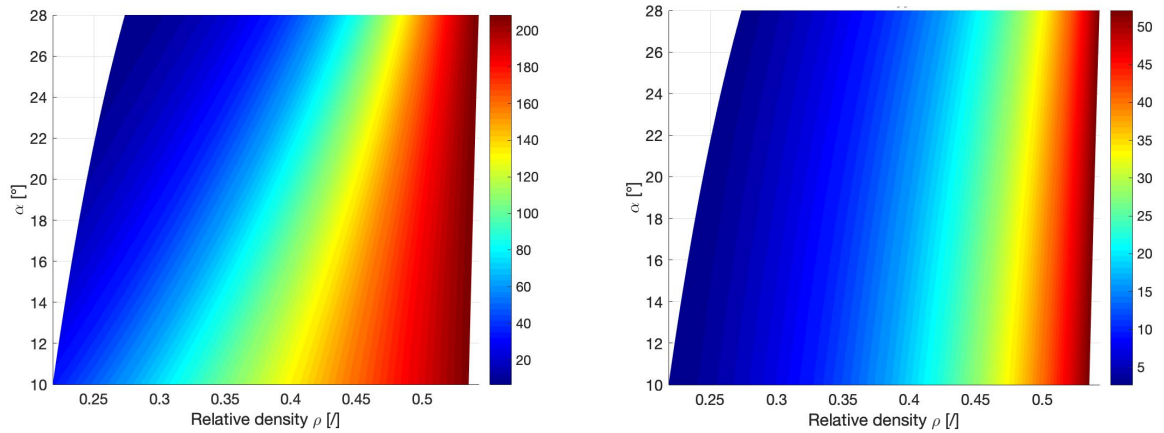


Figure 4.21: Main dimensions and thicknesses for the parametric study.

The equivalent elastic parameters are then obtained as for the re-entrant honeycomb geometry in Figure 4.12. A contour plot of the four homogenized elastic parameters (E_x^0 (upper left in Figure 4.22), E_y^0 (upper right in Figure 4.22), ν_{xy}^0 (lower left in Figure 4.22) and ν_{yx}^0 (lower right in Figure 4.22)) created. The vertical axis represents the re-entrant angle α and the horizontal one shows the relative density.

Looking at the contours, some interesting conclusions can be drawn. The first thing to be noticed is that as the relative density is a function of α , there is a zone where combinations that can not occur leading to a blank zone in the graphs this fact was also visible within the previous parametric study. The second conclusion is that homogenization leads to an easy formulation of the parametric problem. The boundary conditions are easily identified no matter how many branches cross the border of the cell. In the case of the analysis on the RVE, the choice of proper boundary conditions was not self-evident and required particular care. This already mentioned advantage becomes even more pertinent when considering parametric studies where many tests have to be done and where the boundaries of the unit cell can vary.



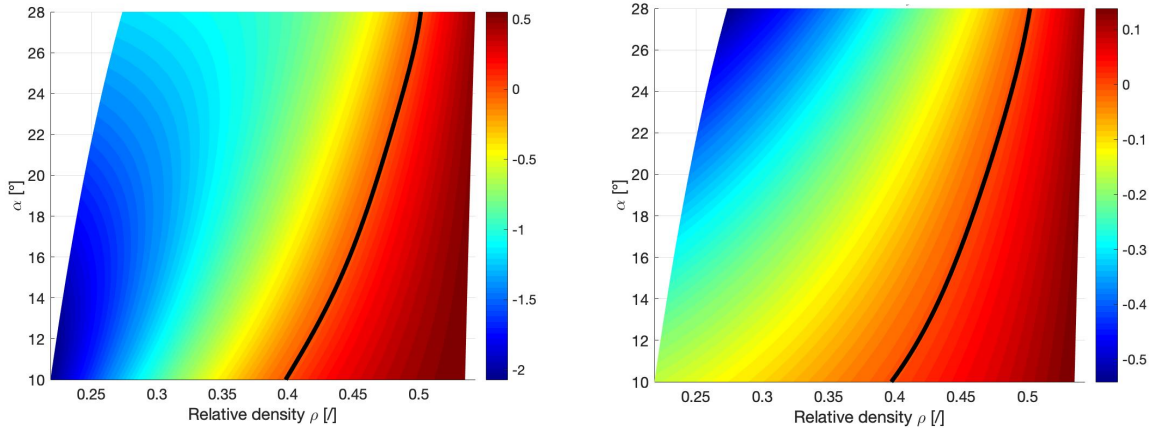


Figure 4.22: Contour plots of the equivalent Young's moduli (first row: E_x^0 left and E_y^0 right) and Poisson's ratio (second row: ν_{xy}^0 left and ν_{yx}^0 right) for the re-entrant triangle.

In terms of the results for the elastic parameters, it can be observed that: the Young's modulus E_x^0 tends to the one of the bulk material for high values of the relative density. The coefficient E_y^0 tends to have smaller values for the studied cases and tends to 70 MPa for increasing values of the relative density. In terms of the Poisson ratio, the bold black line marks the limit between auxetic and traditional behavior: the values of parameters on the left side of these lines lead to auxetic behaviors. Finally, the coefficient ν_{xy}^0 tends to be greater in absolute value compared to ν_{yx}^0 . It can also be seen that the range of value for which the cell behaves auxetically represents roughly two-third of the considered domain.

4.3 Tetra/Hexa-chiral structure

The last geometry to be analyzed is the chiral one. The adjective chiral defines, in general, a structure for which its mirror image cannot be superimposed on the original one. As explained in the introduction, this geometry belongs to another class of auxetic geometries (Figure 1.1). The deformation mechanism occurring during the loading of such cells is no longer based on the re-alignment of the ribs as it was the case for the first two geometries. For the chiral geometries, the deformation of the branches is still present but in addition, the rotation of the central ring is also possible.

The two geometries are represented below and are the so-called "tetra" and "hexa" chiral structures. Their names refer to the number of branches the cell possesses. The two geometries will be studied in parallel. The methodology will be the same as for the previous geometries. Nevertheless, due to the different mechanism of deformation, the applied methodology will be proved to fail, leading to inaccurate results. The choice was, therefore, for the sake of conciseness, to show only the main results.

4.3.1 Geometry of the cell

The two geometries for the unit cell are represented in [Figure 4.23](#)

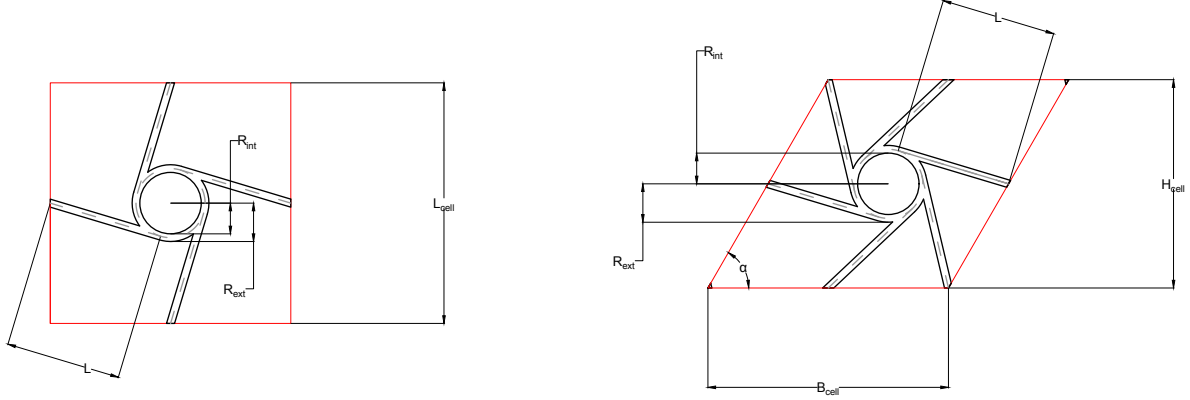


Figure 4.23: Geometry of the chiral cells.

Symbol	Description	Formula	Value
L	Length of the branches	/	30 mm
R_{ext}	Exterior radius of the circular part	/	10 mm
R_{int}	Interior radius of the circular part	/	8 mm
Th	Thickness of the branches	$R_{ext} - R_{int}$	2 mm
In	Mean radius of the circular part	$R_{ext} - \left(\frac{Th}{2}\right)$	9 mm
L_{cell}	Length of a unit cell	$2\sqrt{L^2 + In^2}$	62.6418 mm

Table 4.11: Dimension for the tetra-cell.

Symbol	Description	Formula	Value
L	Length of the branches	/	30 mm
R_{ext}	Exterior radius of the circular part	/	10 mm
R_{int}	Interior radius of the circular part	/	8 mm
Th	Thickness of the branches	$R_{ext} - R_{int}$	2 mm
In	Mean radius of the circular part	$R_{ext} - \left(\frac{Th}{2}\right)$	9 mm
α	Angle of rotation of the cell	$\frac{\pi}{2} - atan\left(\frac{L}{In}\right)$	0.2915 rad
β	/	$-\alpha + (\pi/3)$	0.7557 rad
B_{cell}	Length of the base	$2\sqrt{L^2 + In^2}$	62.6418 mm
H_{cell}	Length of the height	$2(In \cdot \cos(\beta) + L \cdot \sin(\beta))$	54.2494 mm

Table 4.12: Dimension for the hexa-cell.

The main dimensions for the tetra-chiral and hexa-chiral cells can be found respectively in [Table 4.11](#) and [Table 4.12](#). In can be seen that the parameter of internal angle

is no longer present but instead the internal and external radius of the ring becomes the pertinent parameter for auxeticity.

It can directly be noticed that the unit cell for the tetra-chiral geometry will always be a *square* unit cell while the hexa-chiral belongs to the *oblique* Bravais lattice.

4.3.2 Validation of the homogenization

As explained in the introduction of this geometry, only the results for the validation for the homogenization are presented in details. The results of the analyses performed on the metamaterials subjected to a uniform body load with a clamped bottom edge and those of the analysis performed on the equivalent homogenized material are compared in [Figure 4.24](#) for the tetra-chiral material and in [Figure 4.25](#) for the hexa-chiral one in terms of displacement magnitude.

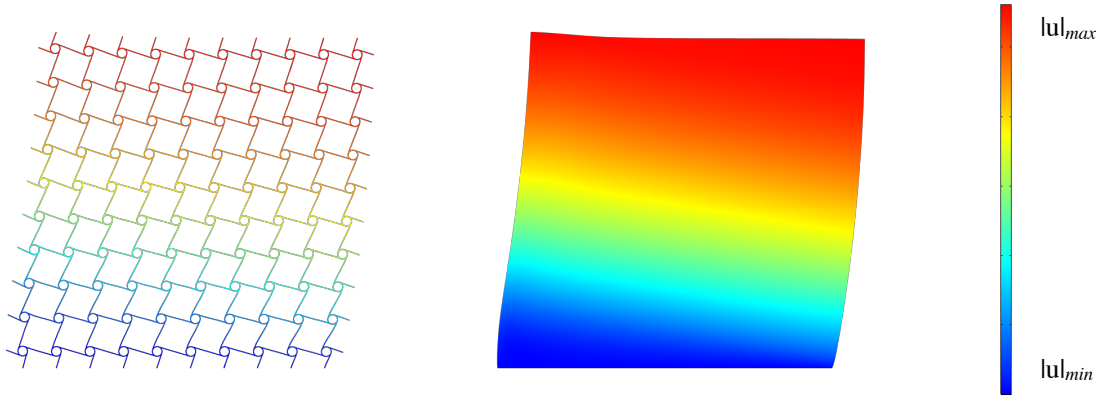


Figure 4.24: Deformed shapes for the two analyses on the tetra-chiral cell.

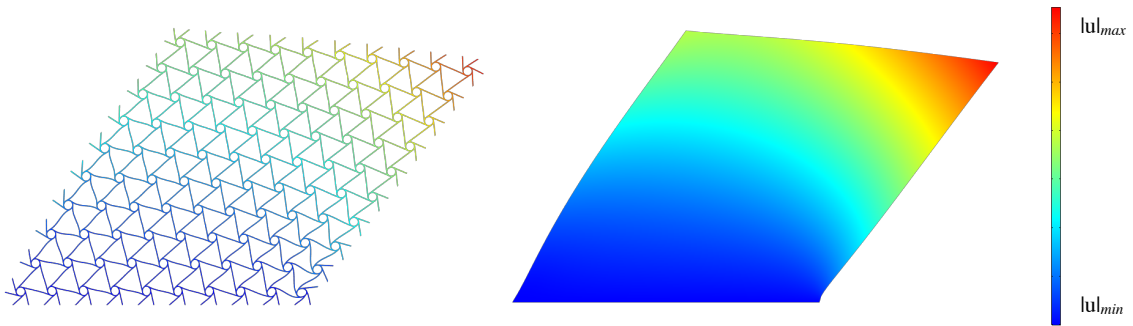


Figure 4.25: Deformed shapes for the two analyses on the hexa-chiral cell.

The results in term magnitude of the horizontal component of displacements are compared in the following table:

Type	u in the real mesh in mm	u in the homogeneous mesh in mm	Relative error [%]
Tetra	1.0915	1.1915	9.16
Hexa	0.0518	0.0431	16.79

Table 4.13: Comparison of the results for the analyses on the chiral geometries.

