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Faculté des sciences

A Categorical Approach to Internal Actions and Semidirect Products

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Année académique 2021-2022

UNIVERSITÉ CATHOLIQUE DE LOUVAIN
FACULTÉ DES SCIENCES
ÉCOLE DE MATHÉMATIQUES

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Master thesis
June 2022

Remerciements

Pour commencer, je voudrais grandement remercier mon promoteur, le Professeur Tim Van der Linden. J'ai fortement apprécié travailler sur le sujet passionnant qu'il m'a proposé et j'ai trouvé particulièrement intéressantes les nombreuses discussions que nous avons échangées. J'aimerais également le remercier pour ses nombreux conseils, autant sur les aspects mathématiques de ce travail que sur la rédaction de celui-ci, ainsi que pour sa disponibilité.

Je remercie également les Professeurs Marino Gran et Pierre-Emmanuel Caprace qui ont accepté d'être les lecteurs de ce mémoire. J'espère que le plaisir qu'ils auront à la lecture de ce travail sera à la hauteur de celui que j'ai eu à le réaliser.

J'aimerais également remercier le Professeur Pascal Lambrechts qui fut mon promoteur pour le projet personnel de fin de bachelier. Bien qu'il n'ait pas directement contribué à ce mémoire, ce projet de fin de bachelier fut mon premier vrai travail scientifique de ce genre et nul doute que ses précieux conseils ont également eu une influence bénéfique sur ce travail-ci.

Finalement, je voudrais aussi remercier tous les professeurs et tous mes amis qui m'ont accompagné durant ces cinq années d'études et avec qui j'ai pu partager ma passion pour les mathématiques.

Contents

Introduction

Actions appear in many places in mathematics, the best-known one being group actions : we see groups acting on sets by permutations, on topological spaces by homeomorphisms, ... in various domains like combinatorics, topology, geometry, ... Vector spaces and modules over a ring are other important examples of actions : indeed, a vector space or a module is nothing more than an abelian group that is acted on by a field or a ring, respectively. However, in this thesis, we will focus on actions where the object that acts and the one that is acted on belong to the same category — hence the name of *internal action* — which excludes vector spaces and modules (in fact, it is possible to recover those examples as well (see Example 2.2.4) but we will not pursue this in detail in this thesis). Among the examples we mentioned above, the only remaining one is groups acting on other groups by automorphisms, but it is of course not the only example of actions we will consider : other actions covered by the theory we will present are ring actions (which will be our second guiding example alongside group actions) and Lie algebra actions by derivations.

We talked a lot about actions in the first paragraph but what exactly is an action ? We will not give a precise answer to this question here since it is not an easy concept to formalise but, intuitively, an action of an object B on another object C can be seen as a collection of bijections of C indexed by the elements of B and that respects the structure of these two objects. For groups, we formalise this by saying that an action of B on C is a group homomorphism ϕ from B to the group $\operatorname{Aut} C$ of automorphisms of C : the fact that ϕ is a group homomorphism means that this action respects the structure of B and the fact that its codomain is $\operatorname{Aut} C$ expresses its compatibility with the structure of C. (Using the natural bijection between $(C^C)^B$, the maps from B to the maps from C to itself, and $C^{(B\times C)}$, the maps from $B \times C$ to C, we can also see actions as maps from the Cartesian product $B \times C$ to C.) However, the general definition of internal action we will present is not really based on this intuitive definition since this definition has a chance to make sense only in concrete categories (i.e. categories of sets equipped with a certain structure, like groups, rings, topological spaces, ...) and is not easy to express in categorical terms (for example, it is not clear what is the suitable replacement for the group $\operatorname{Aut} C$ in other categories). Rather, we will exploit the fact that, via the semidirect product construction, actions correspond to split short exact sequences : we will present a categorical version of this correspondence.

The first chapter of this thesis will be dedicated to establishing the ground for the following chapters by introducing some categorical concepts and giving some useful results. In the second one, we will take a first look at actions in our two guiding examples, groups and commutative rings. In the third one, we will introduce the main tool for our generalisation, the monad $B\flat(-)$. This general notion of internal actions will be presented in the fourth chapter where we will also prove the main result of this work, the equivalence between actions and *split short exact sequences*, which form a key concept in homological algebra. Finally, in the last chapter, we will see how we can apply the theory developed before in some examples.

The results of this thesis are not new but we did follow an approach different from what exists in the literature : we tried to present this theory of internal actions from a self-contained point of view, requiring less categorical background than the original approach. This work can also be seen as a synthesis of notions scattered in various papers and we made explicit some results not developed in detail in the literature. For instance, the main equivalence we will present is the subject of [BJ98] but, in this article, the theory is not explicitly worked out. The results and examples of this document were mainly obtained independently based on indications of my advisor, Professor Tim Van der Linden.

Chapter 1

Preliminary Categorical Notions

In this first chapter, we will present various categorical notions and results which will be used in the rest of this thesis. The first section is focused on a notion of "sameness" of categories, *equivalences of categories*. In the second one, we will present various results about monomorphisms and epimorphisms that we will need in the later chapters. Finally, the third one is an introduction of the main context we will consider, pointed *protomodular* categories.

1.1 Equivalences of Categories

In this section, we will study the concept of equivalence of categories, which is an important tool to compare categories. In the first subsection, we will introduce all the necessary definitions and compare the notions of equivalence and weak equivalence of categories. The second subsection will be dedicated to the link between equivalences of categories and an other important concept in category theory, adjunctions. The results of this second subsection will not be used in the following but we thought they were still interesting since they present a link with adjunctions, which is a central notion in category theory, as said before. This section is principally based on [ML98], Section IV.4 for the first subsection and Section IV.1 for the second one.

1.1.1 Equivalences and Weak Equivalences

In this thesis, we will be interested in saying that two categories are "essentially the same". The first notion that comes to mind for that is the concept of isomorphism of categories but this notion is a bit too strong for what we want. As a reminder, an *isomorphism of categories* between two categories $\mathcal{A}$ and $\mathcal{B}$ is a functor $F\colon \mathcal{A} \to \mathcal{B}$ that is invertible, i.e. such that there is a functor $G\colon \mathcal{B} \to \mathcal{A}$ such that $FG = 1_{\mathcal{B}}$ and $GF = 1_{\mathcal{A}}$. However, asking that these composites are equal to the identities on $\mathcal{A}$ and $\mathcal{B}$ is a strong condition so we would like to find a way to relax it, in a similar way that homotopy equivalence is a relaxation of the concept of homeomorphism in topology. For homotopy equivalences and homeomorphisms, this relaxation is made by asking that both composites of a map and its "inverse" are *homotopic* to the identities instead of being equal to it. In our case, *natural isomorphisms* will play a role which is analogous to these homotopies.

Definition 1.1.1. A *natural isomorphism* is a natural transformation $\eta\colon F \to G$ that is invertible, i.e. such that there is a natural transformation $\varepsilon\colon G \to F$ such that $\varepsilon \circ \eta = 1_F$ and $\eta \circ \varepsilon = 1_G$. We have that such an ε is unique and we denote it by η^{-1}. When this is the case, we write $\eta\colon F \cong G$.

Remark 1.1.2. If $\eta\colon F \cong G$ is a natural isomorphism, then its components are isomorphisms whose inverses are given by the corresponding components of η^{-1}. We easily check that this characterises the natural isomorphisms : a natural transformation is a natural isomorphism if and only if all its components are isomorphisms.

Now that we have defined the natural isomorphisms, we can give the definition of the notion of "sameness" of categories we are interested in, *equivalences of categories*.

Definition 1.1.3. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is an *equivalence of categories* if there is a functor $G\colon \mathcal{B} \to \mathcal{A}$ and natural isomorphisms $\varepsilon\colon FG \cong 1_{\mathcal{B}}$ and $\eta\colon 1_{\mathcal{A}} \cong GF$. The functor G is said to be a *pseudo-inverse* of F. If $\mathcal{A}$ and $\mathcal{B}$ are categories such that there is an equivalence $F\colon \mathcal{A} \to \mathcal{B}$ between them, we write $\mathcal{A} \simeq \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*.

The first property we have about these equivalences of categories is that, as their name suggests, they give us an equivalence relation on categories.

Proposition 1.1.4. *The relation $\simeq$ is an equivalence relation.*

Proof. The reflexivity and the symmetry are almost immediate. A category $\mathcal{A}$ is equivalent with itself via $1_{\mathcal{A}}$ and, if $\mathcal{A} \simeq \mathcal{B}$ via $F\colon \mathcal{A} \to \mathcal{B}$, $G\colon \mathcal{B} \to \mathcal{A}$ and natural isomorphisms $\varepsilon\colon FG \cong 1_{\mathcal{B}}$ and $\eta\colon 1_{\mathcal{A}} \cong GF$, then $\mathcal{B} \simeq \mathcal{A}$ via $G\colon \mathcal{B} \to \mathcal{A}$, $F\colon \mathcal{A} \to \mathcal{B}$ and natural isomorphisms $\eta^{-1}\colon GF \cong 1_{\mathcal{A}}$ and $\varepsilon^{-1}\colon 1_{\mathcal{B}} \cong FG$.

For the transitivity, consider $\mathcal{A} \simeq \mathcal{B}$ and $\mathcal{B} \simeq \mathcal{C}$. By definition, we have functors $F\colon \mathcal{A} \to \mathcal{B}$, $G\colon \mathcal{B} \to \mathcal{A}$ and natural isomorphisms $\varepsilon\colon FG \cong 1_{\mathcal{B}}$ and $\eta\colon 1_{\mathcal{A}} \cong GF$ as well as functors $F'\colon \mathcal{B} \to \mathcal{C}$, $G'\colon \mathcal{C} \to \mathcal{B}$ and natural isomorphisms $\varepsilon'\colon F'G' \cong 1_{\mathcal{C}}$ and $\eta'\colon 1_{\mathcal{B}} \cong G'F'$. Consider the functors $F'F\colon \mathcal{A} \to \mathcal{C}$ and $GG'\colon \mathcal{C} \to \mathcal{A}$. We define $\varepsilon''\colon F'FGG' \to 1_{\mathcal{C}}$ by $\varepsilon''_C = \varepsilon'_C \circ F'\Big(\varepsilon_{G'(C)}\Big)$ for $C \in \mathcal{C}$.

$$\begin{tikzcd} F'FGG'(C) \arrow[r, "F'\left(\varepsilon_{G'(C)}\right)"] \arrow[rr, bend right, "\varepsilon''_C"'] & F'G'(C) \arrow[r, "\varepsilon'_C"] & C \end{tikzcd}$$

Since ε and ε' are natural isomorphisms and F' is a functor, the components of ε'' are isomorphisms and, by Remark 1.1.2, we just need to show the naturality of ε'' to conclude that it is a natural isomorphism. So we must show that the diagram (without the dashed arrow)

$$\begin{tikzcd}
F'FGG'(C) \arrow[r, "F'\left(\varepsilon_{G'(C)}\right)"] \arrow[d, "F'FGG'(f)"] & F'G'(C) \arrow[r, "\varepsilon'_C"] \arrow[d, dashed, "F'G'(f)"] & C \arrow[d, "f"] \\
F'FGG'(C') \arrow[r, "F'\left(\varepsilon_{G'(C')}\right)"'] & F'G'(C') \arrow[r, "\varepsilon'_{C'}"'] & C'
\end{tikzcd}$$

commutes for all $f\colon C \to C'$ in $\mathcal{C}$. By adding the dashed arrow $F'G'(f)\colon F'G'(C) \to F'G'(C')$, we remark that this makes the two new squares commute so the outer rectangle also commutes, as desired. (The left square commutes by naturality of ε and functoriality of F' and the right one commutes by naturality of ε'.) We construct a natural isomorphism $\eta''\colon 1_{\mathcal{A}} \to GG'F'F$ in a similar way ($\eta''_A = G\left(\eta'_{F(A)}\right) \circ \eta_A$ for $A \in \mathcal{A}$) and this shows us that $\mathcal{A} \simeq \mathcal{C}$, finishing the proof. $\square$

Even if we have found the notion we were looking for, in practice, equivalences of categories are a little bit tedious to use since we need to define two functors and two natural isomorphisms. Therefore, we introduce an other definition which is easier to apply in practice, *weak equivalences of categories*.

Definitions 1.1.5.

- A functor $F\colon \mathcal{A} \to \mathcal{B}$ is *full* if, for each A, $A' \in \mathcal{A}$, the map

$$F_{A,A'}\colon \mathcal{A}(A, A') \to \mathcal{B}\Big(F(A), F(A')\Big)\colon f \mapsto F(f)$$

  is surjective and is *faithful* if all these maps $F_{A,A'}$, A, $A' \in \mathcal{A}$, are injective. A functor that is both full and faithful is said to be *fully faithful*.

- A functor $F\colon \mathcal{A} \to \mathcal{B}$ is *essentially surjective* if each object $B \in \mathcal{B}$ is isomorphic to $F(A)$ for some object $A \in \mathcal{A}$.

- A functor $F\colon \mathcal{A} \to \mathcal{B}$ is a *weak equivalence of categories* if it is fully faithful and essentially surjective.

Now, let us look at the connection between these two concepts. First, as its name suggests, weak equivalence is a weaker property than equivalence of categories.

Proposition 1.1.6. *If $F\colon \mathcal{A} \to \mathcal{B}$ is an equivalence of categories, it is also a weak equivalence of categories.*

Proof. By definition, we have a functor $G\colon \mathcal{B} \to \mathcal{A}$ and natural isomorphisms $\varepsilon\colon FG \cong 1_{\mathcal{B}}$ and $\eta\colon 1_{\mathcal{A}} \cong GF$. We must show that F is essentially surjective, full and faithful.

Let B be an object of $\mathcal{B}$. Then $G(B)$ is an object in $\mathcal{A}$ such that $F\big(G(B)\big) = FG(B)$ is isomorphic to B via ε_B. This shows the essential surjectivity of F.