It can be seen from these results that the homogenization fails to give accurate results. The global behavior of the test is reproduced but the displacements are inaccurate. Several articles explained this discrepancy. The mechanism of deformation of the chiral cell can be decomposed into two parts. The first part of the deformation is due to the rotation and distortion of the ring. The second part of the response is achieved through the bending and stretching of the ligaments.

As explained in [6], the lack of explicit description for the internal rotational mechanism is the main cause of inaccuracies in homogenization. Several solutions are presented to overcome this problem. The micropolar homogenization and second-order description are the more popular ones. In [2] the authors also suggest the possibility that the test with periodic boundary condition approach might constrain the system too much with respect to the real mechanism that is taking place in the structured materials.

Chapter 5

Parametric comparison of the two re-entrant geometries

It has been seen in the previous chapter that homogenization is a very efficient procedure to characterize the elastic behavior of micro-structured material with complex geometries. In this chapter, the procedure will be used to study and compare the re-entrant honeycomb and triangle lattices. Before starting this last study on 2D geometries, the cells are going to be redefined by introducing the geometric parameters that will be considered during the comparison.

5.1 Formulation of the problem

In this section, the cells are going to be defined using only three parameters while in chapter 4, the cells were described in a very general case with many variables. The idea is now to simplify the description of the geometries and to describe both the re-entrant triangle and the re-entrant honeycomb with parameters having the same meaning. More precisely, during the study of these geometries, it has been seen that some geometrical features were of particular interest when describing their mechanical behavior. This does not mean that the other parameters presented for the previous geometries are not pertinent, but rather that the parameters that will be now chosen are the more essential in order to design the general features of each geometry.

Three essential parameters have been selected. One of them is the global size of the cell H . The other two are, respectively, the re-entrant angle α and the thickness Th . The choice of three parameters is not arbitrary. The objective of this chapter is to be able to compare the geometries. To do so, the elastic parameters are computed as functions of the re-entrant angle and the thickness as it has been done previously. Furthermore, the comparison are also discussed considering the relative density (function of H , Th , and α) that the different geometries exhibit for a given elastic parameter. The

parametric study is then performed over the whole set of values of Th and α , keeping H equal to 1 millimeter. Finally, the results are compared in terms of the homogenized elastic properties. The geometries with the chosen essential parameters are depicted in [Figure 5.1](#).

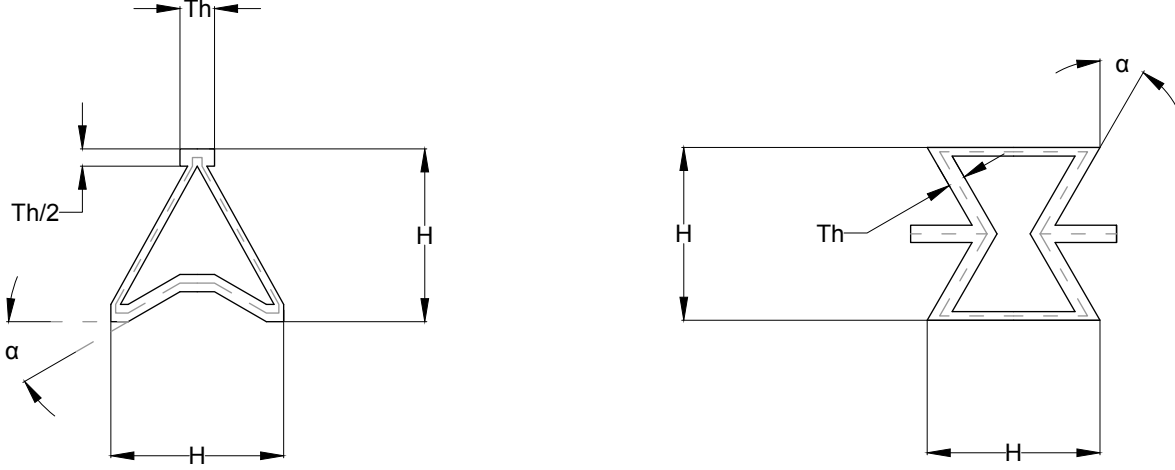


Figure 5.1: Geometry of the two lattices elements for the parametric comparison.

For both geometries, only the two parameters α and Th will vary. It can easily be seen that due to the re-entrant nature of the cells, not all combinations of these parameters are possible. For many combinations, there is the possibility for the branches to overlap. To avoid this problem and risking the evaluation of wrong parameters due to incorrect geometrical definition, a preliminary study is made to specify the range of parameters leading to a correct definition of the cell. The term correct/incorrect used in the paragraph makes reference to geometries without/with overlapping.

For the re-entrant triangles, the overlapping occurs when the inferior arrowhead becomes too close to the upper one. One can express this geometrical feature as: "As the angle α increases, the point located at the top corner of the lower arrowhead will touch the lower boundary of the upper arrowhead". This limit situation for a particular choice of Th is depicted in [Figure 5.2](#) (b).

The equation for the limit case can be expressed as a limit for the tangent of the re-entrant angle.

$$\tan(\alpha) = \frac{(1 - \xi)(1 - 0.5 \cdot \xi)}{0.5 \cdot (1 - 2\xi)} - \frac{0.5\xi}{0.5 - \xi} \quad (5.1)$$

In the previous equation, the variable ξ represents the adimensional ratio Th/H . It is then possible to represent the variation of the relative density as a function of α and ξ . The relative density is the same that has been defined in the equation [4.19](#). The black line on [Figure 5.3](#) represents the limit situation. All the values of α smaller than the

ones described by the black curve are therefore admissible values.



Figure 5.2: Re-entrant triangle: (a) typical geometry; (b) limit geometrical configuration.

For this specific geometry, it was found impossible to find a closed-form expression for the relative density of the unit-cell as a function of α and ξ . The program *COMSOL Multiphysics* was therefore used and a parametric sweep was performed on the geometry for all the values of parameters: $\alpha \in [0, 90]^\circ$ and $\xi \in [0, 0.5]$. The area of the structured material was computed using the *evaluate surface* tool. The area of the unit cell was computed using the formulas of [Table 4.7](#). The ratios between those two values were computed and the values above the black line representing the limit cases were excluded. The contour plot of the relative density as a function of ξ and α is depicted in [Figure 5.3](#)

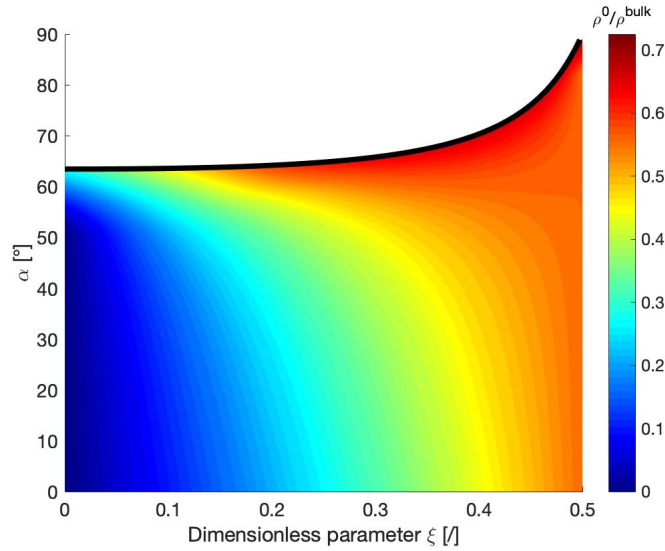


Figure 5.3: Contour plot of the relative mass density ρ^0/ρ^{bulk} for the re-entrant triangle geometry.

One should look carefully at the geometrical definition of the re-entrant triangle. The definition of the latest was created so that when $\xi = 0.5$ the unit cell becomes unique for any values of α . The same problem of interpenetration can also occur for the re-entrant honeycomb geometry. This time the problem can be stated as: " As the re-entrant angle increases the two points within the inner boundary located on the horizontal symmetry axis are getting closer up to the point where they become a unique point". This situation, occuring for a particular choice of Th , is represented in Figure 5.4 (b).



Figure 5.4: Re-entrant honeycomb: (a) typical geometry; (b) limit geometrical configuration.

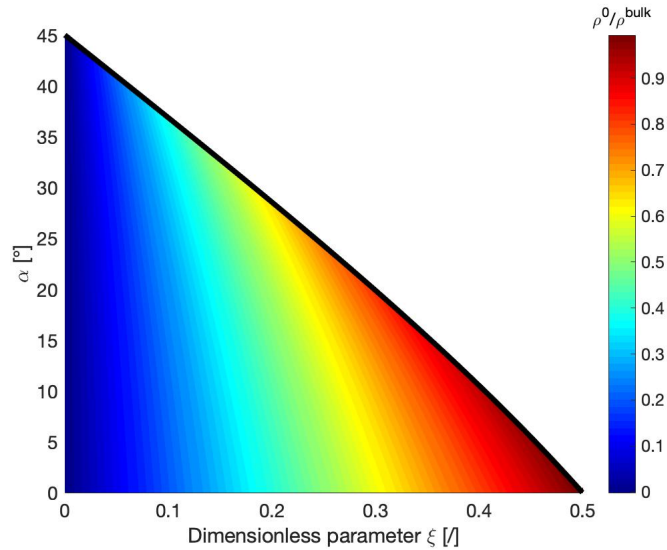


Figure 5.5: Contour plot of the relative mass density ρ^0/ρ^{bulk} for the re-entrant triangle geometry.

The relation for the limit case can be expressed as a limit value for the dimensionless parameter ξ .

$$\xi = \frac{\cos(\alpha) - \sin(\alpha)}{2} \quad (5.2)$$

For this geometry it is possible to find both an expression for the surface of the bulk unit cell and the surface actually occupied by the unit cell. Therefore an analytic closed form expression can be found for the relative density of the cell.

$$\rho = \xi \cdot \frac{2 \cos(\alpha) + 2 - \xi \sin(\alpha) - 4\xi}{2 \cos(\alpha) - \sin(\alpha) - 2\xi} \quad (5.3)$$

The surface of the relative density for the re-entrant honeycomb geometry is depicted in [Figure 5.5](#) as a function of ξ and α .

5.2 Parametric characterization of the geometries

The aim of this section is to discuss the evolution the four elastic parameters (E_x^0, E_y^0, ν_{xy}^0 and ν_{yx}^0) for varying geometrical parameters. As it has been done for the graphic of the relative density, the contour plots are functions of the re-entrant angle (vertical axis) and the dimensionless parameter ξ (horizontal axis). The plots for each elastic parameter are reported side by side for both geometries allowing for a direct comparison.

The steps of the procedure to create these plots are the same as the procedure used to produce [Figure 5.3](#) with an additional step of solving the cell problems. More precisely, for both geometries, a *parametric sweep* was used on *COMSOL Multiphysics* to generate all the possible combinations of cell (even the incorrect ones). For each combination of parameters, the three pertinent cell problems in 2D were solved following the exact same procedure as the ones presented respectively in the chapter devoted to each geometry. The elastic constants could then be retrieved for each combination of the two parameters. Finally, a filter is apply to remove all the combinations that are incorrect using the conditions [Equation 5.1](#) for the re-entrant triangle and [Equation 5.2](#) for the re-entrant honeycomb. [Figure 5.6](#) shows the obtained results in terms of homogenized Young's moduli E_x^0 and E_y^0 for the two geometries.

In terms of the Young's modulus, the general tendency is to increase as ξ gets closer to 0.5. This fact can be explained by the following reasoning: as the dimensionless parameter ξ becomes larger, the branches tend to become thicker and the unit cell tends to restore the bulk material condition. Another interesting conclusion is that the Young's moduli are not very much affected by the variation of the re-entrant angle. It is to say, for a specific choice of ξ , the Young's modulus remains roughly constant for all the values of α .

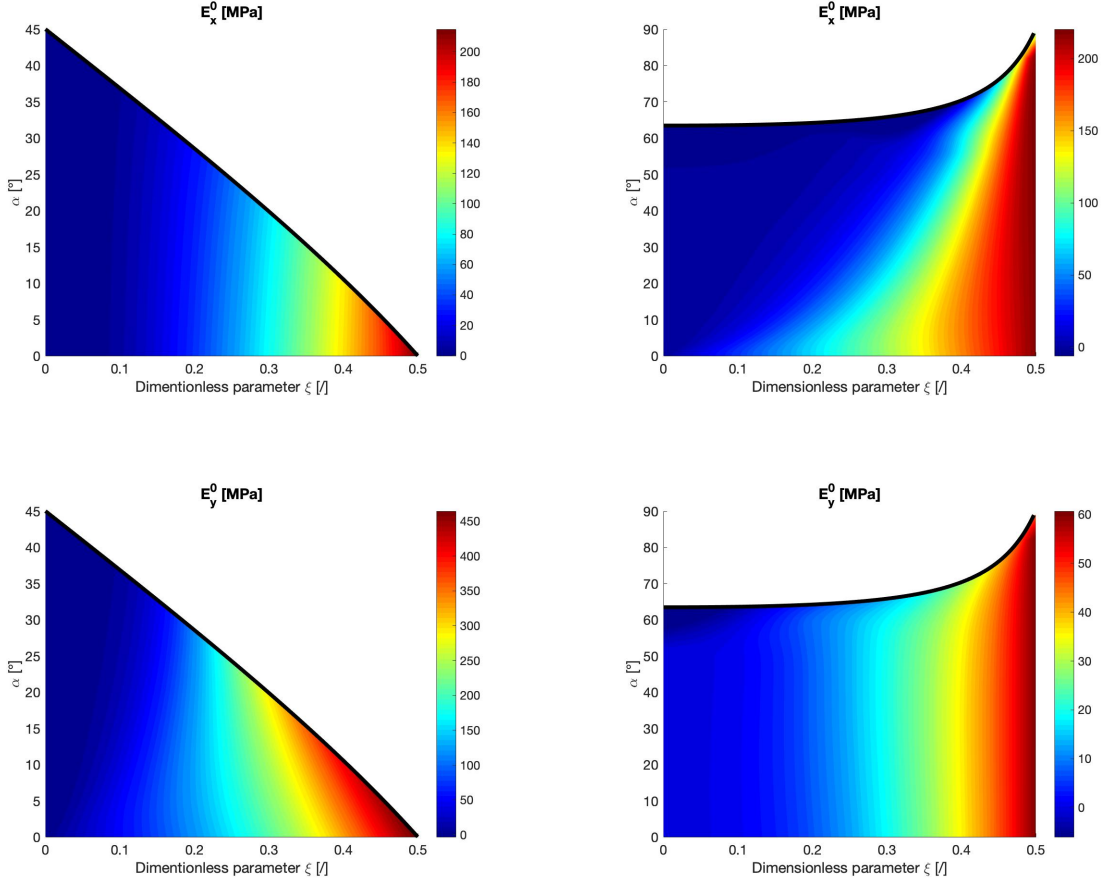


Figure 5.6: Contour plots of the equivalent Young's moduli E_x^0 (first row) and E_y^0 (second row) for the re-entrant honeycomb (left column) and re-entrant triangle (right column).

Comparing now the two geometries (right and left columns of [Figure 5.6](#)), one can observe that the re-entrant honeycomb tends to be stiffer than the re-entrant triangle. While in the honeycomb geometries (left column of [Figure 5.6](#)) E_x^0 tends to be greater than E_y^0 , the inverse can be observed for the triangle geometries (right column of [Figure 5.6](#)).

[Figure 5.7](#) shows the obtained results in terms of homogenized Young's moduli ν_{xy}^0 and ν_{yx}^0 as functions of ξ and α for the two cells.

Considering the variation of the Poisson's ratio, the elastic parameter vary both as a function of the re-entrant angle α and the dimensionless parameter ξ . For all the contour plots in [Figure 5.7](#) the zone above the black line represents the combinations of geometrical parameters leading to auxetic behavior. For the re-entrant honeycomb (left column of [Figure 5.7](#)), the coefficient ν_{xy}^0 tends to reach minimal values for small values of

the dimensionless parameter ξ and high values of the re-entrant angle α . The opposite is observed for the coefficient ν_{yx}^0 . Comparing the magnitude of those two coefficients (still for the re-entrant honeycomb), ν_{xy}^0 tends to be smaller. For the re-entrant triangle (right column in Figure 5.7) the Poisson's ratio ν_{xy}^0 reaches minimal values for combination of low dimensionless parameter ξ and small re-entrant angle α and the opposite holds for ν_{yx}^0 . Comparatively (considering both the left and right columns in Figure 5.7) the re-entrant triangle geometry (right column) tends to exhibit the most auxetic behavior both in terms of values of the Poisson's ratio and of the domain (combination of geometrical parameters).

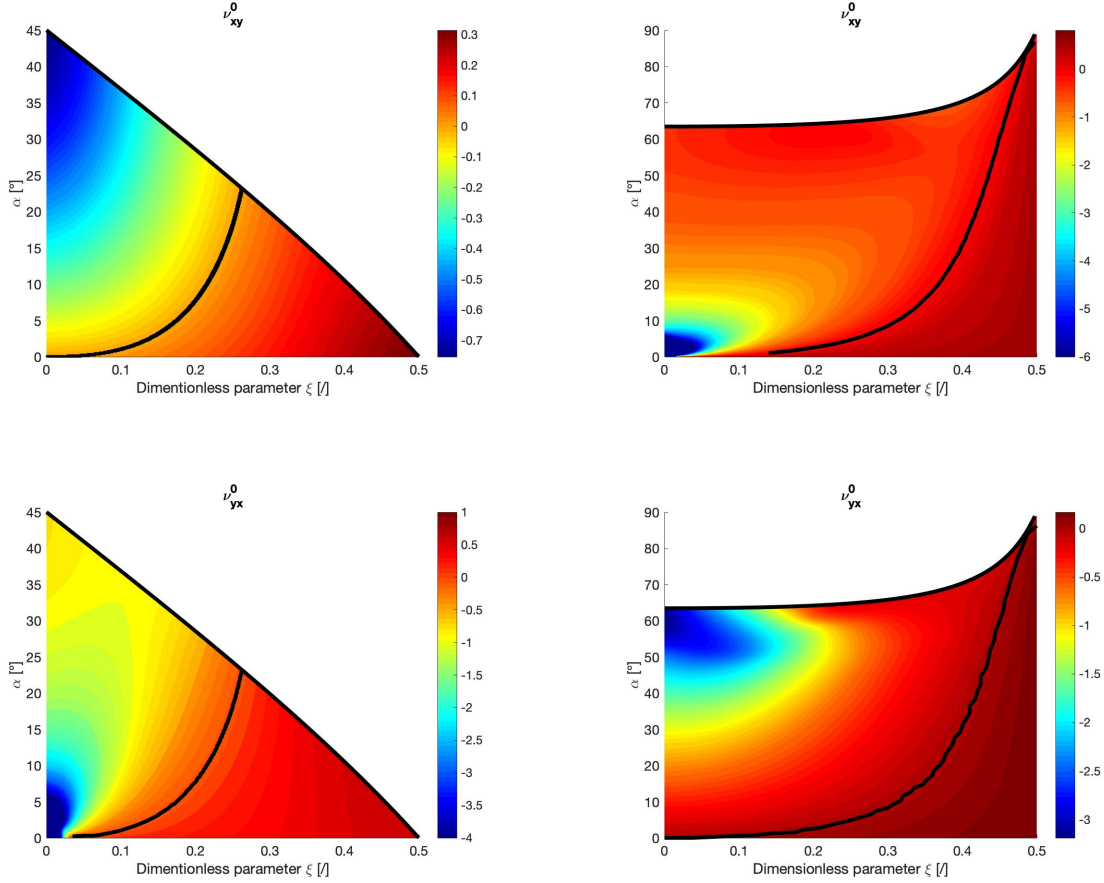


Figure 5.7: Contour plots of the equivalent Poisson's ratio ν_{xy}^0 (first row) and ν_{yx}^0 (second row) for the re-entrant honeycomb (left column) and re-entrant triangle (right column).

Chapter 6

Numerical analyses of auxetic materials with three dimensional periodicity

After having thoroughly studied the problem for the two-dimensional case, this part of the work will now be focused on the three-dimensional periodic auxetic metamaterials. The two geometries of interest are the 3D re-entrant honeycomb and the 3D re-entrant triangle. Those two geometries can be seen as the 3D extensions of the already studied 2D geometries. Since it was one of the first 3D orthotropic auxetic structures presented [31], many works can be found on the 3D re-entrant honeycomb. On the contrary, reviews and comparisons for the 3D re-entrant triangle are very few. It is therefore of interest to describe in detail this particular geometry and to compare it with the re-entrant honeycomb. Throughout this chapter, the two geometries will be studied in parallel with the objective to understand the difference between the behavior of the two geometries.

The characterization of the geometry is done at the geometrical level and then mechanically by evaluating the equivalent elastic parameters. Here again, the theories presented in the first part will be used for the 3D cases. Then, for each geometry, some larger scale experiments similar to those presented for the 2D cases are studied to assess the accuracy of the homogenized elastic matrices. Following the same global procedure as the ones for the 2D cases, the steps are detailed for the first studied geometry (the 3D re-entrant honeycomb) and are then presented in a more synthetically for the other one.

The particular interest of this chapter is to have a solid understanding of the behavior of each cell. This understanding will be needed in order to aboard the next chapter focusing on the test of impact. The mechanical properties used for this entire chapter are the ones already used in the previous part and are recalled in Table 6.1. The reason is that within the simulation of impact, the auxetic structure that will be used will be modeled with these specific characteristics. The procedure to obtain the homogenized

parameters of this chapter will therefore be used in the next chapter.

Parameter	Symbol	Value
Young's modulus	E	480 MPa
Poisson's ratio	ν	0.48
Mass density	ρ^{bulk}	940 kg/m ³

Table 6.1: Mechanical properties of the rigid EVA.

6.1 3D re-entrant honeycomb

6.1.1 Geometry of the cell

The geometry of the 3D re-entrant honeycomb, as stated in the introduction of this chapter, is one of the most reviewed of the 3D re-entrant geometries. In [Figure 6.1](#), the geometry of a unique cell is presented together with an array made of several cells. The 3D re-entrant honeycomb can be seen as a generalization of the already presented two-dimensional geometry: the same re-entrant hexagonal shape is repeated on the two orthogonal planes X-Z and Y-Z, these latter ones are then connected by a central structure. The choice has been made to model the cell with few parameters for the sale of simplicity.

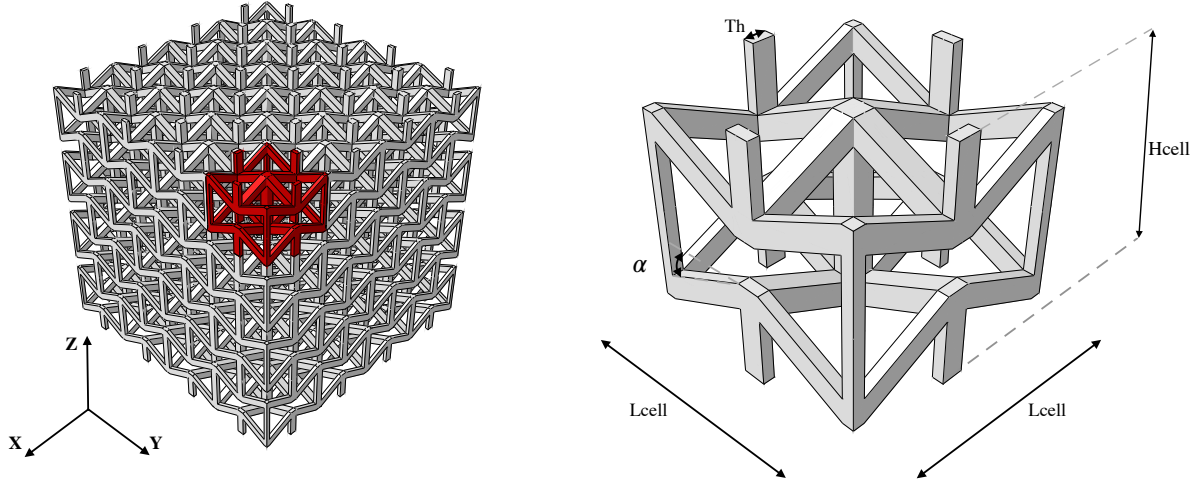


Figure 6.1: Periodic 3D re-entrant honeycomb (left) and its unit cell (right).

The first observation to be made is that the specific choice of the unit cell is arbitrary. The specific choice of one cell over the other, as it has already been explained, does not affect the homogenization procedure. For the rest of the characterization, we fix the parameters to the values summarized in [Table 6.2](#).

Symbole	Description	Value
L_{cell}	Width of the cell in the XY plan	1 mm
H_{cell}	Height of the cell	1 mm
Th	Thickness of the branches	0.1 mm
α	Internal angle	25°

Table 6.2: Main geometrical dimension for the re-entrant honeycomb geometry.

Making reference to [Table 2.1](#) and looking at the geometry, it can be noticed that the unit cell can be either *cubic* or *tetragonal*. For the specific dimensions presented in [Table 6.2](#) the cell is presented as *cubic*. Finally, the relative density can be redefined for the three-dimensional case as:

$$\rho = \frac{\text{Volume filled with the material}}{L_{cell} \cdot L_{cell} \cdot H_{cell}} = \frac{\rho^0}{\rho^{bulk}} \quad (6.1)$$

Considering the specific choice of parameters, the relative density can easily be computed and is equal to 1/10.

6.1.2 Homogenization

The homogenization is then performed on the unit cell presented in [Figure 6.1](#). Formally, in the three-dimensional case, six cell problems need to be solved on the unit cell in order to compute the homogenized coefficients of the stiffness matrix. Nevertheless, considering the specific geometry analyzed, two planes of symmetry (XZ plane and YZ plane) can be identified, as shown in [Figure 6.2](#).

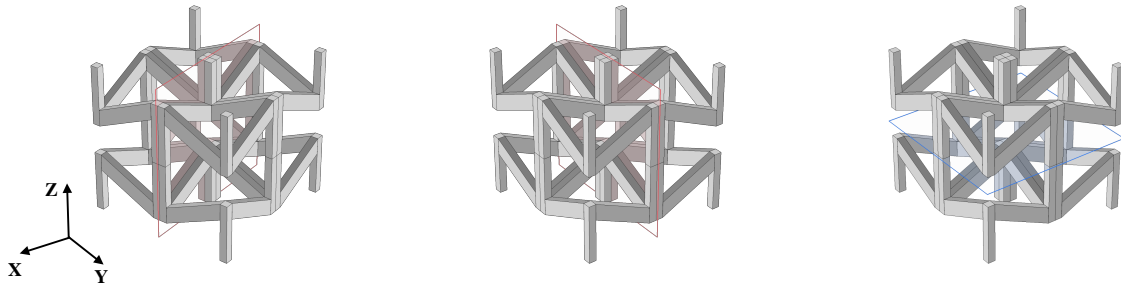


Figure 6.2: Planes of symmetry for the 3D re-entrant honeycomb: XZ (left), YZ (middle) and XY (right).