Let $f, g \in \mathcal{A}(A, A')$, A, $A' \in \mathcal{A}$, such that $F(f) = F(g)$. By applying G to this equality and by naturality of η (see diagram below), we have

$$f = \eta_{A'}^{-1} \circ GF(f) \circ \eta_A = \eta_{A'}^{-1} \circ GF(g) \circ \eta_A = g,$$

which shows that F is faithful.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & GF(A) \arrow[d, "GF(f)"] \\ A' \arrow[r, "\eta_{A'}"'] & GF(A') \end{tikzcd}$$

Finally, let $g \in \mathcal{B}\big(F(A), F(A')\big)$ for some A, $A' \in \mathcal{A}$. The arrow $G(g)$ is an arrow from $GF(A)$ to $GF(A')$ and we can consider $f = \eta_{A'}^{-1} \circ G(g) \circ \eta_A \colon A \to A'$. By naturality of η (see diagram above), we have $GF(f) = \eta_{A'} \circ f \circ \eta_A^{-1} = G(g)$. Since G is faithful (as it is also an equivalence of categories and we proved the faithfulness of equivalences above), we conclude that $F(f) = g$, which shows the fullness of F and finishes the proof. $\square$

As said earlier, we would like to use weak equivalence since it is more practical so we would like to know how much weaker it is compared to equivalence of categories. In fact, as shown in the following result, if we accept the axiom of choice (which we will do in the rest of this thesis), weak equivalence is as strong as the normal one.

Proposition 1.1.7. *Under the axiom of choice, every weak equivalence of categories $F\colon \mathcal{A} \to \mathcal{B}$ is an equivalence of categories.*

Proof. We must construct a functor $G\colon \mathcal{B} \to \mathcal{A}$ and two natural isomorphisms $\varepsilon\colon FG \cong 1_{\mathcal{B}}$, $\eta\colon 1_{\mathcal{A}} \cong GF$.

For each $B \in \mathcal{B}$, there is an object $A \in \mathcal{A}$ such that $F(A) \cong B$ by essential surjectivity of F. Thus we define $G(B) = A$. (Here we use the axiom of choice because we make an arbitrary choice for all $B \in \mathcal{B}$.) For each B, we also fix an isomorphism $\varepsilon_B\colon FG(B) \cong B$ (again using the axiom of choice). Given $g \in \mathcal{B}(B, B')$, B, $B' \in \mathcal{B}$, we must define $G(g)\colon G(B) \to G(B')$. By definition of G on the objects, we have two isomorphisms $\varepsilon_B\colon FG(B) \cong B$ and $\varepsilon_{B'}\colon FG(B') \cong B'$. Moreover, since F is fully faithful, there is a unique $f \in \mathcal{A}\big(G(B), G(B')\big)$ such that $F(f)\colon FG(B) \to FG(B')$ is equal to $\varepsilon_{B'}^{-1} \circ g \circ \varepsilon_B$.

This choice of $F(f)$ makes the diagram

$$\begin{tikzcd} FG(B) \arrow[r, "\varepsilon_B"] \arrow[d, dashed, "F(f)"'] & B \arrow[d, "g"] \\ FG(B') \arrow[r, "\varepsilon_{B'}"'] & B' \end{tikzcd}$$

commute. We set $G(g) = f$ and we must check that these object and arrow assignments form a functor.

For $B \in \mathcal{B}$, $G(1_B)$ is given by $f\colon G(B) \to G(B)$ such that $F(f) = \varepsilon_B^{-1} \circ 1_B \circ \varepsilon_B = 1_B$. Since F is faithful, we conclude that $G(1_B) = f = 1_{G(B)}$.

Let $g \in \mathcal{B}(B, B')$, $g' \in \mathcal{B}(B', B'')$, B, B', $B'' \in \mathcal{B}$. By functoriality of F, we compute

$$\begin{aligned} F\Big(G(g') \circ G(g)\Big) &= FG(g') \circ FG(g) = (\varepsilon_{B''}^{-1} \circ g' \circ \varepsilon_{B'}) \circ (\varepsilon_{B'}^{-1} \circ g \circ \varepsilon_B) \\ &= \varepsilon_{B''}^{-1} \circ (g' \circ g) \circ \varepsilon_B = FG(g' \circ g). \end{aligned}$$

By definition of G on the arrows (or faithfulness of F), this shows that $G(g') \circ G(g) = G(g' \circ g)$, finishing the proof that G is a functor.

For the natural isomorphism $\varepsilon\colon FG \cong 1_{\mathcal{B}}$, we define ε_B for $B \in \mathcal{B}$ as in the second paragraph of the proof. By definition of G on the arrows, we have immediately that the diagram

$$\begin{tikzcd} FG(B) \arrow[r, "\varepsilon_B"] \arrow[d, "FG(g)"'] & B \arrow[d, "g"] \\ FG(B') \arrow[r, "\varepsilon_{B'}"'] & B' \end{tikzcd}$$

commutes for all B, $B' \in \mathcal{B}$ and all $g \in \mathcal{B}(B, B')$. Therefore, ε is indeed a natural isomorphism by Remark 1.1.2 since its components are isomorphisms by construction.

Finally, we must define a natural isomorphism $\eta\colon 1_{\mathcal{A}} \cong GF$. For $A \in \mathcal{A}$, $GF(A)$ is defined such that we have an isomorphism $\varepsilon_{F(A)}\colon FGF(A) \cong F(A)$. Since F is full, we have an arrow $\varphi \in \mathcal{A}\Big(A, GF(A)\Big)$ such that $F(\varphi) = \varepsilon_{F(A)}^{-1}$. We set $\eta_A = \varphi$ and we must check that it is an isomorphism. Again by fullness of F, there is a $\psi\colon GF(A) \to A$ such that $F(\psi) = \varepsilon_{F(A)}$. We compute that $F(\psi \circ \eta_A) = 1_{F(A)}$ and $F(\eta_A \circ \psi) = 1_{FGF(A)}$. By faithfulness of F, this shows that η_A is an isomorphism (and that ψ is its inverse). For the naturality of η, we must show that the diagram

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & GF(A) \arrow[d, "GF(f)"] \\ A' \arrow[r, "\eta_{A'}"'] & GF(A') \end{tikzcd} \tag{1.1}$$

commutes for all A, $A' \in \mathcal{A}$ and all $f \in \mathcal{A}(A, A')$. By applying F, we find the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(\eta_A)=\varepsilon^{-1}_{F(A)}"] \arrow[d, "F(f)"'] & FGF(A) \arrow[d, "FGF(f)"] \\
F(A') \arrow[r, "F(\eta_{A'})=\varepsilon^{-1}_{F(A')}"'] & FGF(A')
\end{tikzcd},$$

which commutes by naturality of ε. Since F is faithful, we deduce that the diagram (1.1) also commutes as desired, finishing the proof. $\square$

1.1.2 Equivalences and Adjunctions

Some readers may have wondered why we wrote $\varepsilon\colon FG \cong 1_{\mathcal{B}}$ and $\eta\colon 1_{\mathcal{A}} \cong GF$ in the definition of equivalence of categories (Definition 1.1.3) instead of $\varepsilon\colon FG \cong 1_{\mathcal{B}}$ and $\eta\colon GF \cong 1_{\mathcal{A}}$ since we easily see that it comes down to the same thing. The reason is that an equivalence of categories is a very special case of a more general and important concept in category theory : adjunctions. For instance, Proposition 3.1.3 will give us an example where the concept of an adjunction occurs in practice. Before defining what an adjunction is, we need some preliminary definitions.

Definition 1.1.8. A *universal arrow from an object $B \in \mathcal{B}$ to a functor $F\colon \mathcal{A} \to \mathcal{B}$* is an arrow $u\colon B \to F(A)$ with $A \in \mathcal{A}$ such that, for any $f\colon B \to F(A')$, $A' \in \mathcal{A}$, there exists a unique $h\colon A \to A'$ such that $F(h) \circ u = f$.

$$\begin{tikzcd}
B \arrow[r, "u"] \arrow[dr, "f"'] & F(A) \arrow[d, dashed, "F(h)"] & A \arrow[d, dashed, "h"] \\
& F(A') & A'
\end{tikzcd}$$

Dually, we define a *universal arrow from a functor $F\colon \mathcal{A} \to \mathcal{B}$ to an object $B \in \mathcal{B}$* as an arrow $u\colon F(A) \to B$ with $A \in \mathcal{A}$ such that, for any $f\colon F(A') \to B$, $A' \in \mathcal{A}$, there exists a unique $h\colon A' \to A$ such that $u \circ F(h) = f$.

$$\begin{tikzcd}
F(A) \arrow[r, "u"] & B & A \\
F(A') \arrow[u, dashed, "F(h)"] \arrow[ur, "f"'] & & A' \arrow[u, dashed, "h"']
\end{tikzcd}$$

Now that we have defined universal arrows, here is a proposition that gives us a list of equivalent conditions a pair of functors can fulfil and, when this is the case, we say that these two functors form an *adjunction*.

Proposition 1.1.9. *Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ be functors. Then the following conditions are equivalent:*

(i) for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$, we have a bijection $\phi_{A,B}\colon \mathcal{B}\big(F(A), B\big) \cong \mathcal{A}\big(A, G(B)\big)$ that is natural in A and B,

(ii) there exists a natural transformation $\eta\colon 1_{\mathcal{A}} \to GF$ such that, for each $A \in \mathcal{A}$, $\eta_A\colon A \to GF(A)$ is universal from A to G,

(iii) there exists a natural transformation $\varepsilon\colon FG \to 1_{\mathcal{B}}$ such that, for each $B \in \mathcal{B}$, $\varepsilon_B\colon FG(B) \to B$ is universal from F to B,

(iv) there exist natural transformations $\eta\colon 1_{\mathcal{A}} \to GF$ and $\varepsilon\colon FG \to 1_{\mathcal{B}}$ such that $\varepsilon_{F(A)} \circ F(\eta_A) = 1_{F(A)}$ for all $A \in \mathcal{A}$ and $G(\varepsilon_B) \circ \eta_{G(B)} = 1_{G(B)}$ for all $B \in \mathcal{B}$ (these equalities are called the counit–unit equations*).*

Given a natural bijection as in (i), we can construct the η and ε that verify (ii), (iii), (iv) by setting $\eta_A = \phi_{A,F(A)}(1_{F(A)})$ for $A \in \mathcal{A}$ and $\varepsilon_B = \phi^{-1}_{G(B),B}(1_{G(B)})$ for $B \in \mathcal{B}$. Conversely, given natural transformations $\eta\colon 1_{\mathcal{A}} \to GF$ and $\varepsilon\colon FG \to 1_{\mathcal{B}}$, we can define the natural bijection ϕ by

$$\phi_{A,B}\colon \mathcal{B}\big(F(A), B\big) \cong \mathcal{A}\big(A, G(B)\big)\colon g \mapsto G(g) \circ \eta_A$$

or by

$$\phi^{-1}_{A,B}\colon \mathcal{A}\big(A, G(B)\big) \cong \mathcal{B}\big(F(A), B\big)\colon f \mapsto \varepsilon_B \circ F(f).$$

Moreover, the two constructions above are each other's inverse. Finally, the natural transformations η and ε given in (iv) verify the universal properties given in (ii) and (iii).

Proof. Omitted (see [ML98, Section IV.1]). □

Definition 1.1.10. When two functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ verify one of the equivalent conditions of Proposition 1.1.9, we say that F *is left adjoint to* G and G *is right adjoint to* F, and we write $F \dashv G$. The natural transformations $\eta\colon 1_{\mathcal{A}} \to GF$ and $\varepsilon\colon FG \to 1_{\mathcal{B}}$ are respectively called the *unit* and the *counit of the adjunction.*

Finally, here is the link between equivalences and adjunctions that motivated our choice of writing $\varepsilon\colon FG \cong 1_{\mathcal{B}}$ and $\eta\colon 1_{\mathcal{A}} \cong GF$ in Definition 1.1.3 as mentioned earlier.

Proposition 1.1.11. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an equivalence of categories with pseudo-inverse $G\colon \mathcal{B} \to \mathcal{A}$ via natural isomorphisms $\varepsilon\colon FG \cong 1_{\mathcal{B}}$ and $\eta\colon 1_{\mathcal{A}} \cong GF$. Then F is left adjoint to G with unit η and counit ε. Moreover, we also have an adjunction in the other direction: F is right adjoint to G with unit ε^{-1} and counit η^{-1}.*

Proof. Given our assumptions, all we have left to check is the universality of every component of η and ε.

Let A be an object in $\mathcal{A}$. We must show that, for all $f\colon A \to G(B)$, $B \in \mathcal{B}$, there is a unique $h\colon F(A) \to B$ such that $G(h) \circ \eta_A = f$.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "f"'] & GF(A) \arrow[d, dashed, "G(h)"] & F(A) \arrow[d, dashed, "h"] \\ & G(B) & B \end{tikzcd}$$

As η_A is an isomorphism (by Remark 1.1.2), we can rewrite the equation $G(h) \circ \eta_A = f$ as $G(h) = f \circ \eta_A^{-1}$ and it is now clear that such an arrow $h\colon F(A) \to B$ exists and is unique since G is fully faithful by Proposition 1.1.6 (we easily see that G is also an equivalence of categories).

Dually, the universality of the components of ε is showed in a similar way.

The other adjunction follows from the fact that G is also an equivalence of categories with pseudo-inverse F via $\eta^{-1}\colon GF \cong 1_{\mathcal{B}}$ and $\varepsilon^{-1}\colon 1_{\mathcal{A}} \cong FG$. □

Remark 1.1.12. Proposition 1.1.11 does not tell us that η and ε taken together verify the counit–unit equations $\varepsilon_{F(A)} \circ F(\eta_A) = 1_{F(A)}$ for all $A \in \mathcal{A}$ and $G(\varepsilon_B) \circ \eta_{G(B)} = 1_{G(B)}$ for all $B \in \mathcal{B}$ (however, one can check that the equivalence constructed in Proposition 1.1.7 verify these equations). Indeed, this is not necessarily the case, here is a counterexample for that. (In particular, this also shows that, if we have a unit and a counit of a given adjunction, they do not necessarily verify the counit–unit equations together.)

Let $F = G = 1_{\mathrm{Ab}}$, where Ab denotes the category of abelian groups with group homomorphisms, and consider $\eta = 1_{1_{\mathrm{Ab}}}\colon 1_{\mathrm{Ab}} \to GF = 1_{\mathrm{Ab}}$ and $\varepsilon\colon FG = 1_{\mathrm{Ab}} \to 1_{\mathrm{Ab}}$ given by $\varepsilon_X\colon X \to X\colon x \to -x$ for $X \in \mathrm{Ab}$. We easily see that η and ε are natural isomorphisms but also that they do not verify the counit–unit equations together.

1.2 Some General Properties of Monomorphisms and Epimorphisms

In this section, we will present various small results about monomorphisms and epimorphisms that we will need in later sections. We split this section into five subsections about kernels and cokernels, split epimorphisms, regular epimorphisms and extremal epimorphisms, respectively.

1.2.1 Kernels and Cokernels

When we have a normal monomorphism, by definition, we know that it is the kernel of some arrow but, sometimes, we are interested to know which arrow it is the kernel of. The following lemma gives us a universal answer to this question.