From this observation, two of the cell problems are redundant as it will be developed in the loading section. Therefore, only four out of the six cell problems must be solved. Finally, it is to be noted that the cell posses also a plane of symmetry parallel to the XY

plane. As for the 2D case, the homogenization procedure is performed through *COMSOL Multiphysics*.

Boundary conditions

The periodic boundary conditions are imposed on each pair of opposite edges lying on the boundary of the unit cell. In addition to this, a point is fixed in order to avoid rigid body motions. The boundary conditions remain the same for the computation of the four pertinent cell problems.

Loading

For each of the four cell problems an *Initial strain state* is imposed. For the general three dimensional case, the initial strain state matrix can be written as:

$$\underline{\underline{\varepsilon}}_0 = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \quad (6.2)$$

The different cell problems will then only differ for the imposed initial strain. The first cell problem will be characterized by a value for $\varepsilon_{xx} = -0.1$ and all the other values equal to zero. At this point, the justification for the reduced number of cell problems to be solved can be detailed. As presented in the geometry of the cell, two of the plans of symmetry (XZ and YZ) can be identified. The presence of these two plans of symmetry is equivalent to the property of invariance under a rotation of 90 degrees around the Z axis. Therefore loading the cell along the X axis only is strictly equivalent to loading the cell along the Y axis only. Solving the cell problems for the loading along X is in conclusion equivalent to solving the cell problem for the loading along the Y axis. Only one of those two cell problems was implemented. The second cell problem to be solved had a loading with only $\varepsilon_{zz} = -0.1$.

The third cell problem was loading with a shear train $\varepsilon_{xy} = \varepsilon_{yx} = -0.05$. The last type of problem to be considered is the one characterized with a loading of $\varepsilon_{xz} = \varepsilon_{zx} = -0.05$ and all the element of the initial strain state equal to zero. With the same reasoning done for the loading along the X axis, the cell problem with loading $\varepsilon_{yz} = \varepsilon_{zy} = -0.05$ is also equivalent to the fourth cell problem. [Table 6.3](#) summarizes the conditions for the different cell problems which allows to compute the displacement fields $\underline{\underline{\chi}}^{ij}$.

Cell problem	ε_{xx}	ε_{yy}	ε_{zz}	$\varepsilon_{xy} = \varepsilon_{yx}$	$\varepsilon_{yz} = \varepsilon_{zy}$	$\varepsilon_{zx} = \varepsilon_{xz}$
$\underline{\chi}^{11}$	-0.1	0	0	0	0	0
$\underline{\chi}^{22}$	0	-0.1	0	0	0	0
$\underline{\chi}^{33}$	0	0	-0.1	0	0	0
$\underline{\chi}^{12}$	0	0	0	-0.05	0	0
$\underline{\chi}^{23}$	0	0	0	0	-0.05	0
$\underline{\chi}^{31}$	0	0	0	0	0	-0.05

Table 6.3: Imposed eigenstrains for each cell problem.

The element of the stiffness matrix can then be computed using [Equation 3.31](#) with a proper scaling of the strains. The stiffness matrix is:

$$\underline{\underline{C}} = \begin{bmatrix} 12.6970 & -2.2392 & -3.2880 & 0 & 0 & 0 \\ -2.2392 & 12.6970 & -3.2880 & 0 & 0 & 0 \\ -3.2880 & -3.2880 & 3.8016 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4163 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1418 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1418 \end{bmatrix} \text{MPa} \quad (6.3)$$

6.1.3 Validation of the homogenization

In this section, the experiments for the validation of the homogenization are presented. The objective is to verify the accuracy of the homogenized stiffness matrix. The quantification of the error is also an interesting data to interpret the results of the analysis of the next chapter.

The comparison and the accuracy assessment are based on two numerical analyses. The first analysis is performed on a real periodic media made of several cells with specific boundary conditions. The second analysis is performed on the homogenized media for which the boundary conditions are replicated and the equivalent load is computed ([Equation 3.23](#)) and applied.

For the first validation, a periodic medium of $4 \times 4 \times 4$ cells is created in *COMSOL Multiphysics*. The geometry used, as explained, is the one described in [Table 6.2](#). The comparison between the homogeneous media and the real one is discussed in terms of total forces transmitted to the bottom face.

The boundary conditions are imposed at the bottom of the mesh and at the top. At the bottom of the real mesh, rollers were put for all the faces, it is to say that the displacement normal to those faces are set equal to zero. This condition is applied with the feature *Roller*. The same boundary conditions are applied to the four external edges.

At the top of the mesh, a displacement of 0.1 mm was imposed downward along the Z axis. [Figure 6.3](#) shows the contour plot of the vertical displacement obtained on the real geometry of the metamaterial and on the equivalent homogenized medium. The reaction force obtained with these two analyses is reported in [Table 6.4](#) together with the relative error which turns out to be very low.

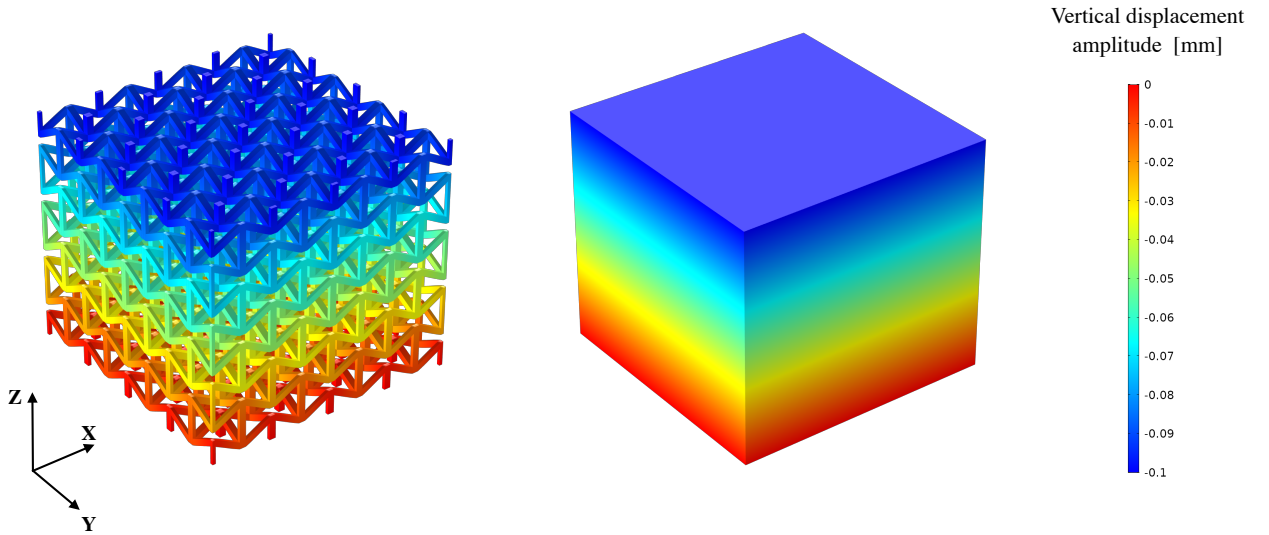


Figure 6.3: First analysis of validation for the homogenization.

Reaction force in the real media	Reaction force in the homogeneous media	Relative error
1.5327 N	1.5229 N	0.573%

Table 6.4: Comparison of the results for the first analysis.

For further validation, a second problem is analyzed with different boundary conditions and the same geometry. The choice is made to clamp the bottom edge and to apply a body force along the X axis. The objective of this second experiment was to simulate a non-uniform state of stress in the homogeneous media and also consider a loading for which the homogenized load vector needs to be computed. In order to assess the accuracy of the homogenized stiffness matrix as well as the capacity for the homogeneous media to replicate the behavior of the real mesh, the displacement in a point within the real mesh was monitored. A point was created as well in the homogeneous

media with the same geometrical coordinates. The comparison will be done comparing the value of displacement for the two corresponding points.

The body load was applied along the X direction with an amplitude of 1000 kN/m^3 . As the body load applied was constant over the whole body, the equivalent load to be applied to the homogeneous media is simply the product of the relative density of the cell and the load applied on the real mesh. As the relative density ρ of the re-entrant honeycomb, for the specific data in [Table 6.2](#), is equal to 0.1, the load to be applied for the validation is equal to 100 kN/m^3 .

[Figure 6.4](#) shows the contour plot of the vertical displacement obtained on the real geometry of the metamaterial and on the equivalent homogenized medium. The reaction force obtained with these two analyses is reported in [Table 6.5](#) together with the relative error which turns out to be very low.

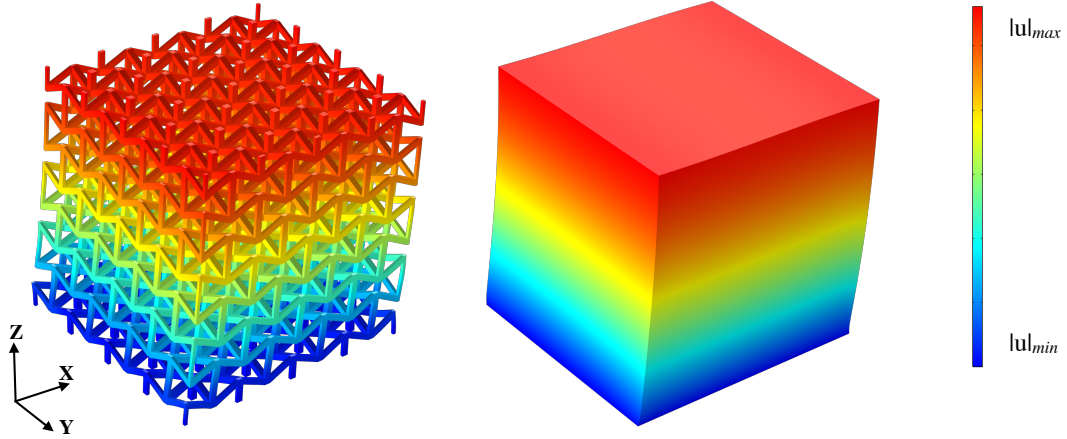


Figure 6.4: Second analysis of validation for the homogenization.

u in the real media	u in the homogeneous media	Relative error
7.8059e-03 mm	7.7305e-03 mm	0.9663 %

Table 6.5: Comparison of the results for the second analysis.

From both the previous examples, it can be seen that the consideration of the homogeneous media leads to quite accurate results together with a simpler formulation.

6.2 3D re-entrant triangle

6.2.1 Geometry of the cell

The second geometry to be analyzed is the 3D re-entrant triangle. The geometry takes inspiration from the 2D re-entrant triangle and is an extension of the three-dimensional geometry. This geometry is by far less common than the re-entrant honeycomb. The interest of this part, therefore, be to present it in detail and to compare it with the re-entrant honeycomb structure.

In [Figure 6.5](#), the periodic structure and the choice of unit cell for the homogenization are represented. The main dimensions of the unit cells are shown in the same figure.

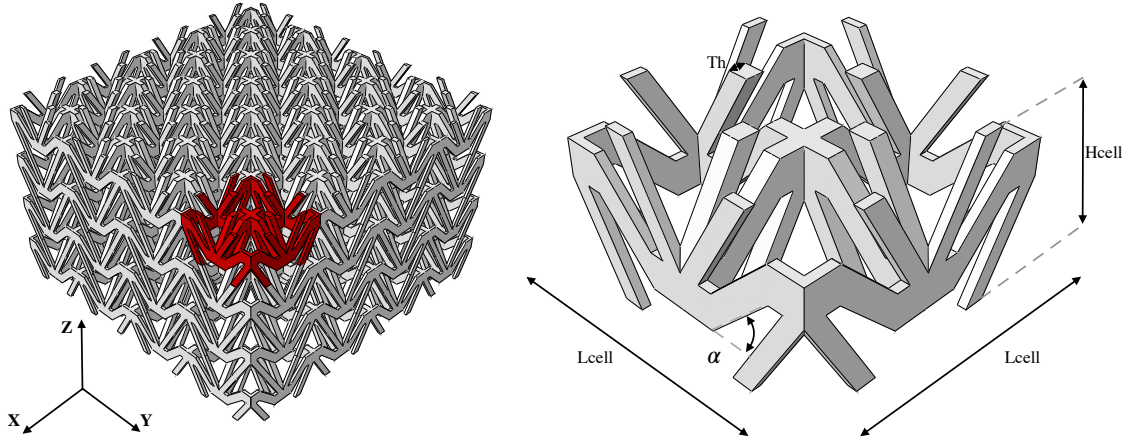


Figure 6.5: Periodic 3D re-entrant triangle geometry (left) and its unit cell (right).

For all the simulations of this chapter, the dimensions of the cell has been chosen respecting two criteria: having the same thickness for the branches and having the same relative density as the re-entrant honeycomb presented in [Table 6.2](#). Since those two conditions do not define uniquely the geometry, the parameters are chosen in order to have a cell of equal height. Finally, the re-entrant angle α is chosen to produce squared sections for the more slender branches. The results in terms of value for the parameters are grouped in [Table 6.6](#)

Symbol	Description	Value
L_{cell}	Width of the cell in the XY plan	1.6 mm
H_{cell}	Height of the cell	1 mm
Th	Thickness of the branches	0.1 mm
α	Re-entrant angle	45°

Table 6.6: Main geometrical dimension for the re-entrant triangle geometry.

6.2.2 Homogenization

The different cell problems are solved for the geometry presented in [Figure 6.1](#). The re-entrant triangle and re-entrant honeycomb are built in a quite similar way in the sense that similar mirror reflections were used to produce the unit cell from the 2D geometry. For this reason, they share in common two planes of symmetry. More specifically, the re-entrant triangle cell possesses an XZ plane of symmetry and a YZ plane of symmetry, as shown in [Figure 6.6](#). A difference from the 3D re-entrant honeycomb unit cell, the one of the re-entrant triangle does not have an XY symmetry.

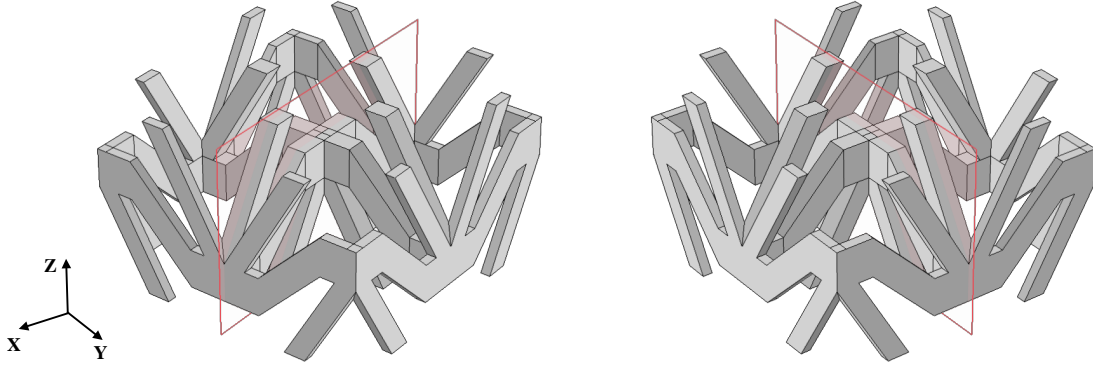


Figure 6.6: Planes of symmetry for the 3D re-entrant triangle: YZ plane of symmetry (left), XZ plane of symmetry (right).

Using the same arguments about the presence of planes of mirror reflection, the number of cell problems needed to be solved can be reduced to four also for the re-entrant triangle geometry.

Boundary conditions

The boundary conditions are also imposed using the *Periodic condition* in a similar way as done for the 3D re-entrant honeycomb unit cell. A point is fixed with zero displacements in all directions to avoid rigid body motions.

The cell problem to solved are subjected to the same imposed initial strains described for the re-entrant honeycomb geometry and summarized in [Table 6.3](#). The same symmetry arguments are used and the cell problems are solved. The homogenized stiffness matrix is then evaluated still using [Equation 3.31](#) and reads:

$$\underline{\underline{C}} = \begin{bmatrix} 4.131 & -2.020 & -1.026 & 0 & 0 & 0 \\ -2.020 & 4.131 & -1.026 & 0 & 0 & 0 \\ -1.026 & -1.026 & 5.348 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.126 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.417 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.417 \end{bmatrix} \text{ MPa} \quad (6.4)$$

To compare the 3D re-entrant honeycomb (3DRHC) and re-entrant triangle (3DRT) geometries, their independent homogenized elastic moduli are reported in [Table 6.7](#) which are computed by inverting the stiffness matrices [Equation 6.3](#) and [Equation 6.4](#)

	$E_x^0 = E_y^0$	E_z^0	ν_{xy}^0	$\nu_{xz}^0 = \nu_{yz}^0$	G_{xy}^0	$G_{xz}^0 = G_{yz}^0$
3DRHC	7.231 MPa	1.734 MPa	-0.516	-1.311	0.416 MPa	0.142 MPa
3DRT	2.684 MPa	4.352 MPa	-0.564	-0.300	0.126 MPa	1.417 MPa

Table 6.7: Independent homogenized elastic parameters for the two geometries considered.

As it can be seen in the previous table, the re-entrant honeycomb is stiffer for solicitations in the X and Y directions while the geometry is more compliant than the re-entrant triangle for loading along the Z axis. In terms of *Poisson's ratios*, the terms ν_{xy}^0 are almost equal while the terms ν_{xz}^0 and ν_{yz}^0 are quite different, the re-entrant honeycomb geometry exhibits a more pronounced auxetic effect. Finally, in terms of shear moduli, the re-entrant triangle tends to be stiffer for shear solicitations within the XZ and YZ planes while it is the opposite for loading in the XY plane.

6.2.3 Validation of the homogenization

The two examples simulated for the previous geometry are reproduced for the re-entrant triangle geometry. The loading and boundary conditions are the same. [Figure 6.7](#) shows the results of the first example, where a uniform u_z was imposed at the top boundary, in terms of contour plots of the vertical displacements obtained on the real geometry and the equivalent homogeneous medium.

The reaction force is obtained with these two analyses is reported in the [Table 6.8](#) together with the relative error which in this case is around 5%.

Reaction force in the real media	Reaction force in the homogeneous media	Relative error
6.6778 N	7.0414 N	5.4449%

Table 6.8: Comparison of the results for the first analysis.

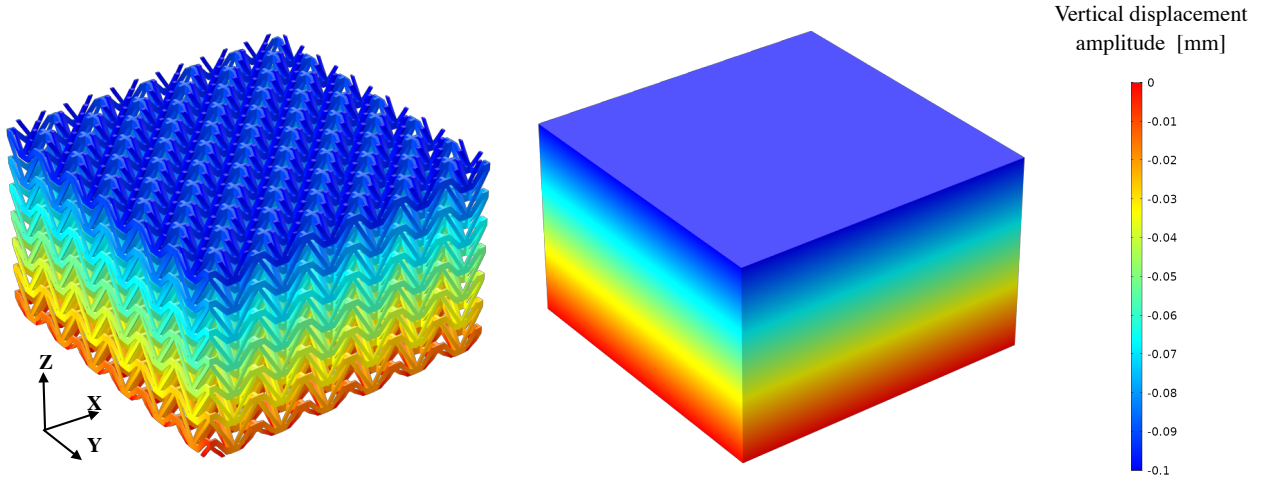


Figure 6.7: First analysis of validation for the homogenization.

The second example, concerning the case of a uniform horizontal body force, leads to the results shown in [Figure 6.8](#). The comparison between the real geometry and the homogenized one is done in terms of the displacement of corresponding points in [Table 6.9](#).

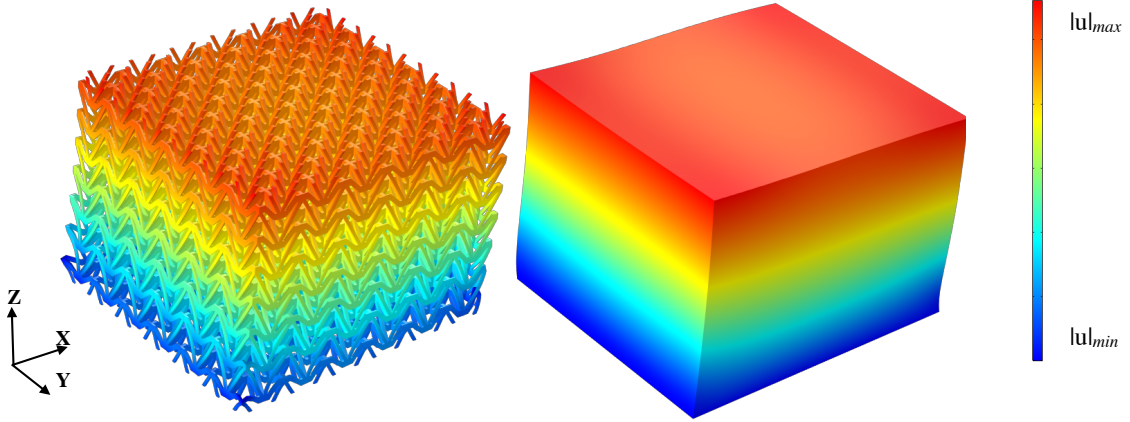


Figure 6.8: Second example of validation for the homogenization.

u [mm] in the real mesh	u [mm] in the homogeneous mesh [mm]	Relative error [%]
1.7894e-03	1.7193e-03	3.9150

Table 6.9: Comparison of the results for the second analysis.

The homogenization for the re-entrant honeycomb leads to less accurate results if one compares the relative errors with those obtained with the 3DRHC. Nevertheless, considering the simplicity of the homogenized model compared to the real one, this procedure still produces pertinent results in terms of accuracy.

Chapter 7

Simulations of impact

7.1 Modelling of the impact

In this last chapter, an experiment at large scale is performed. This last experiment will have a double goal. The first one is to compare the previously presented 3D re-entrant triangle cell to the already well studied re-entrant honeycomb in terms of impact performances. To achieve this first goal, the study presented in [19] is used as a guideline. The second objective is to see if the size of the problem, and therefore the computational cost can be reduced through the use of the homogenization procedure on part of the geometry.

The numerical tests are performed with the finite element code *Abaqus*. To follow the procedure of the paper, the impact is simulated between a rigid sphere and a portion of the protective device made of three layers. The dimensions of the simulation will not especially represent a real problem but rather a significant impact problem. This problem allows to understand better the behavior of the sandwich structure. The chapter will be built with the same procedure of the model on *Abaqus*: the geometry and discretization, the constraints, the loading.

Geometry

The sphere is modeled as a *Discrete rigid* $\rightarrow$ *Shell* with a radius of 0.3 [mm] with a mass of 10 [g]. In order to reduce the size of the model, only half of the sphere is represented. This choice is reasonable in the sense that the sphere is a rigid body and that the contact point are only located on the bottom part.

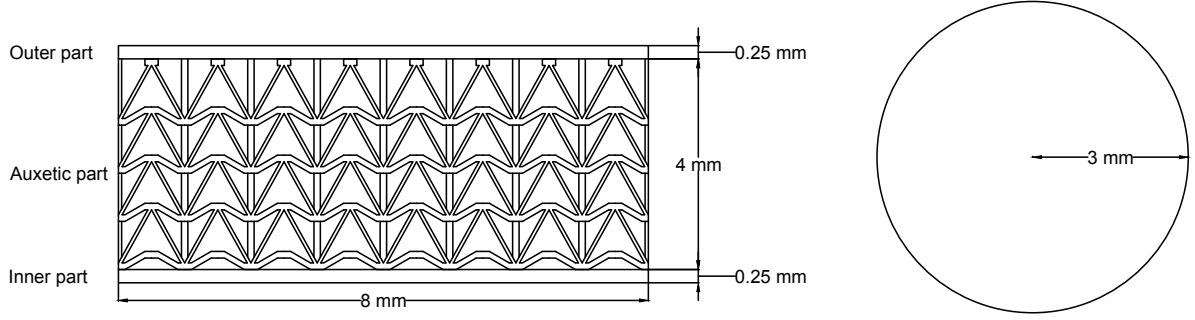


Figure 7.1: Geometry of the mask and sphere the impact problem.

The structure of the portion of the mask is made of three layers. Those layers are modelled as *Deformable* $\rightarrow$ *Solid*. The top and bottom layers of the sandwich structure are parallelepipeds with a height of 0.25 millimeters and a square base of 8 millimeters. Those layers have the purpose to enclose the middle layer made of the studied meta-material. The middle layer will therefore have a square base of 8 millimeters over 8 millimeters and a height of 4 millimeters, see [Figure 7.1](#).

Material properties

The materials used for the analysis will be of two types [\[13\]](#). The one used for part of the mask in contact with the face is flexible ethylene-vinyl acetate (EVA). This choice is made so to guarantee a certain level of comfort. The material flexible EVA can be describe with Ogden model (hyperellastic model). The properties of flexible EVA are reported in [Table 7.1](#).

EVAFlexible		
Mass density	ρ	2000 kg/m ³
Order of the deformation energy		2
	μ_1	7 MPa
	α_1	0.8
	μ_2	2.6 MPa
	α_2	2.6
	D1	0
	D2	0

Table 7.1: Properties of the flexible EVA.