Lemma 1.2.1. *In a pointed category, if we have a normal monomorphism $f\colon X \to Y$ such that its cokernel $q\colon Y \to \operatorname{Coker} f$ exists, then $f = \ker q$.*

Proof. Let $g\colon X \to Z$ such that $f = \ker g$ and $h\colon W \to Y$ such that $q \circ h = 0$. We must show that there is a unique arrow $\varphi\colon W \to X$ such that $f \circ \varphi = h$.

$$\begin{tikzcd}
W \arrow[d, dashed, "\exists!\varphi\,?"'] \arrow[dr, "h"] & & \\
X \arrow[r, "f=\ker g"'] & Y \arrow[r, "q=\operatorname{coker} f"] \arrow[dr, "g"'] & \operatorname{Coker} f \\
& & Z
\end{tikzcd}$$

By the universal property of q, we have an induced arrow $\psi\colon \operatorname{Coker} f \to Z$ such that $\psi \circ q = g$. Moreover, we also have an induced arrow $\varphi\colon W \to X$ such that $f \circ \varphi = h$ since $g \circ h = (\psi \circ q) \circ h = \psi \circ 0 = 0$.

$$\begin{tikzcd}
W \arrow[d, dashed, "\varphi"'] \arrow[dr, "h"] & & \\
X \arrow[r, "f=\ker g"'] & Y \arrow[r, "q=\operatorname{coker} f"] \arrow[dr, "g"'] & \operatorname{Coker} f \arrow[d, dashed, "\psi"] \\
& & Z
\end{tikzcd}$$

Finally, such a factorisation φ is indeed unique since f is monic. $\square$

Now, let us look at some properties of kernels. For that, we will use the following observation that gives us an alternative definition of kernels.

Remark 1.2.2. Until now, we defined the kernel $\ker \alpha$ of an arrow $\alpha\colon A \to B$ by the following universal property : we have $\alpha \circ \ker \alpha = 0$ and, for any arrow $\beta\colon C \to A$ such that $\alpha \circ \beta = 0$, there is a unique arrow $\varphi\colon C \to \operatorname{Ker} \alpha$ such that $\ker \alpha \circ \varphi = \beta$.

$$\begin{tikzcd}
C \arrow[d, dashed, "\exists!\varphi"'] \arrow[dr, "\beta"] & & \\
\operatorname{Ker} \alpha \arrow[r, "\ker \alpha"'] & A \arrow[r, "\alpha"] & B
\end{tikzcd}$$

However, we can also construct this kernel by taking a pullback of α along 0.

$$\begin{tikzcd}
\operatorname{Ker}\alpha = 0 \times_B A \arrow[r, "\ker\alpha"] \arrow[d] & A \arrow[d, "\alpha"] \\
0 \arrow[r] & B
\end{tikzcd}$$

We easily check that these two constructions verify the same universal property.

Dually, we have a similar result to express cokernels in terms of pushouts.

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d] & B \arrow[d, "\operatorname{coker}\alpha"] \\
0 \arrow[r] & \operatorname{Coker}\alpha = 0 +_A B
\end{tikzcd}$$

The first result is about kernels of monomorphisms.

Lemma 1.2.3. *In a pointed category, every monomorphism has a kernel and this kernel is trivial.*

Proof. Let $f\colon X \to Y$ be a monomorphism and let us show that $\kappa\colon 0 \to X$ is a kernel of f. We have immediately $f \circ \kappa = 0$. Given $g\colon Z \to X$ such that $f \circ g = 0$, we must show that g factorises through 0, i.e. $g = 0$ (such a factorisation is automatically unique since 0 is a terminal object).

$$\begin{tikzcd}
Z \arrow[d, dashed, "?"'] \arrow[dr, "g"] & & \\
0 \arrow[r, "\kappa"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

We have $f \circ g = 0_{Z,Y} = f \circ 0_{Z,X}$ so we deduce that $g = 0$ as expected since f is a monomorphism. $\square$

The next result says that, in a certain way, if we take a pullback of an arrow f, it has the "same" kernel as f.

Lemma 1.2.4. *In a pointed category, if we have a pullback*

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

such that p_2 has a kernel $\kappa\colon \operatorname{Ker} p_2 \to X \times_Z Y$, then $p_1 \circ \kappa$ is a kernel of f.

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
\operatorname{Ker} p_2 \arrow[r, "\kappa"] \arrow[d] \arrow[dr, phantom, "(1)"] & X \times_Z Y \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "(2)"] & X \arrow[d, "f"] \\
0 \arrow[r] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

By Remark 1.2.2, (1) is a pullback. Moreover, (2) is also a pullback by assumption so $(1) + (2)$ is again a pullback meaning that $p_1 \circ \kappa \colon \operatorname{Ker} p_2 \to X$ is a kernel of f, again by Remark 1.2.2. ☐

Similarly, if we know that an arrow is a kernel, we can deduce that its image by a pullback is also a kernel.

Lemma 1.2.5. *In a pointed category, every pullback preserves normal monomorphisms. More precisely, if we have a pullback*

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

such that f is the kernel of an arrow $q \colon Z \to W$, then p_2 is a kernel of $q \circ g$.

Proof. Consider the following commutative diagram.

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "(1)"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "(2)"] & Z \arrow[d, "q"] \\
0 \arrow[r] & W
\end{tikzcd}$$

Since f is a kernel of q, (2) is a pullback by Remark 1.2.2. Moreover, (1) is a pullback by assumption so $(1) + (2)$ is a pullback, showing that p_2 is a kernel of $q \circ g$ (again by Remark 1.2.2). ☐

In fact, we will use the dual of this statement, which is the following corollary.

Corollary 1.2.6. *In a pointed category, every pushout preserves normal epimorphisms. More precisely, if we have a pushout*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"'] & X +_Z Y \end{tikzcd}$$

such that f is the cokernel of an arrow $k\colon W \to Z$, then i_2 is a cokernel of $g \circ k$.

Proof. Dual to the previous lemma. $\square$

The following lemma lets us construct pullbacks by taking kernels of a commutative square with a monomorphism.

Lemma 1.2.7. *In a pointed category, if we have a commutative diagram*

$$\begin{tikzcd} K \arrow[r, "\kappa"] \arrow[d, "f"'] \arrow[dr, phantom, "(*)"] & A \arrow[r, "\alpha"] \arrow[d, "g"] & B \arrow[d, tail, "m"] \\ K' \arrow[r, "\kappa'"'] & A' \arrow[r, "\alpha'"'] & B' \end{tikzcd}$$

where κ, κ' are kernels of α, α', respectively, and m is monic, then the square $(*)$ is a pullback.

Proof. Given two arrows $u\colon X \to K'$ and $v\colon X \to K$ such that $\kappa' \circ u = g \circ v$, we must show that we have a unique factorisation $\varphi\colon X \to K$ such that $f \circ \varphi = u$ and $\kappa \circ \varphi = v$.

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!\varphi\,?"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & & \\ & K \arrow[r, "\kappa"] \arrow[d, "f"'] \arrow[dr, phantom, "(*)"] & A \arrow[r, "\alpha"] \arrow[d, "g"] & B \arrow[d, tail, "m"] \\ & K' \arrow[r, "\kappa'"'] & A' \arrow[r, "\alpha'"'] & B' \end{tikzcd}$$

First, we compute

$$m \circ (\alpha \circ v) = (\alpha' \circ g) \circ v = \alpha' \circ (\kappa' \circ u) = 0 \circ u = 0 = m \circ 0$$

so $\alpha \circ v = 0$ since m is monic. Therefore, we obtain a unique factorisation $\varphi\colon X \to K$ such that $\kappa \circ \varphi = v$ since κ is a kernel of α. All we have left to check is the equality $f \circ \varphi = u$. We compute

$$\kappa' \circ (f \circ \varphi) = (g \circ \kappa) \circ \varphi = g \circ v = \kappa' \circ u$$

so we have $f \circ \varphi = u$ as desired since κ' is monic. $\square$

Remark 1.2.8. In the proof of the previous lemma, we did not fully use the assumption that κ' is kernel of α' but only the fact that $\alpha' \circ \kappa' = 0$ and that κ' is monic.

For the end of this subsection on kernels and cokernels, here is a kind of "cancellation property" for cokernels.

Lemma 1.2.9. *In a pointed category, if we have a cokernel $f\colon C \to D$ of an arrow $A \overset{i}{\twoheadrightarrow} B \xrightarrow{m} C$ where i is epic, then f is also a cokernel of m.*

Proof. First, since we have $(f \circ m) \circ i = 0 = 0 \circ i$ and i is epic, we obtain $f \circ m = 0$ as wanted. Now, we must show that, if we have $g\colon C \to X$ such that $g \circ m = 0$, then there is a unique $\varphi\colon D \to X$ such that $\varphi \circ f = g$.

$$\begin{tikzcd} A \arrow[r, two heads, "i"] & B \arrow[r, "m"] & C \arrow[r, "f"] \arrow[dr, "g"'] & D \arrow[d, dashed, "\exists! \varphi ?"] \\ & & & X \end{tikzcd}$$

Since $g \circ (m \circ i) = 0 \circ i = 0$, this follows directly from the fact that f is a cokernel of $m \circ i$. $\square$

1.2.2 Split Epimorphisms

In this subsection, we will present some results are about split epimorphisms : the first one tells us that it is a stronger notion than regular epimorphisms and the second one gives us a necessary and sufficient condition to have an isomorphism.

Lemma 1.2.10. *If we have a split epimorphism $r\colon X \to Y$ with splitting $s\colon Y \to X$, then $r = \mathrm{coeq}(1_X, s \circ r)$. In particular, every split epimorphism is a regular epimorphism.*

Proof. First, we have $r \circ 1_X = r = 1_Y \circ r = r \circ (s \circ r)$ as wanted. Now, given $u\colon X \to Z$ such that $u = u \circ 1_X = u \circ (s \circ r)$, we must show that we have a unique factorisation $\varphi\colon Y \to Z$ such that $\varphi \circ r = u$.

$$\begin{tikzcd} X \arrow[r, shift left, "1_X"] \arrow[r, shift right, "s \circ r"'] & X \arrow[r, shift left, "r"] \arrow[dr, "u"'] & Y \arrow[l, shift left, "s"] \arrow[d, dashed, "\exists! \varphi ?"] \\ & & Z \end{tikzcd}$$

We easily see that $\varphi = u \circ s$ is a valid factorisation since $u = u \circ (s \circ r)$ and such a factorisation is unique as required since r is an epimorphism. $\square$

Lemma 1.2.11. *If an arrow is both a monomorphism and a split epimorphism, then it is an isomorphism. Dually, if an arrow is both a epimorphism and a split monomorphism, then it is an isomorphism.*

Proof. Let $u\colon X \to Y$ be such an arrow and $v\colon Y \to X$ be a splitting of u. We will show that v is the inverse of u. By assumption, we already have $u \circ v = 1_Y$ so we only have to check the identity $v \circ u = 1_X$. We compute

$$u \circ (v \circ u) = 1_Y \circ u = u = u \circ 1_X$$

so $v \circ u = 1_X$ as wanted since u is monic, finishing the proof. $\square$

1.2.3 Regular Epimorphisms

Similarly to the case of normal monomorphisms we saw in Lemma 1.2.1, we are also sometimes interested to know which pair of arrows a regular epimorphism is the co-equaliser of. In this case, a universal answer is given by its kernel pair (see the following definition).

Definition 1.2.12. The *kernel pair* $\big(p_1\colon \mathrm{Eq}(f) \to X, p_2\colon \mathrm{Eq}(f) \to X\big)$ of an arrow $f\colon X \to Y$ is the pullback of f along itself.

$$\begin{tikzcd}
\mathrm{Eq}(f) \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Lemma 1.2.13. *If we have a regular epimorphism $f\colon X \to Y$ such that its kernel pair $\big(p_1\colon \mathrm{Eq}(f) \to X, p_2\colon \mathrm{Eq}(f) \to X\big)$ exists, then $f = \mathrm{coeq}(p_1, p_2)$.*

Proof. Let u, $v\colon W \to X$ be such that $f = \mathrm{coeq}(u, v)$ and $g\colon X \to Z$ be such that $g \circ p_1 = g \circ p_2$. Let us show that there is a unique $\varphi\colon Y \to Z$ such that $\varphi \circ f = g$.

$$\begin{tikzcd}
W \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & & \\
& \mathrm{Eq}(f) \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \arrow[dr, bend left, "g"] & \\
& X \arrow[r, "f"'] \arrow[rr, bend right, "g"'] & Y \arrow[r, dashed, "\exists! \varphi ?"'] & Z
\end{tikzcd}$$

By the universal property of the kernel pair (p_1, p_2), we have a uniquely induced arrow $\psi\colon W \to \mathrm{Eq}(f)$ such that $p_1 \circ \psi = u$ and $p_2 \circ \psi = v$. Then we compute

$$g \circ u = (g \circ p_1) \circ \psi = (g \circ p_2) \circ \psi = g \circ v$$

so there is an arrow $\varphi\colon Y \to Z$ such that $\varphi \circ f = g$ since $f = \mathrm{coeq}(u, v)$.

$$\begin{tikzcd}
W \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & & \\
& \mathrm{Eq}(f) \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \arrow[ddr, bend left, "g"] & \\
& X \arrow[r, "f"'] \arrow[drr, bend right, "g"'] & Y \arrow[dr, dashed, "\varphi"'] & \\
& & & Z
\end{tikzcd}$$

Finally, since f is epic (because it is a coequaliser), our factorisation φ is unique as desired. $\square$

In concrete categories like Set, Grp, $\ldots$, the kernel pair of an arrow $f\colon X \to Y$ is a subobject of the product $X \times X$ equipped with the restrictions of the two projections of this product. A property of this subobject $\mathrm{Eq}(f)$ of $X \times X$ is that it contains the diagonal $\{(x, y) \in X \times X \mid x = y\}$. In other words, the arrow $\Delta\colon X \to \mathrm{Eq}(f)\colon x \mapsto (x, x)$ is well defined and splits both projections of the kernel pair. This result is in fact a general result, as shown in the following lemma. (In Set, Grp, $\ldots$, the existence of such a common splitting Δ follows from the fact that $\mathrm{Eq}(f)$ is a reflexive relation. In fact, the existence of this splitting can be used to define the categorical notion of reflexive relation (see Definitions 4.1.2) so the following lemma tells us that every kernel pair is a reflexive relation.)