The parameters in [Table 7.1](#) define the elastic energy: μ_1, μ_2 , are material moduli while α_1 and α_2 are coefficients used to define the deviatoric elastic energy. The two parameters D_1 and D_2 are parameters used to define the volumetric elastic energy.

The material for the external layer will be the same as the one used for the meta-material and will be the rigid EVA; the material is modelled as linear elastic with the parameters of [Table 7.2](#)

EVARigid		
Mass density	ρ	940 kg/m ³
Young's modulus	E	480 MPa
Poisson's ratio	ν	0.48

Table 7.2: Properties of the rigid EVA.

The properties of these material are the same of those in [\[7\]](#). The rigid EVA material ([Table 7.2](#)) was already used for the experiment in all the previous chapters and has a linear-elastic behavior.

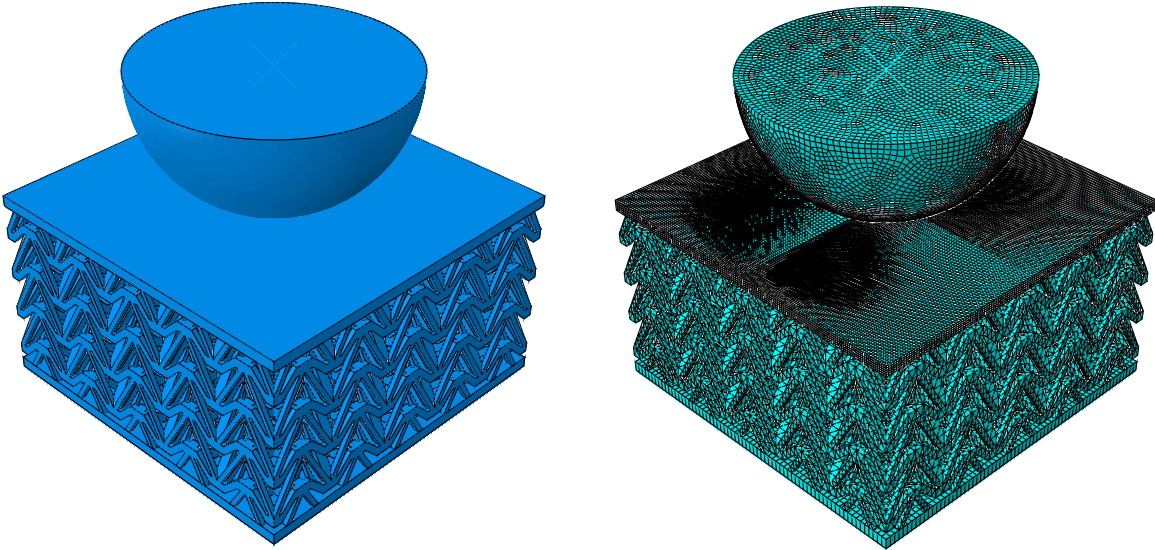


Figure 7.2: Assembly of the model on the left and visualization of the mesh for the resolution on the right.

Interaction and boundary conditions

Four specific interactions zones need to be specifically detailed, namely, the interaction between the internal part of the mask and the face, the interaction between the auxetic structure and respectively the internal and external part of the mask, and finally the interaction between the sphere and the external part.

For the first interaction between the mask and the face, a *Coupling* interaction is imposed between the inner face (in magenta in [Figure 7.3](#)) and a *Reference Point*. The total reaction force transmitted by the device will therefore be stocked in the behavior

of this reference point. This point is fully constrained.

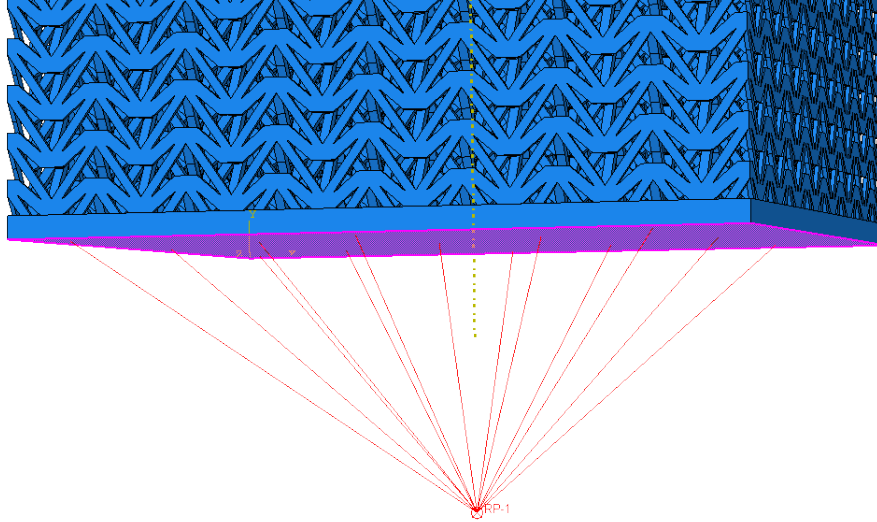


Figure 7.3: Interaction between the face and the mask device through the coupling interaction and definition of the reference point.

Between the mask and the two plates, an interaction of type *Tie* is imposed. This condition suggests that the whole structure is made of a unique component with a complete collaboration between the different parts (outer part, auxetic structure, and inner part).

The last interaction is the one between the sphere and the outer part of the mask. The modeling of this contact was made through the use of *General contact (Explicit)*. The *Normal Behavior* is configured as an "*Hard*" *contact* while the *Tangential Behavior* is assigned with an isotropic *Penalty* coefficient of 0.2. This interaction is imposed with the option *All* with self*. The contact between the external part and the sphere as well as contact within the auxetic structure are, therefore, governed by those properties.

Load

The part of the geometry represented in the problem is considered as a part of the mask far enough from its boundaries. The behavior of the four lateral faces can therefore be described with the *Symmetry/Anti-symmetry/Encaster* boundary conditions. The latest retracts the normal translation and the rotation around the axis parallel to the surface considered.

Mesh

Due to the localized nature of the phenomenon, the construction of the mesh has been designed to provide precision in the most deformed parts. The critical zone is located at the interface between the sphere and the outer part of the mask. The mesh on this part is produced to be quite coarse on the border and to become finer and finer as it gets closer to the middle point (impact point). For both the plate and the sphere, the mesh is generated with the tool *Seed* with *Bias factor* the latest was oriented from the outer towards the center for the plate and from the top to the bottom for the sphere.

For the sphere, bilinear rigid quadrilateral finite elements are used (R3D4). For the inner and outer parts of the mask, height nodes linear bricks (C3D8R) are used. Finally, for the auxetic part, because the geometry is more complex, ten nodes modified quadratic tetrahedrons (C3D10M) elements are chosen.

Because of the complexity and the size of the model, the mesh, even in the contact region cannot be very refined to limit the computational cost.

Impact analysis

The analysis is performed in two steps, the first one is the initial, and the second one is created as a *Dynamic, Explicit*. As the precise duration of the impact is unknown, the total time of the experiment *Time period* is imposed equal to 10 milliseconds. To reduce the time of computation, the results are only saved in the output file every 0.01 milliseconds. In the second step, an initial speed of 1.5 meters per second is imposed on the *Reference Point* representing the position and the mass of the rigid body which represents the sphere.

7.2 Results of the analysis and comparison

As described before, the analysis of the impact is performed using for the internal layer of the mask either metamaterial with the re-entrant triangle geometry and the re-entrant honeycomb geometry or an homogeneous bulk material. The purpose of these three analyses is to compare the efficiency of the different solutions.

In [Figure 7.4](#) and [7.5](#), the undeformed (left) and the deformed configuration at the time of inversion of the movement of the sphere are represented for the full geometry and half of the geometry (re-entrant triangle). The contours in [Figure 7.4](#) represent the displacement amplitude. The time at which the picture is taken is the precise moment where the inversion of the motion of the sphere started. It corresponds to minimal kinetic energy and maximal total force transmitted.

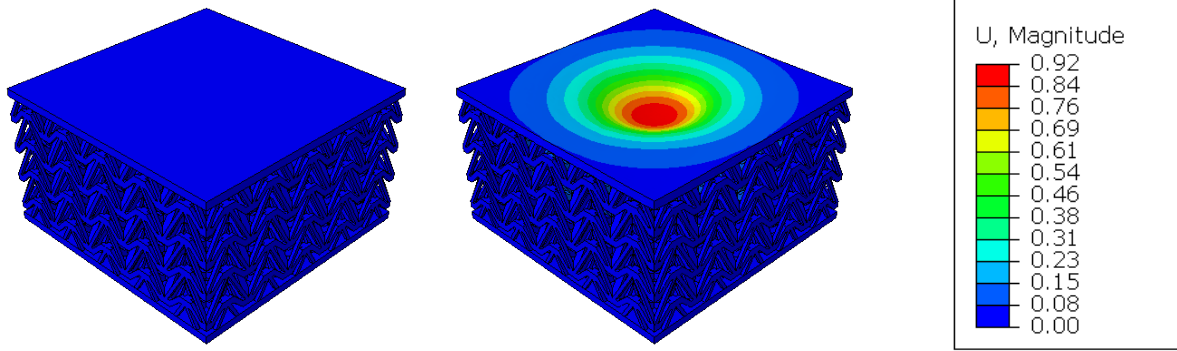


Figure 7.4: Undeformed configuration (left) and the deformed configuration (right) for the portion of the mask considered with the re-entrant triangle geometry at the time $t = 0.87$ ms. The scale for the amplitude of the displacement is in mm.

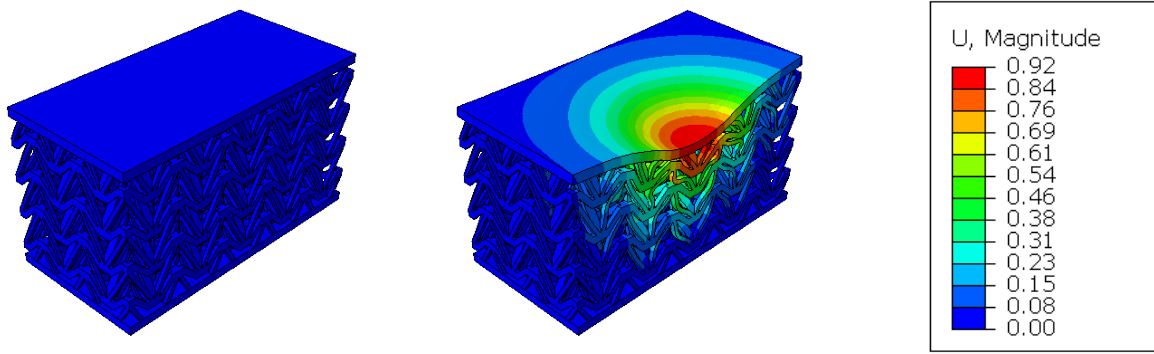


Figure 7.5: Undeformed configuration (left) and the deformed configuration (right) for half of the portion of the mask considered with the re-entrant triangle geometry at the time $t = 0.87$ ms. The scale for the amplitude of the displacement is in mm.

To better understand the mechanism of deformation for the two metamaterial lattices, the deformation in the central section of the considered portion of the mask is represented in [Figure 7.6](#) for different time steps of the analysis. The different instants are: the time just before the impact $t = 0$ ms, the time when the motion of the sphere starts to invert for the re-entrant triangle $t = 0.87$ ms and, the time when the deformation is maximal for the re-entrant honeycomb $t = 1.4$ ms.

Comparing the different time steps of the deformation for the two geometries, several remarks can already be made on their respective behaviors under impact. Looking at the deformation at $t = 8.7$ ms, two important remarks can already be made. The first one is that in the re-entrant honeycomb geometry, a larger number of cells participate to the deformation mechanism. In the re-entrant triangle, the deformations are more local and therefore do not take advantage of all the resources available.

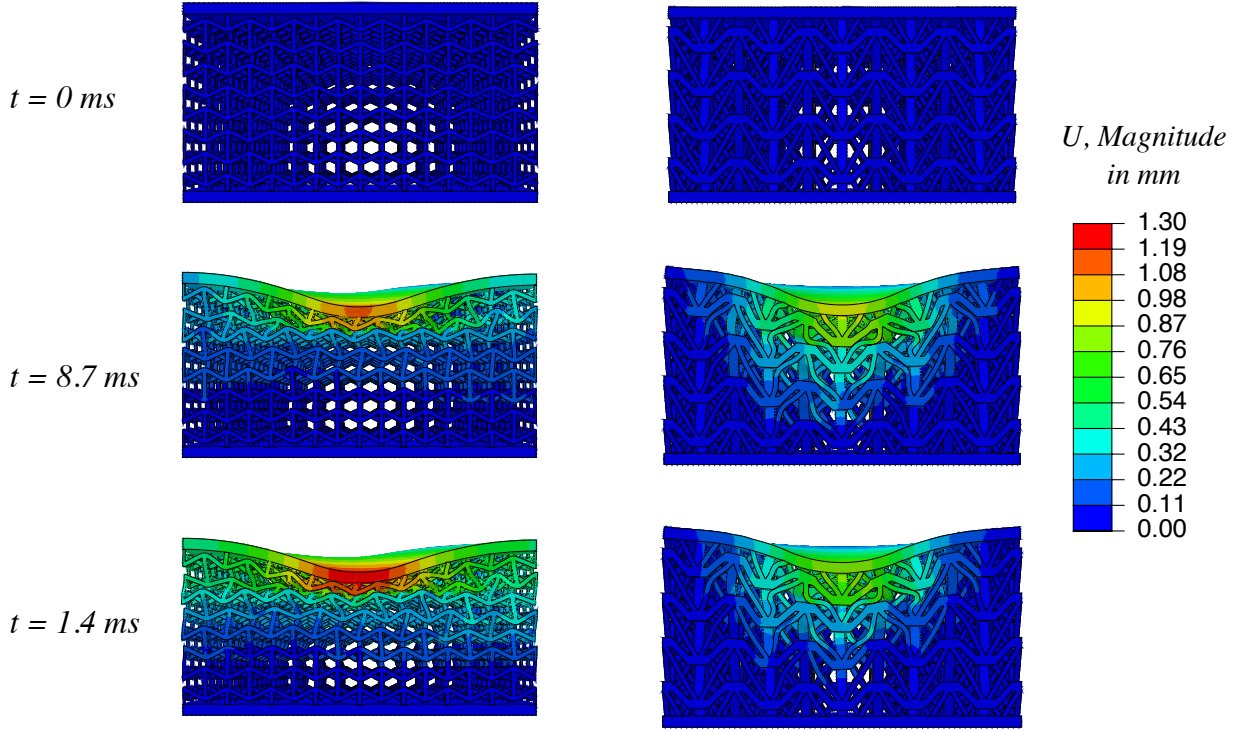


Figure 7.6: Comparison of the deformation mechanism for the two geometries.

The second remark concerns the instability of the cells. For the re-entrant triangle geometry, the thinner diagonal elements of the two cells directly under the impact zone instabilize and strongly bend. The deformation develop until the contact of the upper branches with the upper part of the cell located below. This mode of deformation will induce an increase in the global stiffness of the sandwich structure. For the re-entrant honeycomb geometry, a certain number of cell instabilize with a more diffuse gradual deformation.

To further compare the geometry, the plot of the total force transmitted by the rigid body to the protective device is represented as a function of the time in [Figure 7.7](#). The comparison is made between the re-entrant triangle, the re-entrant honeycomb, and a block a material with the same properties of the material used to model the auxetic structures ([Table 7.2](#)). The layer of homogeneous material is of the same height of the ones of auxetic materials. The interest was to appreciate the advantage of such auxetic structures and confirm the augmented resistance properties announced in the introduction part.

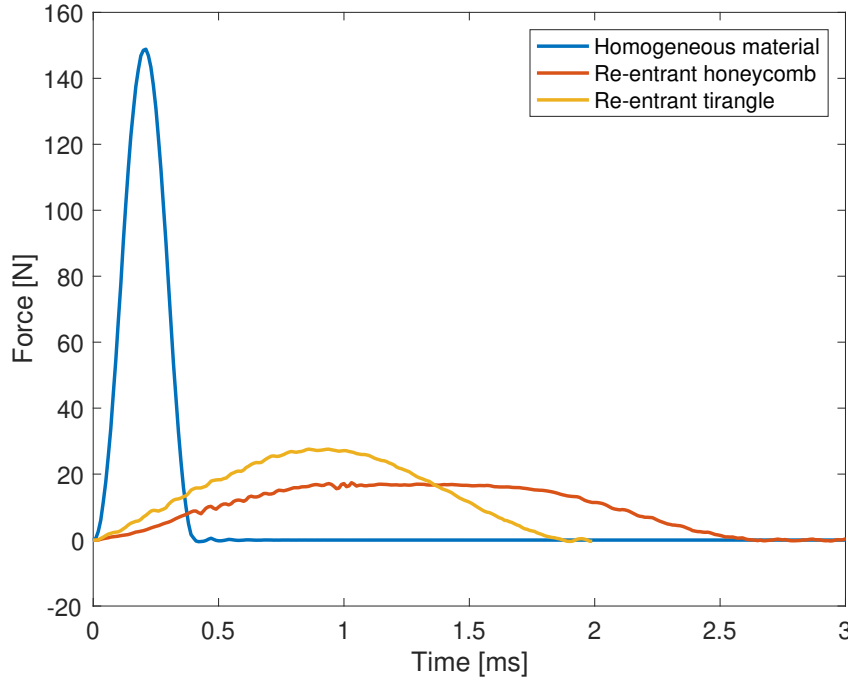


Figure 7.7: Evolution in time of the force transmitted by the mask to the face for the three geometries studied.

Material	t_{pic} [ms]	F_{pic} [N]
Homogeneous	0.19	152.18
Re-entrant honeycomb	1.00	17.39
Re-entrant triangle	0.86	27.57

Table 7.3: Maximal reaction occurring during the impact.

It can immediately be observed that the analysis with the homogeneous material gives rise to a much higher force in comparison to the two auxetic structures. The force transmitted through the re-entrant honeycomb is reduced by 88%. For the new re-entrant triangle geometry, the reduction was smaller and equal to 81%.

For the comparison between the auxetic structures, the re-entrant honeycomb showed to be more efficient in terms of impact absorption. This result is aligned with the previous analysis. The re-entrant triangle was showed to have a higher stiffness for the solicitations along the Z axis. The transmitted force was therefore expected to be higher. Comparatively, in terms of maximal values, the re-entrant honeycomb showed a reduction of 36% in comparison to the re-entrant triangle. Nevertheless, as explained in the introduction, the behavior of the auxetic structure is by far superior to those which do

not belong to the class of auxetic materials.

As a final comparison between the two geometries, the stresses transmitted to the inner part of the protective device are considered. The maximal compression stress transmitted by the inner face of the protective device is transmitted to the face of the athlete. The minimization of this stress is, therefore, a crucial aspect when one needs to design a protective device. This result is aligned with the fact that the re-entrant triangle was globally stiffer than the other geometry.

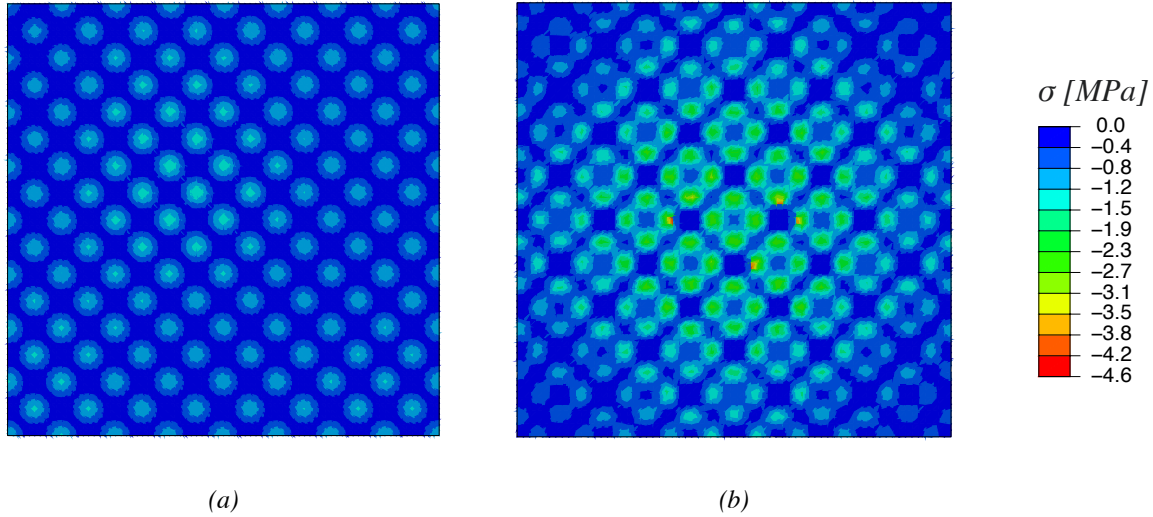


Figure 7.8: Maximum compression stress on the inner surface of protective devices for the re-entrant honeycomb (a) and the re-entrant triangle (b).

The contour plots in [Figure 7.8](#) represent the maximum compression stress at the moment of inversion of the motion for the rigid sphere for the re-entrant honeycomb ([Figure 7.8](#) a) and the re-entrant triangle ([Figure 7.8](#) b). It can be seen that these values also show better performances for the re-entrant honeycomb. The maximum for the latest is about -1.9 MPa while pics of -2.7 MPa are observed for the re-entrant honeycomb.

7.3 Results of the analysis with the use of the homogenization

In this last section, the possibility of replacing part of the geometry with an equivalent homogenized media is studied. As explained above, the complexity of the geometry as well as the size of the problem calls for the use of a quite dense mesh and therefore leads to simulation taking up to three days. The procedure of homogenization presented throughout this work could possibly be a useful tool to make these analyses faster. As it

was used previously, the homogenization procedure allows to compute the homogenized elastic parameters from a given periodic structure. The idea is, therefore, to replace part of the complex auxetic geometry with an equivalent homogenized block. This operation will allow to reduce the number of elements as well as the complexity of the mesh and finally the time of analysis.

Three cases are studied to assess the possibility of using the above-mentioned procedure. The case describe in in Section 7.2 with the whole geometry represented is used as a reference point for the other analysis. For the second analysis, the whole geometry is replaced by the homogeneous media. Other analyses are focused on a hybrid representation where the part closer to the impact is fully described while the part on the border and on the bottom is replaced by the equivalent homogeneous media.

Within the Abaqus model, a new *Material* was added. The latest was described as elastic orthotropic. In the Equation 7.1, the elastic properties retrieved from the homogenization procedure are defined. As the material is no longer isotropic, an orientation is also assigned to the homogenized medium. Note that the equivalent stiffness in Equation 7.1 differs from the one obtained in Chapter 6 (Equation 6.4) as the geometry of the metamaterial have been slightly changes by doubling the thicknesses of the branches.

$$\underline{\underline{C}} = \begin{bmatrix} 8.849 & -4.380 & -1.929 & 0 & 0 & 0 \\ -4.380 & 8.849 & -1.929 & 0 & 0 & 0 \\ -1.929 & -1.929 & 1.116 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.874 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.195 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.195 \end{bmatrix} \text{MPa} \quad (7.1)$$

In Figure 7.10 the different curves of force versus time are represented. The curve of the homogenized medium is able to approximate the initial stiffness but quickly diverge from the real case, giving a stiffer response. At peak value, the force produced by the homogenized medium is greater by 38.49% in comparison with the real model. This increase of stiffness can be explained by looking at the mechanism of deformation of the re-entrant triangle geometry (Figure 7.7). During the impact most of the cells taking part in the deformation instabilize. The homogeneous media is unable to reproduce those mechanisms as the response is only global. The instabilities of the real geometry reduce the global stiffness of the protective device, giving rise to a smaller peak reaction force in comparison to the homogeneous medium. In terms of computational time, the study on the completely homogenized media only needed 2.19% of the total time needed by the real geometry (see Table 7.4).

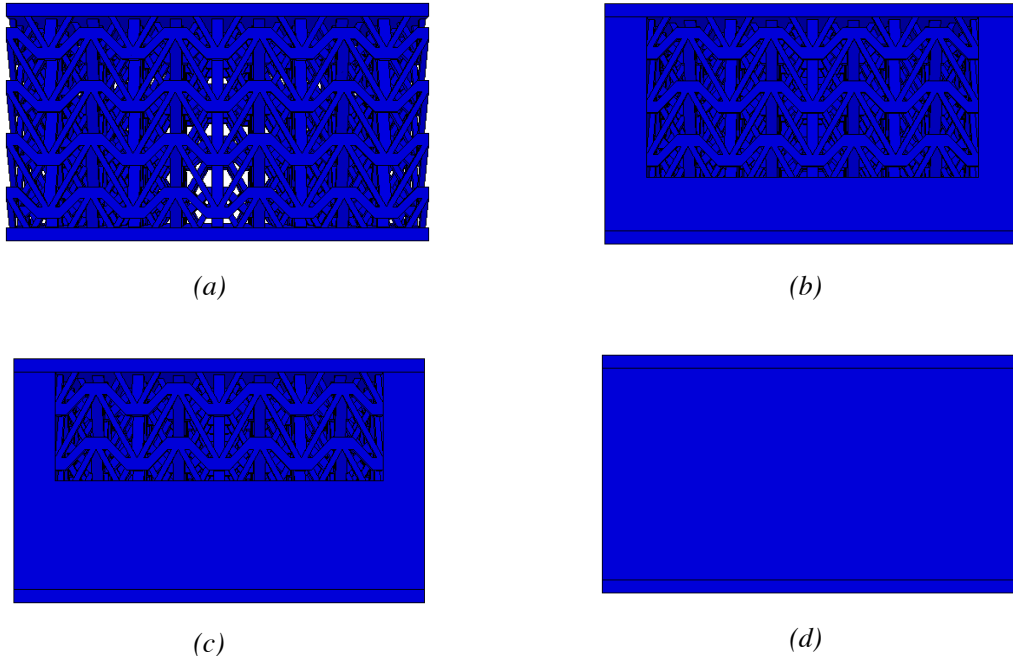


Figure 7.9: Different analyzed problems: the full geometry (a), the first mixed (homogeneous and structured material) case (b), the second mixed case (c), fully homogenized geometry (e).