Lemma 1.2.14. *The projections p_1, $p_2\colon \mathrm{Eq}(f) \to X$ of a kernel pair are split epimorphisms with a unique common splitting, i.e. there exists a unique arrow $\Delta\colon X \to \mathrm{Eq}(f)$ such that $p_1 \circ \Delta = p_2 \circ \Delta = 1_X$.*

Proof. Let $\big(p_1\colon\ \mathrm{Eq}(f) \to X, p_2\colon\ \mathrm{Eq}(f) \to X\big)$ be the kernel pair of an arrow $f\colon X \to Y$. Applying the universal property of the kernel pair on the diagram below, we obtain a unique splitting $\Delta\colon X \to \mathrm{Eq}(f)$ of p_1 and p_2 as desired.

$$\begin{tikzcd}
X \arrow[dr, dashed, "\exists!\Delta"] \arrow[drr, equal, bend left] \arrow[ddr, equal, bend right] & & \\
& \mathrm{Eq}(f) \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$\square$

Thanks to this result, we will show that the kernel pair of a regular epimorphism forms a pushout involving this regular epimorphism (see Corollary 1.2.16 below for the precise statement). For that, we will first need the following lemma which tells us that, under some assumptions, a common splitting like in the previous lemma gives us a pushout.

Lemma 1.2.15. *Consider a commutative square*

$$\begin{tikzcd}
X \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, two heads, "f"] \\
Y \arrow[r, two heads, "f"'] & Z
\end{tikzcd}$$

where f is a coequaliser of p_1 and p_2. If p_1 and p_2 are split epimorphisms with a common splitting $\Delta\colon Y \to X$, then this square is a pushout.

Proof. Let u, $v\colon Y \to A$ such that $u \circ p_1 = v \circ p_2$. We must show that we have a unique factorisation $\varphi\colon Y \to Z$ such that $\varphi \circ f = u$ and $\varphi \circ f = v$.

$$\begin{tikzcd}
X \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_1"'] & Y \arrow[l, shift left, "\Delta"] \arrow[d, "f"] \arrow[ddr, bend left, "v"] & \\
Y \arrow[r, "f"'] \arrow[u, shift right, "\Delta"'] \arrow[drr, bend right, "u"'] & Z \arrow[dr, dashed, "\exists!\varphi?"] & \\
& & A
\end{tikzcd}$$

Since p_1 and p_2 are both split by Δ, we have $u = u \circ (p_1 \circ \Delta) = v \circ (p_2 \circ \Delta) = v$ so we can apply the universal property of the coequaliser f of p_1 and p_2 to have a uniquely induced arrow $\varphi\colon Y \to Z$ such that $\varphi \circ f = u = v$, showing that we have a pushout. $\square$

Corollary 1.2.16. *Let* $\big(p_1\colon \mathrm{Eq}(f) \to X, p_2\colon \mathrm{Eq}(f) \to X\big)$ *be the kernel pair of a regular epimorphism* $f\colon X \to Y$*. Then the square*

$$\begin{tikzcd} \mathrm{Eq}(f) \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, two heads, "f"] \\ X \arrow[r, two heads, "f"'] & Y \end{tikzcd}$$

is a pushout.

Proof. By Lemmas 1.2.13 and 1.2.14, we can apply the previous result (Lemma 1.2.15). □

The next lemma, which comes from [ML98, Section VI.6], gives us a tool to identify certain coequalisers. It is based on the following definitions.

Definitions 1.2.17.

- A *fork* is a triple (u, v, e) of arrows where u, v are two parallel arrows and e coequalises u and v, i.e. $e \circ u = e \circ v$.

$$\begin{tikzcd} A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[r, "e"] & C \end{tikzcd}$$

- A *split fork* is a fork $(u\colon A \to B, v\colon A \to B, e\colon B \to C)$ such that there exist two arrows $s\colon C \to B$ and $t\colon B \to A$ that satisfy the equalities $e \circ s = 1_C$, $u \circ t = 1_B$ and $v \circ t = s \circ e$ with u, v and e. We say that s and t *split* the fork (u, v, e).

$$\begin{tikzcd} A \arrow[r, shift left=2, "u"] \arrow[r, shift right=2, "v"'] & B \arrow[l, "t" description] \arrow[r, shift left, "e"] & C \arrow[l, shift left, "s"] \end{tikzcd}$$

Lemma 1.2.18. *If* (u, v, e) *is a split fork, then* e *is a coequaliser of* u *and* v*.*

$$\begin{tikzcd} A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[r, "e"] & C \end{tikzcd}$$

Proof. By definition of fork, we already know that e coequalises u and v so we just need to check the universal property of the coequaliser.

Let $f\colon B \to D$ such that $f \circ u = f \circ v$. We must show that there is a unique $\varphi\colon C \to D$ such that $\varphi \circ e = f$.

$$\begin{tikzcd} A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[r, "e"] \arrow[dr, "f"'] & C \arrow[d, dashed, "\exists!\varphi\,?"] \\ & & D \end{tikzcd}$$

Let $s\colon C \to B$ and $t\colon B \to A$ be arrows that split our fork (u, v, e). We set $\varphi = f \circ s$ and, by definition of split fork, we have

$$\varphi \circ e = (f \circ s) \circ e = f \circ (v \circ t) = (f \circ u) \circ t = f \circ 1_B = f$$

as wanted. Moreover, since e is a split epimorphism (split by s), we also have the uniqueness of φ as desired. □

Our last result of this subsection compares regular epimorphisms with again an other kind of epimorphisms, the *strong epimorphisms*.

Lemma 1.2.19. *If $f\colon A \to B$ is a regular epimorphism, then it is* strongly epic*, i.e., if we have a commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

where m is a monomorphism, then there is a unique $d\colon B \to C$ such that $d \circ f = g$ and $m \circ d = h$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[dl, dashed, "\exists! d"'] \\ C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

Proof. Let u, $v\colon X \to A$ be arrows such that f is their coequaliser. Then we have $m \circ (g \circ u) = (h \circ f) \circ u = h \circ (f \circ v) = m \circ (g \circ v)$ so $g \circ u = g \circ v$ since m is monic. Therefore, by the universal property of f, there is a unique $d\colon B \to C$ such that $d \circ f = g$.

$$\begin{tikzcd} X \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[dl, dashed, "\exists! d"'] \\ & C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

In order to finish the proof, we must show that we also have $m \circ d = h$. We compute $(m \circ d) \circ f = m \circ g = h \circ f$ so we obtain $m \circ d = h$ as expected since f is epic. □

1.2.4 Extremal Epimorphisms

In this subsection, we introduce a new kind of epimorphism which is a weaker version of regular epimorphisms but has other interesting properties, as we will see.

Definition 1.2.20. An arrow $f\colon A \to B$ is *extremally epic* when, for any monomorphism $m\colon M \to B$ for which there exists $f'\colon A \to M$ such that $m \circ f' = f$, then m is an isomorphism.

$$\begin{tikzcd} & M \arrow[d, tail, "m"] \\ A \arrow[ur, "f'"] \arrow[r, "f"'] & B \end{tikzcd}$$

First, let us show that this notion is weaker than regular epimorphisms.

Lemma 1.2.21. *Every regular epimorphism is extremally epic.*

Proof. Let $f\colon B \to C$ be the coequaliser of $u\colon A \to B$ and $v\colon A \to B$. Given a factorisation $B \xrightarrow{f'} M \overset{m}{\rightarrowtail} C$ of f where m is monic, we must show that m is an isomorphism. Since $m \circ (f' \circ u) = f \circ u = f \circ v = m \circ (f' \circ v)$ and m is monic, we have $f' \circ u = f' \circ v$ so, by the universal property of the coequaliser f, we obtain a unique arrow $\varphi\colon C \to M$ such that $\varphi \circ f = f'$.

$$\begin{tikzcd} & & M \arrow[d, tail, shift left, "m"] \\ A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[ur, "f'"] \arrow[r, "f"'] & C \arrow[u, dashed, shift left, "\varphi"] \end{tikzcd}$$

Now, we compute $(m \circ \varphi) \circ f = m \circ f' = f$ so $m \circ \varphi = 1_C$ since f is epic. Therefore, we conclude that m is an isomorphism by Lemma 1.2.11 since it is both monic and split epic, finishing the proof. □

However, as we have said earlier, these extremal epimorphisms have some nice properties, like this composition property.

Lemma 1.2.22. *If a composite $A \xrightarrow{f} B \xrightarrow{g} C$ is extremally epic, so is g.*

Proof. Let $m\colon M \to C$ be a monomorphism for which there exists $g'\colon B \to M$ such that $m \circ g' = g$. Then $m \circ (g' \circ f) = g \circ f$ so m is an isomorphism since $g \circ f$ is extremally epic, finishing the proof.

$$\begin{tikzcd} & & M \arrow[d, tail, "m"] \\ A \arrow[urr, bend left, "g' \circ f"] \arrow[r, "f"'] & B \arrow[ur, "g'"] \arrow[r, "g"'] & C \end{tikzcd}$$

□

Since every split epimorphism is extremally epic by Lemmas 1.2.10 and 1.2.21, our last result can be seen as a refinement of Lemma 1.2.11.

Lemma 1.2.23. *If an arrow is both a monomorphism and an extremal epimorphism, then it is an isomorphism.*

Proof. Let $f\colon A \to B$ be monic and extremally epic. Since f is monic, we have the commutative diagram

$$\begin{tikzcd} & A \arrow[d, tail, "f"] \\ A \arrow[ur, equal] \arrow[r, "f"'] & B \end{tikzcd}$$

so we deduce that f is an isomorphism by using the fact that f is extremally epic. $\square$

1.3 Protomodularity

In this section, we will define *protomodularity* in a pointed context and prove some of its properties. We are interested in protomodularity since it has a strong connection with internal actions : in a certain sense, in a pointed protomodular category, the middle object of a split short exact sequence (see Definition 1.3.9) is "generated" by its extremities, which is the first step allowing us to define the concept of *semidirect product* (see Definition 4.2.2) which plays an important role in the theory of internal actions. One important thing about protomodularity is its relation with the *Split Short Five Lemma*, which we will explore in the second subsection of this section. Finally, in the last subsection, we will see a first application of the notions we developed before : we will show that, in a pointed protomodular context, the categories of points and split short exact sequences we will define in this subsection are "the same" in the sense of Section 1.1.

1.3.1 Definition and First Properties

The definition of protomodularity we will use is expressed in terms of *jointly extremally epic* arrows so let us define this notion.

Definition 1.3.1. A pair of arrows $f\colon A \to C$ and $g\colon B \to C$ with same codomain C is *jointly extremally epic* when, for any monomorphism $m\colon M \to C$ for which there exist $f'\colon A \to M$ and $g'\colon B \to M$ such that $m \circ f' = f$ and $m \circ g' = g$, then m is an isomorphism.

$$\begin{tikzcd} & M \arrow[d, tail, "m"] & \\ A \arrow[ur, "f'"] \arrow[r, "f"'] & C & B \arrow[ul, "g'"'] \arrow[l, "g"] \end{tikzcd}$$

Before moving to the definition of protomodularity, here are some small results to show that being jointly extremally epic is a stronger property than being jointly epic.

Lemma 1.3.2. *If the equaliser of a pair of arrows $u, v \colon C \to D$ is an isomorphism, then $u = v$.*

Proof. Let $m \colon M \xrightarrow{\cong} C$ be an equaliser of u and v. Then $u = (u \circ m) \circ m^{-1} = (v \circ m) \circ m^{-1} = v$. □

Lemma 1.3.3. *In a category with equalisers, if $f \colon A \to C$ and $g \colon B \to C$ are jointly extremally epic, then they are jointly epic.*

Proof. Let $u, v \colon C \to D$ be such that $u \circ f = v \circ f$ and $u \circ g = v \circ g$. We must show that $u = v$. First, we take an equaliser $m \colon M \to A$ of u and v, which is a monomorphism. As $u \circ f = v \circ f$ and $u \circ g = v \circ g$, by the universal property of m, there are unique $f' \colon A \to M$ and $g' \colon C \to M$ such that $m \circ f' = f$ and $m \circ g' = g$.

$$\begin{tikzcd}
 & M \arrow[d, tail, "m"] & \\
A \arrow[ur, "f'"] \arrow[r, "f"'] & C \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] & B \arrow[ul, "g'"'] \arrow[l, "g"] \\
 & D &
\end{tikzcd}$$

Since f and g are jointly extremally epic, we deduce that m is an isomorphism and we can conclude by Lemma 1.3.2. □

Since we are talking about jointly epic arrows, let us quickly show a small lemma about them that we will need later.

Lemma 1.3.4. *If we have a jointly epic pair of arrows $f, g \colon A \to B$ and an epimorphism $h \colon B \to C$, then $h \circ f$ and $h \circ g$ are jointly epic.*

Proof. Let $u, v \colon C \to D$ be such that $u \circ (h \circ f) = v \circ (h \circ f)$ and $u \circ (h \circ g) = v \circ (h \circ g)$. Since f and g are jointly epic, we have $u \circ h = v \circ h$ and we can conclude that $u = v$ since h is an epimorphism. □

We can now define the concept of *protomodularity* in a pointed context. Our definition is not the standard one but we chose this definition since it is relatively easy to verify, as we will do in some examples just after. See [JMT02, Section 2.4] to see that this definition is equivalent to the usual ones. (This paper uses coproducts to express the definition we will give but the existence of these coproducts is in fact not necessary. They chose this approach since they go through an adjunction using coproducts (the one we will show in Proposition 3.1.7) and their main goal is semi-abelian categories (see Section 4.1) which have binary coproducts by definition.)

Definition 1.3.5. A pointed category $\mathcal{C}$ is *protomodular* if it has finite limits and, for any split epimorphism $\alpha\colon A \to C$ with splitting β, its kernel $\kappa\colon K \to A$ is jointly extremally epic with β.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"] \end{tikzcd}$$

Examples 1.3.6. First, let us remark that the following examples have indeed finite limits. For Grp, CRng, ..., it suffices to equip the corresponding limits in Set with the suitable structure. For the abelian categories (see [VdL20, Definition 5.1]), it is also known that finite limits exist (we can show that we have pullbacks and a terminal object or that finite products and equalisers exist). For $\mathsf{Set}_*^{\mathrm{op}}$ (the dual of the category of pointed sets), it means that Set_* must have finite colimits, which is the case (we can show that coproducts and coequalisers exist).

- In Grp, consider a split epimorphism with its kernel

  $$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"] \end{tikzcd}$$

  and let us prove that κ and β are jointly extremally epic. Given κ', β' and m as on the following commutative diagram

  $$\begin{tikzcd} & & M \arrow[d, tail, "m"] & \\ 0 \arrow[r] & K \arrow[ur, "\kappa'"] \arrow[r, "\kappa"'] & A \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"] \arrow[ul, "\beta'"'] \end{tikzcd},$$

  we must show that m is an isomorphism. Since m is a monomorphism by assumption, we just need to prove its surjectivity. First, up to isomorphism, we can assume that κ is the inclusion of the canonical kernel $\operatorname{Ker}\alpha = \{a \in A \mid \alpha(a) = 1\}$ of α in A. For an arbitrary $a \in A$, consider $c = \alpha(a) \in C$ and $k = a\beta(c)^{-1} \in A$. As $\alpha(k) = \alpha(a)(\alpha \circ \beta \circ \alpha)(a)^{-1} = \alpha(a)\alpha(a)^{-1} = 1$, k is in fact in K. Finally, using the commutativity of the diagram above and the fact that m is a group homomorphism, we compute

  $$m\Big(\kappa'(k)\beta'(c)\Big) = \kappa(k)\beta(c) = k\beta(c) = a\beta(c)^{-1}\beta(c) = a$$

  and, as $a \in A$ was arbitrary, this shows the surjectivity of m and concludes the proof.

- A proof for CRng is similar to the proof in Grp.

- In an abelian category, since every epimorphism is a cokernel (by definition of abelian category) and each cokernel is a cokernel of its kernel (by the dual of Lemma 1.2.1), if we consider a split epimorphism with its kernel, we have in fact a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd} .$$

Then, this short exact sequence is isomorphic to

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\iota_1"] & K \oplus C \arrow[r, shift left, "\pi_2"] & C \arrow[l, shift left, "\iota_2"] \arrow[r] & 0 \end{tikzcd}$$

where ι_1 and ι_2 are the inclusion of K and C in the coproduct $K \oplus C$, respectively, and π_2 is the projection of the product $K \oplus C$ on C (see [VdL20, Corollary 6.10]). So, in order to conclude that κ and β are jointly extremally epic, we need to show that this is the case for ι_1 and ι_2. Given κ', β' and m as on the following commutative diagram

$$\begin{tikzcd} & & M \arrow[d, tail, "m"] & & \\ 0 \arrow[r] & K \arrow[ur, "\kappa'"] \arrow[r, "\iota_1"'] & K \oplus C \arrow[r, shift left, "\pi_2"] & C \arrow[l, shift left, "\iota_2"] \arrow[r] \arrow[ul, "\beta'"'] & 0 \end{tikzcd} ,$$

we must show that m is an isomorphism. By the universal property of the coproduct, we have a unique $\varphi\colon K \oplus C \to M$ such that $\varphi \circ \iota_1 = \kappa'$ and $\varphi \circ \iota_2 = \beta'$.

$$\begin{tikzcd} & & M \arrow[d, tail, shift left, "m"] & & \\ 0 \arrow[r] & K' \arrow[ur, "\kappa'"] \arrow[r, "\iota_1"] & K \oplus C \arrow[u, dashed, shift left, "\varphi"] \arrow[r, shift left, "\pi_2"] & C \arrow[l, shift left, "\iota_2"] \arrow[r] \arrow[ul, "\beta'"'] & 0 \end{tikzcd}$$

Now, consider the diagram below.

$$\begin{tikzcd} & & K \oplus C & & \\ 0 \arrow[r] & K \arrow[ur, "\iota_1"] \arrow[r, "\iota_1"'] & K \oplus C \arrow[u, dashed, shift right, "m\circ\varphi"] \arrow[u, dashed, shift left, "1_{K\oplus C}"'] & C \arrow[l, "\iota_2"] \arrow[r] \arrow[ul, "\iota_2"'] & 0 \end{tikzcd}$$

We compute $(m \circ \varphi) \circ \iota_1 = m \circ \kappa' = \iota_1$ and $(m \circ \varphi) \circ \iota_2 = m \circ \beta' = \iota_2$ so $m \circ \varphi = 1_{K \oplus C}$ by uniqueness of the universal property of the coproduct. Therefore, by Lemma 1.2.11, we conclude that m is an isomorphism since it is monic by assumption, which finishes the proof. (Let us note that we only used the fact that we had a coproduct to prove that the inclusions are jointly extremally epic, not that we had a biproduct or that we were in an abelian category.)

- For $\mathrm{Set}_*^{\mathrm{op}}$, we will show the dual statement in Set_* (in order to avoid heavy notations, we will omit the base points of our objects when confusion is unlikely):

given a split monomorphism α with splitting β and a cokernel κ as follows

$$\begin{tikzcd} C \arrow[r, shift left, "\alpha"] & A \arrow[l, shift left, "\beta"] \arrow[r, "\kappa"] & K \arrow[r] & 0 \end{tikzcd},$$

we must show that β and κ are jointly extremally monic, i.e. that, given β', κ' and an epimorphism m as in the following commutative diagram

$$\begin{tikzcd} C \arrow[r, shift left, "\alpha"] & A \arrow[l, shift left, "\beta"] \arrow[r, "\kappa"] \arrow[d, two heads, "m"] & K \arrow[r] & 0 \\ & M \arrow[ul, "\beta'"] \arrow[ur, "\kappa'"'] & & \end{tikzcd},$$

m is an isomorphism. First, up to isomorphism, we can assume that K is given by $K = (A/\operatorname{Im}\alpha, [*_A])$, where the quotient by $\operatorname{Im}\alpha$ means that all elements of $\operatorname{Im}\alpha$ are identified in K and $*_A$ is the base point of A, and κ is given by the canonical projection of this quotient. Since m is an epimorphism, it is surjective and we only need to show its injectivity to conclude that it is an isomorphism. Let a, $a' \in A$ such that $m(a) = m(a')$. We have $\kappa(a) = (\kappa' \circ m)(a) = (\kappa' \circ m)(a') = \kappa(a')$ so either a and a' are in $\operatorname{Im}\alpha$ or $a = a'$. In the second case, we have directly what we wanted. In the other case, let c, $c' \in C$ be such that $a = \alpha(c)$ and $a' = \alpha(c')$. Then $c = (\beta \circ \alpha)(c) = (\beta' \circ m)(a) = (\beta' \circ m)(a') = (\beta \circ \alpha)(c') = c'$ so $a = \alpha(c) = \alpha(c') = a'$. Combining these two cases, we deduce that m is injective and thus an isomorphism. This shows that $\mathrm{Set}_*^{\mathrm{op}}$ is protomodular.

Let us now show some results that hold in pointed protomodular categories.

The first one tells us that, in a pointed protomodular category, each split epimorphism is a cokernel of its kernel (which exists since a pointed protomodular category has finite limits by definition).

Lemma 1.3.7. *In a pointed protomodular category, let $\alpha\colon A \to C$ be a split epimorphism with splitting β and let $\kappa\colon K \to A$ be a kernel of α. Then α is a cokernel of κ. In particular, in a pointed protomodular category, every split epimorphism is a normal epimorphism.*

Proof. First, we have immediately $\alpha \circ \kappa = 0$ since κ is a kernel of α.

Let $f\colon A \to B$ be an arrow such that $f \circ \kappa = 0$. We must show that there is a unique $\varphi\colon C \to B$ such that $\varphi \circ \alpha = f$.

$$\begin{tikzcd} K \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] \arrow[dr, "f"'] & C \arrow[l, shift left, "\beta"] \arrow[d, dashed, "\exists!\varphi\,?"] \\ & & B \end{tikzcd}$$

We set $\varphi = f \circ \beta$. Then we have $(\varphi \circ \alpha) \circ \kappa = \varphi \circ 0 = 0 = f \circ \kappa$ and $(\varphi \circ \alpha) \circ \beta = \varphi \circ 1_C = \varphi = f \circ \beta$. Therefore, $\varphi \circ \alpha = f$ since κ and β are jointly epic by Lemma 1.3.3 (combined with the fact that we are in a pointed protomodular category). Moreover, if we have another arrow $\psi\colon C \to B$ such that $\psi \circ \alpha = f = \varphi \circ \alpha$, then $\psi = \varphi$ since α is an epimorphism. □

Example 1.3.8. In Mon, consider the arrow $\alpha\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}\colon (m, n) \mapsto m + n$. It admits a splitting $n \in \mathbb{N} \mapsto (n, 0) \in \mathbb{N} \times \mathbb{N}$ and its kernel is $\operatorname{Ker} \alpha = 0$. However, α is not a cokernel of its kernel since the cokernels of 0 are isomorphisms. In particular, we deduce that Mon is not protomodular.

Let us introduce a new definition which will allow us to reformulate the previous lemma.

Definition 1.3.9. A *split short exact sequence* in a pointed category $\mathcal{C}$ is a triple (κ, α, β) where $\kappa\colon K \to A$, $\alpha\colon A \to C$ and $\beta\colon C \to A$ are arrows in $\mathcal{C}$ such that κ is a kernel of α, α is a cokernel of κ and $\alpha \circ \beta = 1_C$.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

As said just before, we can reformulate Lemma 1.3.7 in terms of split short exact sequences. This corollary is the crucial result that will let us show an equivalence of categories at the end of the section.

Corollary 1.3.10. *In a pointed protomodular category, each split epimorphism gives rise to a split short exact sequence by taking its kernel. More precisely, if $\alpha\colon A \to C$ is a split epimorphism with splitting β and $\kappa\colon K \to A$ is a kernel of α, then*

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

is a split short exact sequence.

Proof. Follows directly from Lemma 1.3.7. □

The next result is a refinement of Lemma 1.3.7 since every split epimorphism is a regular epimorphism (see Lemma 1.2.10).

Proposition 1.3.11. *In a pointed protomodular category, every regular epimorphism is a normal epimorphism.*

Proof. Let $f\colon X \to Y$ be a regular epimorphism and let us consider its kernel pair $\big(p_1\colon \operatorname{Eq}(f) \to X, p_2\colon \operatorname{Eq}(f) \to X\big)$ as well as the kernel $\kappa\colon \operatorname{Ker} p_2 \to \operatorname{Eq}(f)$ of p_2 (let us recall that $\mathcal{C}$ is finitely complete by definition of protomodularity). By Lemma 1.2.14,

p_2 is a split epimorphism (with splitting Δ) so it is a cokernel of its kernel κ by Lemma 1.3.7.

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} p_2 \arrow[r, "\kappa"] & \mathrm{Eq}(f) \arrow[r, shift left, "p_2"] \arrow[d, "p_1"'] & X \arrow[l, shift left, "\Delta"] \arrow[r] \arrow[d, "f"] & 0 \\ & & X \arrow[r, "f"'] & Y & \end{tikzcd}$$

Therefore, by Corollary 1.2.6, since our square is a pushout by Corollary 1.2.16, f is a cokernel of $p_1 \circ \kappa$, finishing the proof. $\square$

As a reminder, as shown in Lemma 1.2.3, every monomorphism has a trivial kernel and the following proposition tells us that the converse is also true in a pointed protomodular context.

Proposition 1.3.12. *In a pointed protomodular category, if an arrow $f\colon X \to Y$ has a trivial kernel, then it is a monomorphism.*

Proof. Consider the kernel pair $\big(p_1\colon \mathrm{Eq}(f) \to X, p_2\colon \mathrm{Eq}(f) \to X\big)$ of f and take a kernel $\kappa\colon \operatorname{Ker} p_2 \to \mathrm{Eq}(f)$ of p_2. By Lemma 1.2.4, we have $\operatorname{Ker} p_2 = \operatorname{Ker} f = 0$. By Lemma 1.2.14, the projections p_1 and p_2 are split epimorphisms with a common splitting $\Delta\colon X \to \mathrm{Eq}(f)$ so, by definition of protomodularity (Definition 1.3.5), κ and Δ are jointly extremally epic. Applying this property on the commutative diagram

$$\begin{tikzcd} & X \arrow[d, tail, "\Delta"] & \\ \operatorname{Ker} p_2 = 0 \arrow[ur, "0"] \arrow[r, "\kappa"'] & \mathrm{Eq}(f) \arrow[r, shift left, "p_2"] & X \arrow[l, shift left, "\Delta"] \arrow[ul, "1_X"'] \end{tikzcd}$$

where Δ is monic since it is split (by p_1 and p_2), we deduce that Δ is an isomorphism and, since it splits both p_1 and p_2, these two arrows p_1 and p_2 are equal and are isomorphisms. Therefore, the square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback since it is isomorphic to the pullback of the kernel pair (p_1, p_2) of f and its universal property expresses precisely that f is monic, finishing the proof. $\square$

The fact that we can recognise monomorphisms by their kernels allows us to show the following result.

Corollary 1.3.13. *In a pointed protomodular category, pullbacks reflect monomorphisms, i.e., if we have a pullback*

$$\begin{tikzcd} P \arrow[r, tail, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

where p_2 is a monomorphism, then f is also a monomorphism.

Proof. By Lemma 1.2.3, the kernel of p_2 is trivial and, by Lemma 1.2.4, $\operatorname{Ker} f = \operatorname{Ker} p_2$.

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r, tail, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

Therefore, the kernel of f is also trivial so f is monic by Proposition 1.3.12. □

1.3.2 The Split Short Five Lemma

We will now show that the Split Short Five Lemma, which is an important result since it allows us to retrieve some important results of abelian categories in a non-additive context, holds in pointed protomodular categories.

Theorem 1.3.14 (Split Short Five Lemma)**.** *In a pointed protomodular category, consider the following commutative diagram where κ, κ' are kernels of α, α', respectively, $\alpha \circ \beta = 1_C$, $\alpha' \circ \beta' = 1_{C'}$ and u, w are isomorphisms.*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, "u", "\cong"'] & B \arrow[r, shift left, "\alpha"] \arrow[d, "v"] & C \arrow[l, shift left, "\beta"] \arrow[d, "w", "\cong"'] \\ 0 \arrow[r] & A' \arrow[r, "\kappa'"] & B' \arrow[r, shift left, "\alpha'"] & C' \arrow[l, shift left, "\beta'"] \end{tikzcd}$$

Then v is also an isomorphism.

Remarks 1.3.15.

- By Lemma 1.3.7, α and α' are cokernels of their kernels κ and κ' so both lines of our diagram are in fact split short exact sequences.

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, "u", "\cong"'] & B \arrow[r, shift left, "\alpha"] \arrow[d, "v"] & C \arrow[l, shift left, "\beta"] \arrow[r] \arrow[d, "w", "\cong"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "\kappa'"] & B' \arrow[r, shift left, "\alpha'"] & C' \arrow[l, shift left, "\beta'"] \arrow[r] & 0
\end{tikzcd}
$$

- The splitting β' is not necessary since we can construct a splitting of α' that makes the diagram commute by setting $\beta' = v \circ \beta \circ w^{-1}$.