To overcome the problem presented here above, the idea is to substitute only part of the geometry with the homogenized media. Doing so, the mechanisms of instabilities are still represented by the remaining cells while the parts of the geometry less solicited are modeled by rather the equivalent homogenized medium. The results of the two analyses with the mixed geometry shown in [Figure 7.9\(b\)](#) and [Figure 7.9\(c\)](#) are represented in [Figure 7.10](#) with the yellow and purple lines respectively. It can be seen the objective was reached as the error on the peak force is only of approximately 5% while saving almost half of the computational time.

Type of model	CPU time [days/hours/minutes]	Relative CPU time [%]
Real metamaterial	1/5/45	100
Case1	0/13/46	46.31
Case2	0/11/44	39.46
Homogenized model	0/0/39	2.19

Table 7.4: Comparison for the duration of the analyses with the use of homogenization.

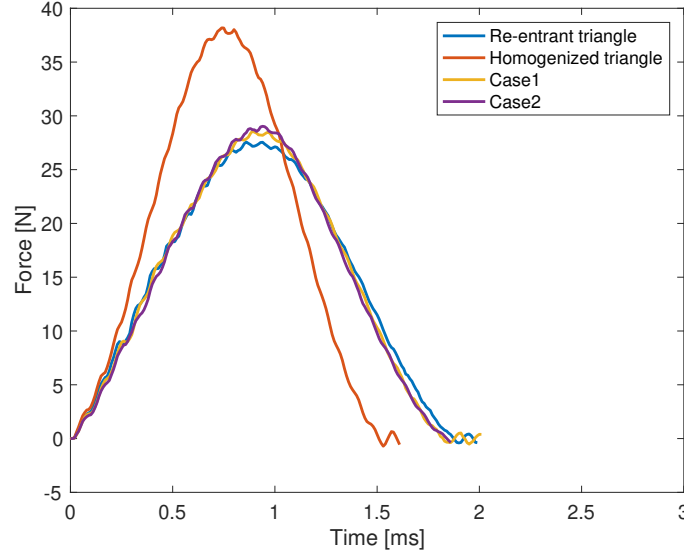


Figure 7.10: Evolution in time of the force transmitted by the mask to the face for the three cases studied.

The analysis with the equivalent homogeneous media is performed for the re-entrant honeycomb as well. The procedure for the setup of the analysis is the same as the one explained for the re-entrant triangle geometry. The stiffness matrix used to characterize the homogenized media is described in [Equation 6.3](#).

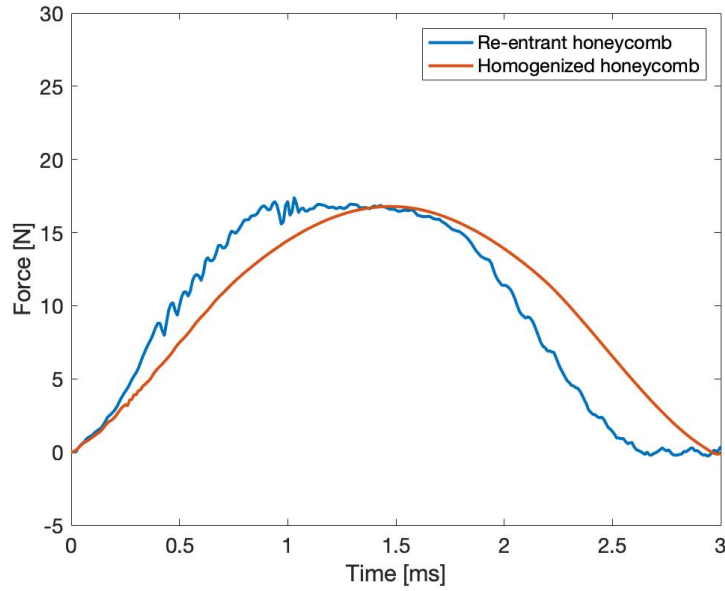


Figure 7.11: Evolution in time of the force transmitted by the mask to the face for the complete re-entrant honeycomb geometry (blue) and the homogenized medium (red).

In [Figure 7.11](#), the responses of the two analyses in terms of total transmitted force are represented. The results for this geometry are by far better as the error on the peak force is only of 3.61 %. The fact that the analyses for the re-entrant honeycomb are more accurate can also be explained by looking at [Figure 7.7](#). Unlike the re-entrant triangle geometry, the re-entrant honeycomb geometry deforms in a more collaborative way. Instabilities still occur during the impact simulation. The fact that the response involves more cells and that the response of the systems is more spread over all the cells explain why the homogenization procedure gives more accurate results.

As the procedure of homogenization showed better results for the case of re-entrant honeycomb, the rest of the comparative analyses are done for this geometry. The total force transmitted through the protective device as a function of the displacement of the rigid sphere is shown in [Figure 7.12](#) for the re-entrant honeycomb and for the equivalent homogenized material.

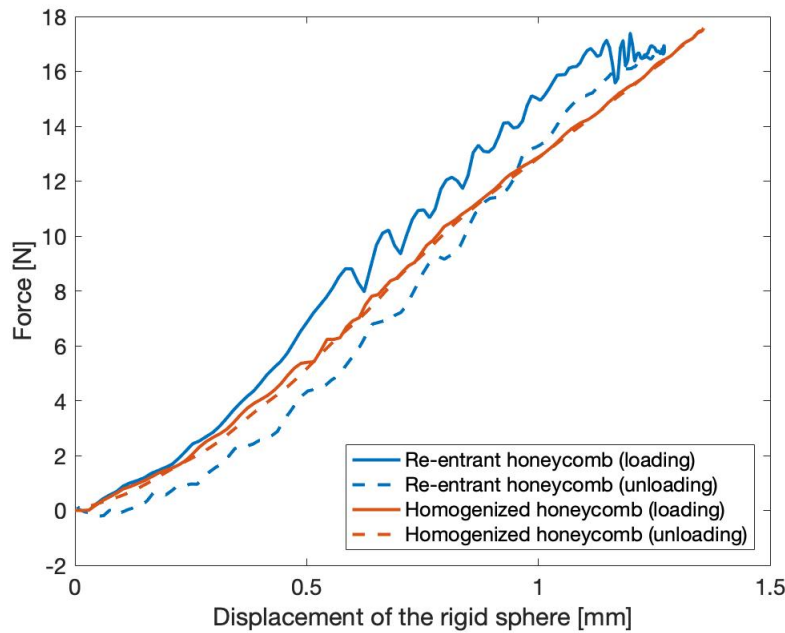


Figure 7.12: Total force transmitted through the protective device as function of the displacement of the rigid sphere for the two analysis.

On [Figure 7.12](#) the initial stiffness of the homogenized media was found to be a very good approximation of the real mesh. Along the loading, the two curves remain very similar even though a reduction of stiffness is observed for the homogenized media.

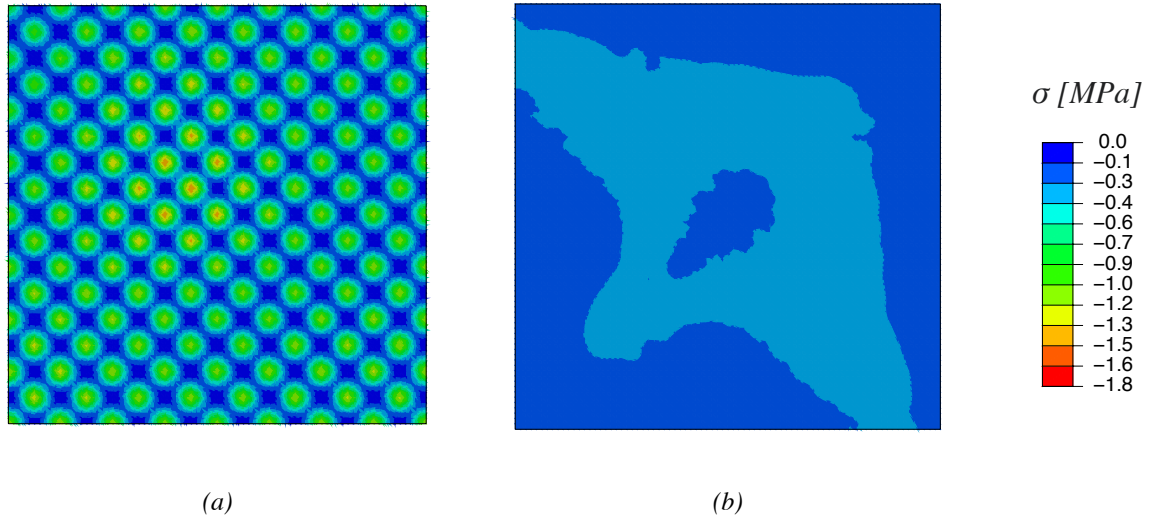


Figure 7.13: Maximum compression stress on the inner surface of protective devices for the re-entrant honeycomb (a) and the homogenized honeycomb(b).

Finally, the maximum compression are compared in [Figure 7.13](#). It can be seen that even though the global responses of the homogenized problem are very close to the real response ([Figure 7.11](#) and [Figure 7.12](#)), the local response in terms of compression stress completely differs. In fact, the homogenized material predict a maximum compression which is only 21% of the real one. It is to be noted that the apparent non-symmetry on [Figure 7.13](#) (b) is only due to the poorness of the mesh.

Chapter 8

Conclusion and further developments

The present work is oriented toward addressing two issues. The first one concerned the extensive study of the 3D re-entrant triangle and the comparison with the 3D re-entrant honeycomb. The comparison is achieved keeping in mind the final scope of these structures, namely: their particularly interesting response to impact. The second topic concerns the theory of homogenization with the goal of describing the auxetic structures. Furthermore, the final objective was to assess the possibility for such a procedure to reduce the complexity and computational time of a real scale problem among which impact simulation is one.

The first part of this work is focused on theoretical aspects that will be used throughout this work. The specific vocabulary of periodic structure is presented and the main typologies for 2D and 3D microstructures are described. Then the theory of homogenization is formulated for the specific case of solid mechanics. The main assumptions are presented and, step by step, the equations are developed to finally obtain a close form equation for the coefficients of the homogenized stiffness matrix.

The objective concerning the description of the re-entrant honeycomb and the re-entrant triangle started with the study of 2D lattices. The theory of homogenization is addressed for the 2D plane strain problem. The results are showed to be of reasonable reliability. The results achieved with the homogenized re-entrant honeycomb as a function of the number of cells showed to be more precise but demonstrated a slower convergence than with the re-entrant triangle geometry. A study of the influence of the number of cells on the accuracy of the model is run. Comparing the homogenized stiffness with the one numerically computed on lattices with a certain number of cells, one can observe that in the case of re-entrant honeycomb the homogeneous media is more compliant while for the re-entrant triangle the opposite is observed. Therefore a general rule cannot be established. At the end of this part, a complete parametric study is performed. The results for the two cells are compared as functions of two specific

parameters. The homogenization procedure showed to be very efficient for this type of characterization.

The three-dimensional geometries are then presented as an extension of the two-dimensional ones. The geometries are detailed and a specific choice for the parameters is done so to obtain two cells of identical relative density. The equivalent homogenized stiffness matrix is obtained through the resolution of the four independent cell problems for symmetric 3D cells. Then two analyses are set up. The first one is with an imposed displacement type of loading and the force is monitored. For the second experiment the force is applied and the displacement is the quantity of interest. For both experiments, the homogenized media showed a good agreement compared to the real case. For this part, a tendency is also observed for the homogenized re-entrant honeycomb to be more compliant and for the re-entrant triangle to be stiffer. Nevertheless, no general rule could be formulated. Due to the duration of each analysis, the study on the influence of the number of cells could not be achieved.

The last part of the work is focused on the impact simulation. The simulations are performed in two distinct steps. The first step has as main objective to compare three sandwich structures. These three structures only differ by the middle layer that is: for the first one the homogeneous bulk material, for the second the 3D re-entrant honeycomb structure made of the same bulk material and, lastly, the 3D re-entrant triangle also made with the same bulk material. The experiments showed that the two metamaterials had by far a better behavior considering the total transmitted force. Between the two geometries it can be added that for the specific choice of parameters, the re-entrant honeycomb showed a slightly better behavior. A preliminary parametric study on the second geometry (re-entrant triangle) could have allowed one to better control the influence of each parameter and choosing a better set of parameters that, with the same equivalent density could have performed better. This nevertheless showed the interest of this structure but also the large variety of geometries inside the same class of re-entrant auxetic structures.

When using complex micro-structured materials, one of the critical aspects of the impact simulation is the computational cost. With the homogenized matrices being already computed, the idea is to progressively replace the auxetic structure with the homogenized equivalent media. This would allow one to retrieve a more simple model and reduce the computational cost of the simulation. A first experiment was done by replacing completely the auxetic with the equivalent homogenized media. This analysis showed that with the homogenized medium, one can predict correctly the initial stiffness of the whole structure and with less accuracy the peak force while reducing drastically the computational time. Increasing layer of cells by layer of cells, the results in terms of forces as a function of time becomes closer to the real one while the time of computation also became closer to the full analysis one.

Some critical aspects and additional studies could have been added. The first comment to be made is that the impact test is a very local phenomenon, instabilities and plasticity are therefore unavoidable. Homogenization instead is only valid for the linear elastic case which obviously represents a limitation. The second critical point is the size of the mesh which, for limitations related to the computational time, is chosen quite coarse. Studies with finer mesh and study on the influence of the mesh on the accuracy of the real and homogeneous simulation are still missing.

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