Proof (of Theorem 1.3.14). First, let us show that v is a monomorphism. In order to do this, by Lemma 1.3.12, it suffices to show that $\operatorname{Ker} v = 0$. Let us take the kernels of u, v and w. By commutativity of the diagram, we have induced arrows between them and, since u and w are isomorphisms, their kernels are trivial.

$$
\begin{tikzcd}
& 0 \arrow[r, dashed, "\overline{\kappa}"] \arrow[d, tail, "\ker u"] & \operatorname{Ker} v \arrow[r, dashed, shift left, "\overline{\alpha}"] \arrow[d, tail, "\ker v"] & 0 \arrow[l, dashed, shift left, "\overline{\beta}"] \arrow[d, tail, "\ker w"] \\
0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, "u", "\cong"'] & B \arrow[r, shift left, "\alpha"] \arrow[d, "v"] & C \arrow[l, shift left, "\beta"] \arrow[d, "w", "\cong"'] \\
0 \arrow[r] & A' \arrow[r, "\kappa'"] & B' \arrow[r, shift left, "\alpha'"] & C' \arrow[l, shift left, "\beta'"]
\end{tikzcd}
$$

Let us show that $\overline{\kappa}$ is a kernel of $\overline{\alpha}$. Given an arrow $f\colon D \to \operatorname{Ker} v$ such that $\overline{\alpha} \circ f = 0$, we must show that there is a unique arrow $\varphi\colon D \to 0$ such that $\overline{\kappa} \circ \varphi = f$. We have $\alpha \circ (\ker v \circ f) = (\ker w \circ \overline{\alpha}) \circ f = \ker w \circ 0 = 0$ so we have an induced arrow $\psi\colon D \to A$ such that $\kappa \circ \psi = \ker v \circ f$ since κ is a kernel of α. Moreover, since $\kappa' \circ (u \circ \psi) = (v \circ \kappa) \circ \psi = v \circ (\ker v \circ f) = 0$ and κ' is monic (as it is a kernel), we have $u \circ \psi = 0$ so the universal property of $\ker u$ gives us a factorisation $\varphi\colon D \to 0$ such that $\ker u \circ \varphi = \psi$.

$$
\begin{tikzcd}
& D \arrow[d, dashed, "\varphi"] \arrow[dr, "f"] \arrow[dd, dashed, bend right, "\psi"'] & & \\
& 0 \arrow[r, "\overline{\kappa}"] \arrow[d, tail, "\ker u"] & \operatorname{Ker} v \arrow[r, shift left, "\overline{\alpha}"] \arrow[d, tail, "\ker v"] & 0 \arrow[l, shift left, "\overline{\beta}"] \arrow[d, tail, "\ker w"] \\
0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, "u", "\cong"'] & B \arrow[r, shift left, "\alpha"] \arrow[d, "v"] & C \arrow[l, shift left, "\beta"] \arrow[d, "w", "\cong"'] \\
0 \arrow[r] & A' \arrow[r, "\kappa'"] & B' \arrow[r, shift left, "\alpha'"] & C' \arrow[l, shift left, "\beta'"]
\end{tikzcd}
$$

We compute $\ker v \circ (\overline{\kappa} \circ \varphi) = (\kappa \circ \ker u) \circ \varphi = \kappa \circ \psi = \ker v \circ f$ so $\overline{\kappa} \circ \varphi = f$ as wanted since $\ker v$ is monic. Finally, its uniqueness follows from the uniquenesses given by the universal properties of $\ker u$ and κ. (Since the codomain of ϕ is 0, its uniqueness is immediate but let us remark that the reasoning above remains valid if u (or w) is not an isomorphism and, in this case, the codomain of φ is not necessarily 0.)

Since its kernel is trivial, by Lemma 1.3.12, $\overline{\alpha}$ is monic. Moreover, we remark that $\overline{\alpha}$ is a split epimorphism with splitting $\overline{\beta}$, either by checking by hand or by noticing that $\overline{\alpha} \circ \overline{\beta}$ is an arrow from 0 to itself so it must be its identity (the advantage of the first method being that it remains valid if w (or u) is not an isomorphism). Therefore, by Lemma 1.2.11, $\overline{\alpha}$ is an isomorphism so $\operatorname{Ker} v = 0$ as expected.

Now that we know that v is monic, since κ' and β' are jointly extremally epic by protomodularity, the existence of the commutative diagram

$$\begin{tikzcd}
 & B \arrow[d, tail, "v"] & \\
A' \arrow[ur, "\kappa \circ u^{-1}"] \arrow[r, "\kappa'"'] & B' & C' \arrow[ul, "\beta \circ w^{-1}"'] \arrow[l, "\beta'"]
\end{tikzcd}$$

lets us conclude that v is an isomorphism, finishing the proof. □

We also have a similar result where, instead of requiring that the left arrow u is an isomorphism, we just assume that it is an extremal epimorphism and this lets us deduce that the middle arrow v is also extremally epic.

Corollary 1.3.16. *In a pointed protomodular category, consider the following commutative diagram where κ, κ' are kernels of α, α', respectively, $\alpha \circ \beta = 1_C$, $\alpha' \circ \beta' = 1_{C'}$, u is extremally epic and w is an isomorphism.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, two heads, "u"] & B \arrow[r, shift left, "\alpha"] \arrow[d, "v"] & C \arrow[l, shift left, "\beta"] \arrow[d, "w", "\cong"'] \\
0 \arrow[r] & A' \arrow[r, "\kappa'"] & B' \arrow[r, shift left, "\alpha'"] & C' \arrow[l, shift left, "\beta'"]
\end{tikzcd}$$

Then v is also extremally epic.

Proof. We must show that, if we have a factorisation $B \xrightarrow{v'} I \overset{m}{\rightarrowtail} B'$ of v where m is a monomorphism, then m is an isomorphism.

First, we take a kernel $j\colon J \to I$ of $\alpha' \circ m\colon I \to C'$. Since $(\alpha' \circ m) \circ (v' \circ \kappa) = \alpha' \circ v \circ \kappa = (w \circ \alpha) \circ \kappa = 0$ and $\alpha' \circ (m \circ j) = 0$, by the universal properties of $j = \ker(\alpha' \circ m)$ and $\kappa' = \ker \alpha'$, we have induced arrows $u'\colon A \to J$ and $m'\colon J \to A'$ such that $j \circ u' = v' \circ \kappa$ and $\kappa' \circ m' = m \circ j$. Let us remark that, since $\kappa' \circ m' = m \circ j$ is monic by composition,

m' is also monic.

$$
\begin{tikzcd}
0 \arrow[rr] & & A \arrow[rr, "\kappa"] \arrow[dl, dashed, "u'"'] \arrow[dd, two heads, "u"] & & B \arrow[r, shift left, "\alpha"] \arrow[dl, "v'"'] \arrow[dd, "v"] & C \arrow[l, shift left, "\beta"] \arrow[dd, "w", "\cong"'] \\
0 \arrow[r] & J \arrow[rr, "j"] \arrow[dr, dashed, tail, "m'"'] & & I \arrow[dr, tail, "m"] & & \\
0 \arrow[rr] & & A' \arrow[rr, "\kappa'"'] & & B' \arrow[r, shift left, "\alpha'"] & C' \arrow[l, shift left, "\beta'"]
\end{tikzcd}
$$

Finally, since u is extremally epic by assumption, m' is an isomorphism so all we have left to do is to apply the Split Short Five Lemma (Theorem 1.3.14) on the diagram

$$
\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "j"] \arrow[d, "m'", "\cong"'] & I \arrow[r, shift left, "\alpha' \circ m"] \arrow[d, "m"] & C' \arrow[l, shift left, "\widetilde{\beta}"] \arrow[d, equal] \\
0 \arrow[r] & A' \arrow[r, "\kappa'"] & B' \arrow[r, shift left, "\alpha'"] & C' \arrow[l, shift left, "\beta'"]
\end{tikzcd}
$$

where $\widetilde{\beta} = v' \circ \beta \circ w^{-1}$ (we easily check that $(\alpha' \circ m) \circ \widetilde{\beta} = 1_{C'}$ and $m \circ \widetilde{\beta} = \beta' = \beta' \circ 1_{C'}$) to conclude that m is an isomorphism and finish the proof. $\square$

Here is another direct consequence of the Split Short Five Lemma which gives us an easy-to-use tool to identify pullbacks.

Corollary 1.3.17. *If we have a commutative diagram*

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, "u", "\cong"'] & B \arrow[r, shift left, "\alpha"] \arrow[d, "v"] & C \arrow[l, shift left, "\beta"] \arrow[d, "w"] \arrow[r] & 0 \\
0 \arrow[r] & A' \arrow[r, "\kappa'"] & B' \arrow[r, shift left, "\alpha'"] & C' \arrow[l, shift left, "\beta'"] \arrow[r] & 0
\end{tikzcd}
$$

where both lines are split short exact sequences and u is an isomorphism, then the right square with the horizontal arrows pointing to the right is a pullback.

Proof. We will show that the comparison arrow $\varphi \colon B \to P$ between this right square and the pullback $(P, p_1 \colon P \to B', p_2 \colon P \to C)$ of α' and w is an isomorphism, which will prove that the right square itself is a pullback and finish the proof.

$$\begin{tikzcd}
B \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "v"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & C \arrow[d, "w"] \\
& B' \arrow[r, "\alpha'"'] & C'
\end{tikzcd}$$

By Lemma 1.2.4, we have a kernel $\ker p_2 \colon A' \to P$ of p_2 such that $p_1 \circ \ker p_2 = \kappa'$ so we obtain the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, "u", "\cong"'] & B \arrow[r, shift left, "\alpha"] \arrow[d, "\varphi"] & C \arrow[l, shift left, "\beta"] \arrow[d, equal] \\
0 \arrow[r] & A' \arrow[r, "\ker p_2"] & P \arrow[r, shift left, "p_2"] & C \arrow[l, shift left, "\delta"]
\end{tikzcd}$$

where $\delta = \varphi \circ \beta$. Both right squares commutes by construction of φ and δ so we just need to check that δ is a splitting of p_2 and that the left square commutes in order to use the Split Short Five Lemma (Theorem 1.3.14) and conclude that φ is an isomorphism as desired, finishing the proof.

- We compute $p_2 \circ \delta = p_2 \circ (\varphi \circ \beta) = \alpha \circ \beta = 1_C$ as expected.
- Let us recall that p_1 and p_2 are jointly monic as the projections of a pullback so we just need to show that we have $p_1 \circ (\varphi \circ \kappa) = p_1 \circ (\ker p_2 \circ u)$ and $p_2 \circ (\varphi \circ \kappa) = p_2 \circ (\ker p_2 \circ u)$. Both members of the second equation are zero since $p_2 \circ \varphi = \alpha$ and κ is a kernel of α and, for the first one, we compute

  $$p_1 \circ (\varphi \circ \kappa) = v \circ \kappa = \kappa' \circ u = (p_1 \circ \ker p_2) \circ u$$

  as wanted.

$\square$

Let us note that, in the proof of the first corollary, we only need the protomodularity to use the Split Short Five Lemma so we could have stated this result in an arbitrary category where the Split Short Five Lemma holds (and that has a zero object and finite limits). However, as we will show in the next proposition, pointed protomodular categories are exactly the pointed finitely complete categories where the Split Short Five Lemma holds. In fact, they are sometimes defined in these terms : see [JMT02, Section 2.3], for example.

Proposition 1.3.18. *If the Split Short Five Lemma holds in a pointed category with finite limits, i.e. if, for each commutative diagram*

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, "u", "\cong"'] & B \arrow[r, shift left, "\alpha"] \arrow[d, "v"] & C \arrow[l, shift left, "\beta"] \arrow[d, "w", "\cong"'] \\
0 \arrow[r] & A' \arrow[r, "\kappa'"] & B' \arrow[r, shift left, "\alpha'"] & C' \arrow[l, shift left, "\beta'"]
\end{tikzcd}
$$

where κ, κ' are kernels of α, α', respectively, $\alpha \circ \beta = 1_C$, $\alpha' \circ \beta' = 1_{C'}$ and u, w are isomorphisms, v is also an isomorphism, then this category is protomodular in the sense of Definition 1.3.5.

Proof. Given the assumptions of the statement, we just have to check that, for each split epimorphism $\alpha\colon A \to C$ with splitting β, its kernel $\kappa\colon K \to A$ is jointly extremally epic with β. Let $\kappa'\colon K \to A'$, $\beta'\colon C \to A'$ and $m\colon A' \to A$ be such that m is monic, $m \circ \kappa' = \kappa$ and $m \circ \beta' = \beta$. We have to show that m is an isomorphism to finish the proof.

$$
\begin{tikzcd}
& A' \arrow[d, tail, "m"] & \\
K \arrow[ur, "\kappa'"] \arrow[r, "\kappa"'] & A \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"] \arrow[ul, "\beta'"']
\end{tikzcd}
$$

Let us define $\alpha' = \alpha \circ m$. To show that m is an isomorphism, we just need to show that we can apply the Split Short Five Lemma on the following commutative diagram, the only missing assumption being the fact that κ' must be a kernel of α'.

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\kappa'"] \arrow[d, equal] & A' \arrow[r, shift left, "\alpha'"] \arrow[d, tail, "m"] & C \arrow[l, shift left, "\beta'"] \arrow[d, equal] \\
0 \arrow[r] & K \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"]
\end{tikzcd}
$$

Let $f\colon D \to A'$ be an arrow such that $\alpha' \circ f = 0$. By the universal property of κ, since $\alpha \circ (m \circ f) = \alpha' \circ f = 0$, we have a factorisation $\varphi\colon D \to K$ such that $\kappa \circ \varphi = m \circ f$.

$$
\begin{tikzcd}
& D \arrow[r, "f"] \arrow[d, dashed, "\varphi"'] & A' \arrow[d, tail, "m"] \arrow[dr, "\alpha'"] & \\
0 \arrow[r] & K \arrow[ur, "\kappa'"] \arrow[r, "\kappa"'] & A \arrow[r, "\alpha"'] & C
\end{tikzcd}
$$

We have $m \circ (\kappa' \circ \varphi) = \kappa \circ \varphi = m \circ f$ so $\kappa' \circ \varphi = f$ as wanted since m is monic. Finally, the uniqueness of the factorisation φ follows from the one given by the universal property of κ, showing that κ' is a kernel of α' and finishing the proof. $\square$

Remark 1.3.19. Let us note that this proof uses a slightly weaker version of the Split Short Five Lemma : in the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, "u", "\cong"'] & B \arrow[r, shift left, "\alpha"] \arrow[d, "v"] & C \arrow[l, shift left, "\beta"] \arrow[d, "w", "\cong"'] \\ 0 \arrow[r] & A' \arrow[r, "\kappa'"] & B' \arrow[r, shift left, "\alpha'"] & C' \arrow[l, shift left, "\beta'"] \end{tikzcd}$$

of the statement of the Split Short Five Lemma, we can require that v is monic in addition to the other conditions. Therefore, the protomodularity condition can also be expressed using this weaker form of the Split Short Five Lemma. Moreover, if we look back to the proof of the Split Short Five Lemma (Theorem 1.3.14), we remark that the difficult part was to show that v is a monomorphism so we see that the equivalence between this weaker version of the Split Short Five Lemma and the definition of protomodularity we use (Definition 1.3.5) is quite straightforward. (Let us remark again that it is this weaker version that is used in the proof of Corollary 1.3.16.)

1.3.3 Points and Split Short Exact Sequences

As said in the introduction of the section, we will present here the categories of points and split short exact sequences and show that they are equivalent in a pointed protomodular context.

Definition 1.3.20. Let $\mathcal{C}$ be a category and C an object in $\mathcal{C}$. We define the *category of points (or pointed objects) over* C, denoted by Pt_C (or $\mathrm{Pt}_C(\mathcal{C})$ if the underlying category $\mathcal{C}$ is ambiguous) as follows :

- its objects are triples (A, α, β), called *points (or pointed objects) over* C, with $A \in \mathcal{C}$, $\alpha \colon A \to C$ and $\beta \colon C \to A$ such that $\alpha \circ \beta = 1_C$,
- an arrow from (A, α, β) and (B, γ, δ) in Pt_C is an arrow $f \colon A \to B$ in $\mathcal{C}$ such that $\alpha = \gamma \circ f$ and $f \circ \beta = \delta$ (making the following diagram commute in both directions),

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, shift left, "\alpha"] & & B \arrow[dl, shift left, "\gamma"] \\ & C \arrow[ul, shift left, "\beta"] \arrow[ur, shift left, "\delta"] & \end{tikzcd}$$

- the composition and the identities in Pt_C are given by those of $\mathcal{C}$.

Remark 1.3.21. The category $\mathrm{Pt}_C(\mathcal{C})$ we just defined can also be defined using two other important constructions of categories : *slices* and *coslices.*

The *(slice) category of objects over* C, denoted by $(\mathcal{C} \downarrow C)$, is given as follows :

- its objects are are couples (A,α), called *objects (or slices) over* C, with $A\in\mathcal{C}$ and $\alpha\colon A\to C$,
- an arrow from (A,α) to (B,β) in $(\mathcal{C}\downarrow C)$ is an arrow $f\colon A\to B$ in $\mathcal{C}$ such that $\beta\circ f=\alpha$,

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & B \arrow[dl, "\beta"] \\ & C & \end{tikzcd}$$

- the composition and the identities in Pt_C are given by those of $\mathcal{C}$.

The *(coslice) category of objects under* C, denoted by $(C\downarrow\mathcal{C})$, is defined dually. (We have that $(C\downarrow\mathcal{C})$ is isomorphic to $(\mathcal{C}^{\mathrm{op}}\downarrow C)^{\mathrm{op}}$.)

The link with $\mathrm{Pt}_C(\mathcal{C})$ is the following : we can easily check that $\mathrm{Pt}_C(\mathcal{C})$ is isomorphic to the categories $\big(1_C\downarrow(\mathcal{C}\downarrow C)\big)$ and $\big((C\downarrow\mathcal{C})\downarrow 1_C\big)$.

Definition 1.3.22. Let $\mathcal{C}$ be a pointed category and C an object in $\mathcal{C}$. We define the *category of split short exact sequences over* C, denoted by $\mathrm{SSES}(C)$ (or $\mathrm{SSES}_{\mathcal{C}}(C)$ if $\mathcal{C}$ is ambiguous), as follows :

- its objects are *split short exact sequences over* C, i.e. split short exact sequences (see Definition 1.3.9) of the form

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd},$$

- an arrow from $(\kappa\colon K\to A,\alpha\colon A\to C,\beta\colon C\to A)$ to $(\lambda\colon L\to B,\gamma\colon B\to C,\delta\colon C\to B)$ in $\mathrm{SSES}(C)$ is a couple $(f\colon K\to L, g\colon A\to B)$ of arrows in $\mathcal{C}$ such that $g\circ\kappa=\lambda\circ f$, $\alpha=\gamma\circ g$ and $g\circ\beta=\delta$ (making the following diagram commute),

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\kappa"] \arrow[d, "f"] & A \arrow[r, shift left, "\alpha"] \arrow[d, "g"] & C \arrow[l, shift left, "\beta"] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & L \arrow[r, "\lambda"] & B \arrow[r, shift left, "\gamma"] & C \arrow[l, shift left, "\delta"] \arrow[r] & 0 \end{tikzcd} \tag{1.2}$$

- the composition and the identities in $\mathrm{SSES}(C)$ are given by those of $\mathcal{C}$ componentwise.

We will now show that, in a protomodular context, we have an equivalence of categories between points and split short exact sequences. In fact, this equivalence is quite simple : to go from points to split short exact sequences, we simply take a kernel and, conversely, each split short exact sequence gives us a point if we "forget" its "left part".

Proposition 1.3.23. *For a pointed protomodular category $\mathcal{C}$ and $C \in \mathcal{C}$, the constructions above give us an equivalence between the category* Pt_C *of points over C and the category* $\mathrm{SSES}(C)$ *of split short exact sequences over C.*

Proof. We will show that the functor $F\colon \mathrm{SSES}(C) \to \mathrm{Pt}_C$ that "forgets" the "left part" of the sequences is a weak equivalence of categories. More precisely, F sends an object (κ, α, β) to the object $(A = \operatorname{dom} \alpha, \alpha, \beta)$ and an arrow (f, g) to the arrow g. (We easily see that it indeed forms a functor.)

Given $(A, \alpha, \beta) \in \mathrm{Pt}_C$, we choose a kernel $\kappa\colon K \to A$ of α and consider the triple (κ, α, β). By Corollary 1.3.10, (κ, α, β) is an object of $\mathrm{SSES}(C)$ (α is a split epimorphism with splitting β by definition of Pt_C). Finally, $F(\kappa, \alpha, \beta) = (A, \alpha, \beta)$, which shows the essential surjectivity of F.

Now, let us consider an arrow $g\colon F(\kappa, \alpha, \beta) \to F(\lambda, \gamma, \delta)$ with (κ, α, β) and $(\lambda, \gamma, \delta)$ as in the diagram below. To finish the proof, we must show that there is a unique arrow $(f, g')\colon (\kappa, \alpha, \beta) \to (\lambda, \gamma, \delta)$ such that $F(f, g') = g$. Of course, we must set $g' = g$. Moreover, as $\gamma \circ (g \circ \kappa) = \alpha \circ \kappa = 0$, there is a unique $f\colon K \to L$ such that $\lambda \circ f = g \circ \kappa$ by the universal property of the kernel λ.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\kappa"] \arrow[d, dashed, "\exists! f"] & A \arrow[r, shift left, "\alpha"] \arrow[d, "g"] & C \arrow[l, shift left, "\beta"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r, "\lambda"] & B \arrow[r, shift left, "\gamma"] & C \arrow[l, shift left, "\delta"] \arrow[r] & 0
\end{tikzcd}$$

This gives us the unique arrow from (κ, α, β) to $(\lambda, \gamma, \delta)$ that is sent to g. $\square$

Chapter 2

Actions and Semidirect Products of Groups and Commutative Rings

In this chapter, we will look at actions of two algebraic structures, groups and commutative rings, and see their similarities, which will motivate us to define a general notion of actions in the next chapters. The main similarity we will explore here is that their corresponding categories (see the following for the precise definition of these categories) are equivalent to categories of split short exact sequences as defined in Definition 1.3.22.

2.1 Actions and Semidirect Products of Groups

A classical source for the content of this section is [ML63, Section IV.3].

Let us start with the well-known actions of groups. Here is a little reminder to fix our definitions and notations.

Reminder 2.1.1. Let C be a group. A *C-group* is a couple (K, ϕ) where K is a group and $\phi\colon C \to \operatorname{Aut} K$ is a group homomorphism, called *action (of C on K)*. For $c \in C$ and $k \in K$, we write ${}^{c}k \coloneqq \phi(c)(k)$ if the action ϕ is understood.

We will be interested in studying the C-groups as a whole and not individually so we define their category as follows.

Definition 2.1.2. Let C be an object in Grp (i.e. a group). We define the *category of C-groups*, denoted by Grp^C, as follows :

- its objects are C-groups (K, ϕ),
- an arrow from (K, ϕ) and (L, ψ) in Grp^C is an arrow $f\colon K \to L$ in Grp that is *equivariant*, i.e. such that $f({}^{c}k) = {}^{c}f(k)$ for all $c \in C$ and all $k \in K$,
- the composition and the identities in Grp^C are given by those of Grp.

Finally, before showing the equivalence of categories between Grp^C and $\mathrm{SSES}_{\mathrm{Grp}}(C)$ announced in the introduction of the chapter, let us recall what the *semidirect product of groups* is since we will need it.

Reminder 2.1.3. Given a C-group (K, ϕ), we have a group structure on the set $K \times C$ given by $(k, c)(k', c') = \big(k\,{}^c k', cc'\big)$ for (k, c), $(k', c') \in K \times C$. This group is called the *semidirect product of K by C (relative to the action ϕ)* and is denoted by $K \rtimes_\phi C$ (or $K \rtimes C$ if the action ϕ is understood).

We now have all the necessary tools to describe the equivalence between the categories Grp^C and $\mathrm{SSES}_{\mathrm{Grp}}(C)$ (where C is a fixed group). Given a C-group (K, ϕ), we associate it to the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\kappa"] & K \rtimes_\phi C \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

where $\kappa(k) = (k, 1)$ for $k \in K$, $\alpha(k, c) = c$ for $(k, c) \in K \rtimes_\phi C$ and $\beta(c) = (1, c)$ for $c \in C$. Conversely, if we have a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\lambda"] & B \arrow[r, shift left, "\gamma"] & C \arrow[l, shift left, "\delta"] \arrow[r] & 0 \end{tikzcd} ,$$

we can see λ as the inclusion of the canonical kernel $\mathrm{Ker}\,\gamma \leq B$ of γ in B and thus this split short exact sequence induces an action ϕ of C on K given by the formula $\phi(c)(k) = \delta(c)k\delta(c)^{-1}$ for $c \in C$ and $k \in K$.

Proposition 2.1.4. *Given a group C, the constructions above give us an equivalence between the category Grp^C of C-groups and the category $\mathrm{SSES}_{\mathrm{Grp}}(C)$ of split short exact sequences over C.*

Proof. Let us construct a weak equivalence $F\colon \mathrm{Grp}^C \to \mathrm{SSES}_{\mathrm{Grp}}(C)$.

For $(K, \phi) \in \mathrm{Grp}^C$, we define $F(K, \phi)$ as the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\kappa"] & K \rtimes_\phi C \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

where $\kappa(k) = (k, 1)$ for $k \in K$, $\alpha(k, c) = c$ for $(k, c) \in K \rtimes_\phi C$ and $\beta(c) = (1, c)$ for $c \in C$. (We easily see that it is indeed a split short exact sequence.)

Given $f\colon (K, \phi) \to (L, \psi)$ in Grp^C, we construct an arrow (f', g) from $F(K, \phi) = (\kappa, \alpha, \beta)$ to $F(L, \psi) = (\lambda, \gamma, \delta)$ as follows: we set $f' = f$ and define

$$g(k, c) = \big(f(k), c\big) \quad \text{for } (k, c) \in K \rtimes_\phi C. \tag{2.1}$$

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\kappa"] \arrow[d, "f'"] & K \rtimes_\phi C \arrow[r, shift left, "\alpha"] \arrow[d, "g"] & C \arrow[l, shift left, "\beta"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r, "\lambda"] & L \rtimes_\psi C \arrow[r, shift left, "\gamma"] & C \arrow[l, shift left, "\delta"] \arrow[r] & 0
\end{tikzcd}$$

Using the equivariance of f, we have, for all (k, c), $(k', c') \in K \rtimes_\phi C$,

$$\begin{aligned} g\Big((k, c)(k', c')\Big) &= g(k\,{}^c k', cc') = \Big(f(k\,{}^c k'), cc'\Big) = \Big(f(k) f({}^c k'), cc'\Big) \\ &= \Big(f(k)\,{}^c f(k'), cc'\Big) = \Big(f(k), c\Big)\Big(f(k'), c'\Big) = g(k, c) g(k', c'), \end{aligned}$$

showing that g is indeed a group homomorphism. Finally, we compute that

$$(g \circ \kappa)(k) = \Big(f(k), 1\Big) = (\lambda \circ f')(k) \quad \text{for all } k \in K,$$

$$(\gamma \circ g)(k, c) = c = \alpha(k, c) \quad \text{for all } (k, c) \in K \rtimes_\phi C$$

and

$$(g \circ \beta)(c) = \Big(f(1), c\Big) = (1, c) = \delta(c) \quad \text{for all } c \in C$$

which lets us conclude that (f', g) is an arrow from $F(K, \phi)$ to $F(L, \psi)$ in $\mathrm{SSES}_{\mathrm{Grp}}(C)$.

By construction, the fact that F is fully faithful follows almost immediately, we just need to check that, given an arrow $(f, g)\colon F(K, \phi) \to F(L, \psi)$ in $\mathrm{SSES}_{\mathrm{Grp}}(C)$ as below, f is indeed an arrow from (K, ϕ) to (L, ψ) in Grp^C and that $F(f) = (f, g)$.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\kappa"] \arrow[d, "f"] & K \rtimes_\phi C \arrow[r, shift left, "\alpha"] \arrow[d, "g"] & C \arrow[l, shift left, "\beta"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r, "\lambda"] & L \rtimes_\psi C \arrow[r, shift left, "\gamma"] & C \arrow[l, shift left, "\delta"] \arrow[r] & 0
\end{tikzcd}$$

Since $g \circ \kappa = \lambda \circ f$ and $g \circ \beta = \delta$, we see that g can only be given as in (2.1) (because $(k, c) = (k, 1)(1, c)$ for all $(k, c) \in K \rtimes_\phi C$) so we will have $F(f) = (f, g)$ as wanted once we will have shown that f is an arrow in Grp^C. Moreover, this must define a group homomorphism so, for all $k \in K$ and all $c \in C$,

$$g\Big((1, c)(k, 1)\Big) = g({}^c k, c) = \Big(f({}^c k), c\Big)$$

and

$$g(1, c) g(k, 1) = \Big(f(1), c\Big)\Big(f(k), 1\Big) = \Big(1, c\Big)\Big(f(k), 1\Big) = \Big({}^c f(k), c\Big)$$

should be equal. Thus we must have $f({}^c k) = {}^c f(k)$ for all $k \in K$ and all $c \in C$, which means that f is equivariant, i.e. an arrow in Grp^C.

Finally, let us show that, for all object $(\lambda, \gamma, \delta)$ in $\mathrm{SSES}_{\mathrm{Grp}}(C)$ of the form

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\lambda"] & B \arrow[r, shift left, "\gamma"] & C \arrow[l, shift left, "\delta"] \arrow[r] & 0 \end{tikzcd},$$

there exists an object (K, ϕ) in Grp^C such that $F(K, \phi)$ is isomorphic to $(\lambda, \gamma, \delta)$. Remark that an isomorphism in $\mathrm{SSES}_{\mathrm{Grp}}(C)$ is an arrow (u, v) such that u and v are isomorphisms of groups. With that in mind, we have that, by the essential uniqueness of the kernel, $(\lambda, \gamma, \delta)$ is isomorphic to the split short exact sequence $(\iota\colon \operatorname{Ker}\gamma \to B, \gamma, \delta)$ where $\operatorname{Ker}\gamma = \{b \in B \mid \gamma(b) = 1\}$ is the canonical kernel of γ and ι is its inclusion in B. (An isomorphism between these two split short exact sequences is given by $(\varphi, 1_B)$ where $\varphi\colon L \to \operatorname{Ker}\gamma$ is the unique isomorphism such that $\iota \circ \varphi = \lambda$.) We let K be equal to $\operatorname{Ker}\gamma$ and define $\phi\colon C \to \operatorname{Aut} K$ by $\phi(c)(k) = \delta(c)k\delta(c)^{-1}$ for $c \in C$ and $k \in K$. As it is just a conjugation composed with a homomorphism, it is indeed an action of C on K (as $\operatorname{Ker}\gamma \lhd B$, it is indeed well defined). We must now show that we have an isomorphism (f, g) between (ι, γ, δ) and $F(K, \phi) = (\kappa, \alpha, \beta)$ in $\mathrm{SSES}_{\mathrm{Grp}}(C)$ as on the following diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\kappa"] \arrow[d, dashed, "f"] & K \rtimes_\phi C \arrow[r, shift left, "\alpha"] \arrow[d, dashed, "g"] & C \arrow[l, shift left, "\beta"] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \operatorname{Ker}\gamma \arrow[r, "\iota"] & B \arrow[r, shift left, "\gamma"] & C \arrow[l, shift left, "\delta"] \arrow[r] & 0 \end{tikzcd}.$$

As $K = \operatorname{Ker}\gamma$, we let f be equal to 1_K. To finish the proof, we must construct an isomorphism $g\colon K \rtimes_\phi C \to B$ such that

$$g(k, 1) = (g \circ \kappa)(k) = (\iota \circ f)(k) = k \quad \text{for all } k \in K = \operatorname{Ker}\gamma, \tag{2.2}$$

$$g(1, c) = (g \circ \beta)(c) = \delta(c) \quad \text{for all } c \in C \tag{2.3}$$

and

$$(\gamma \circ g)(k, c) = \alpha(k, c) = c \quad \text{for all } (k, c) \in K \rtimes_\phi C. \tag{2.4}$$

As $\{(k, 1)\}_{k\in K} \cup \{(1, c)\}_{c\in C} \subset K \rtimes_\phi C$ is a generating set of $K \rtimes_\phi C$ (for all $(k, c) \in K \rtimes_\phi C$, $(k, c) = (k, 1)(1, c)$), the equations (2.2) and (2.3) uniquely determine our homomorphism g. (Since $\{(k, 1)\}_{k\in K} \cap \{(1, c)\}_{c\in C} = \{1\}$, we have no contradiction in the construction of g and we check that this definition gives us a group homomorphism.) For the last equation (2.4), we compute that, for all $(k, c) \in K \rtimes_\phi C$,

$$\begin{aligned} (\gamma \circ g)(k, c) &= (\gamma \circ g)(k, 1)(\gamma \circ g)(1, c) && \text{since } \gamma \circ g \text{ is a homomorphism} \\ &= \gamma(k)\gamma\big(\delta(c)\big) && \text{by (2.2) and (2.3)} \\ &= c && \text{as } k \in K = \operatorname{Ker}\gamma \text{ and } \gamma \circ \delta = 1_C \end{aligned}$$

as wanted.

For the injectivity, we have

$$\begin{aligned}\operatorname{Ker} g &= \{(k,c) \in K \rtimes_\phi C \mid g(k,c) = 1\} = \{(k,c) \in K \rtimes_\phi C \mid k\delta(c) = 1\} \\ &= \{(k,c) \in K \rtimes_\phi C \mid k = \delta(c)^{-1}\}.\end{aligned}$$

Since K is a subgroup, we must search for which $c \in C$ we have $\delta(c) \in K = \operatorname{Ker}\gamma$. As $\gamma \circ \delta = 1_C$, we have $c = \gamma\big(\delta(c)\big) = 1$ so

$$\operatorname{Ker} g = \{(k,c) \in K \rtimes_\phi C \mid c = 1, k = \delta(c)^{-1}\} = \{(1,1)\}$$

and g is indeed injective.

Finally, given $b \in B$, consider $k = b(\delta \circ \gamma)(b)^{-1} \in B$. We compute

$$\gamma(k) = \gamma(b)(\gamma \circ \delta \circ \gamma)(b)^{-1} = \gamma(b)\gamma(b)^{-1} = 1$$

so we have a $k \in K = \operatorname{Ker}\gamma$ and a $c = \gamma(b) \in C$ such that

$$g(k,c) = k\delta(c) = \Big(b(\delta \circ \gamma)(b)^{-1}\Big)(\delta \circ \gamma)(b) = b,$$

which shows the surjectivity of g and lets us conclude that g is an isomorphism, finishing the proof. $\square$

Corollary 2.1.5. Grp^C *is also equivalent to* $\mathrm{Pt}_C(\mathrm{Grp})$.

Proof. It suffices to combine this result with Proposition 1.3.23 and the fact that we showed the protomodularity of Grp in Examples 1.3.6. $\square$

Remark 2.1.6. Let us consider a split short exact sequence in Grp

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

where A is abelian. As K and C inject themselves in A, they are also abelian so our sequence lies in fact in the full subcategory Ab of Grp. We showed in the proof of the previous proposition that A is isomorphic to the semidirect product $\operatorname{Ker}\alpha \rtimes_\phi C$ where ϕ is given by $\phi(c)(k) = \beta(c)k\beta(c)^{-1}$ for $c \in C$ and $k \in \operatorname{Ker}\alpha$ (the product $\beta(c)k\beta(c)^{-1}$ being made in A). As A is abelian, ϕ is a trivial action (i.e. $\phi(c) = 1_{\operatorname{Ker}\alpha}$ for all $c \in C$) so A is isomorphic to the direct product $\operatorname{Ker}\alpha \times C \cong K \times C$. In particular, all semidirect products in Ab are trivial (i.e. isomorphic to a direct product). Moreover, as Ab is an abelian category, this product $K \times C$ can be completed to a biproduct. Therefore, our short exact sequence is also split on the left, i.e. κ is a split monomorphism (a splitting of κ can be given by taking a cokernel of β)(see [VdL20, Corollary 6.10]).

2.2 Actions and Semidirect Products of Commutative Rings

Our second example concerns actions of commutative rings. First, let us recall that CRng is the category of commutative rings (not necessarily with a unit) with ring homomorphisms.

The definition of action of commutative rings uses the notion of *tensor product of commutative rings* so here is its definition.

Definition 2.2.1. For A, $B \in$ CRng, we define their *tensor product* $A \otimes B$ as the tensor product $A \otimes_{\mathbb{Z}} B$ of $\mathbb{Z}$-modules and we turn it into a commutative ring by defining its multiplication by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ on the elementary tensors and extending it by bilinearity.

We can now define actions of commutative rings and their category.

Definition 2.2.2. Let R be a commutative ring. An *R-ring* is a couple (A, ϕ) where A is a commutative ring and $\phi\colon R \otimes A \to A$ is a ring homomorphism, called *action (of R on A)*, such that, for all r, $s \in R$ and all a, $a' \in A$, $\phi(rs \otimes a) = \phi\big(r \otimes \phi(s \otimes a)\big)$ and $\phi(r \otimes aa') = \phi(r \otimes a)a' = a\phi(r \otimes a')$. For $r \in R$ and $a \in A$, we write $r \cdot a \coloneqq \phi(r \otimes a)$ if the action ϕ is understood.

Definition 2.2.3. Let R be an object in CRng (i.e. a commutative ring). We define the *category of R-rings*, denoted by CRng^R, as follows:

- its objects are R-rings (A, ϕ),
- an arrow from (A, ϕ) and (B, ψ) in CRng^R is an arrow $f\colon A \to B$ in CRng that is *equivariant*, i.e. such that $f(r \cdot a) = r \cdot f(a)$ for all $r \in R$ and all $a \in A$,
- the composition and the identities in CRng^R are given by those of CRng.

Before we go to the equivalence between CRng^R and $\mathrm{SSES}_{\mathrm{CRng}}(R)$, let us see a particular example of R-ring.

Example 2.2.4. If A is an abelian ring, i.e. if $aa' = 0$ for all a, $a' \in A$ (as its multiplication is trivial, A is essentially an abelian group), then a structure of R-ring on A is the same as a structure of R-module: the linearity of the action corresponds to the distributivity of the scalar multiplication, we have $rs \cdot a = r \cdot (s \cdot a)$ by assumption for both structures and $r \cdot aa' = (r \cdot a)a' = a(r \cdot a')$ is always verified since the multiplication in A is trivial and the action is linear (all three products of these equalities are 0). Moreover, we easily see that an arrow between two abelian R-rings (i.e. two R-rings (A, ϕ), (B, ψ) with A and B being abelian rings) is equivariant if and only if it is a linear map between these two R-rings seen as R-modules. More precisely, using these facts, we can show that ${}_R\mathrm{Mod}$, the category of R-modules with linear maps, is isomorphic to the full subcategory of CRng^R whose objects are abelian rings. (It is an isomorphism and not simply an equivalence because every abelian group admits a unique trivial multiplication.) (As mentioned in the introduction of the thesis, this example gives us an indication of how "external" actions can sometimes be seen as "internal" ones.)

Similarly to Grp, we will need the *semidirect product* to construct our equivalence. The following proposition gives us a commutative ring structure on the abelian group $R \oplus A$ and it is this commutative ring that we will define as the *semidirect product*.

Proposition 2.2.5. *Given an R-ring (A, ϕ), the formula $(r, a)(s, b) = (rs, ab+r\cdot b+s\cdot a)$ defines a multiplication on the abelian group $R\oplus A$ that turns it into a commutative ring.*

Proof. Using the commutativity of R and A, we easily see that this formula defines a commutative operation.

The distributivity of this multiplication over the addition follows from the fact that the action ϕ is bilinear. Indeed, using this property, we compute, for (r, a), (s, b), $(t, c) \in R \oplus A$,

$$\begin{aligned}
(r,a)\big((s,b)+(t,c)\big) &= (r,a)(s+t, b+c) = \big(r(s+t), a(b+c) + r\cdot(b+c) + (s+t)\cdot a\big) \\
&= \big(rs+rt, (ab+ac) + (r\cdot b + r\cdot c) + (s\cdot a + t\cdot a)\big) \\
&= (rs, ab + r\cdot b + s\cdot a) + (rt, ac + r\cdot c + t\cdot a) \\
&= (r,a)(s,b) + (r,a)(t,c).
\end{aligned}$$

By commutativity, we also have the right distributivity.

For the associativity, we compute, for (r, a), (s, b), $(t, c) \in R \oplus A$,

$$\begin{aligned}
\big((r,a)(s,b)\big)(t,c) &= (rs, ab + r\cdot b + s\cdot a)(t,c) \\
&= \big(rst, (ab + r\cdot b + s\cdot a)c + rs\cdot c + t\cdot(ab + r\cdot b + s\cdot a)\big) \\
&= \big(rst, abc + (r\cdot b)c + (s\cdot a)c + rs\cdot c + t\cdot ab + t\cdot(r\cdot b) + t\cdot(s\cdot a)\big)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
(r,a)\big((s,b)(t,c)\big) &= (r,a)(st, bc + s\cdot c + t\cdot b) \\
&= \big(rst, abc + a(s\cdot c) + a(t\cdot b) + r\cdot bc + r\cdot(s\cdot c) + r\cdot(t\cdot b) + st\cdot a\big).
\end{aligned}$$

By rearranging the terms and using the fact that the action verifies $rs\cdot a = r\cdot(s\cdot a)$ and $r\cdot aa' = (r\cdot a)a' = a(r\cdot a')$ for all r, $s \in R$ and all a, $a' \in A$, we see that these two expressions are equal as wanted. □

Definition 2.2.6. The commutative ring defined in the previous proposition is called the *semidirect product of A by R (relative to the action ϕ)* and is denoted by $R \ltimes_\phi A$ (or $R \ltimes A$ if the action ϕ is understood).

Finally, let us describe the equivalence of categories between CRng^R and $\mathrm{SSES}_{\mathrm{CRng}}(R)$ before showing it in the next proposition. To a R-ring (A, ϕ), we associate the split short exact sequence

$$
\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\kappa"] & R \ltimes_\phi A \arrow[r, shift left, "\alpha"] & R \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}
$$

where $\kappa(a) = (0, a)$ for $a \in A$, $\alpha(r, a) = r$ for $(r, a) \in R \ltimes_\phi A$ and $\beta(r) = (r, 0)$ for $r \in R$. Conversely, given a split short exact sequence

$$
\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\lambda"] & C \arrow[r, shift left, "\gamma"] & R \arrow[l, shift left, "\delta"] \arrow[r] & 0 \end{tikzcd} ,
$$

we can again see λ as the inclusion of the canonical kernel of γ in C and it induces an action ϕ of R on A by the formula $\phi(r \otimes a) = \delta(r)a$.

Proposition 2.2.7. *Given a commutative ring R, the constructions above give us an equivalence between the category CRng^R of R-rings and the category $\mathrm{SSES}_{\mathrm{CRng}}(R)$ of split short exact sequences over R.*

Proof. Let us construct a weak equivalence $F \colon \mathrm{CRng}^R \to \mathrm{SSES}_{\mathrm{CRng}}(R)$.

For $(A, \phi) \in \mathrm{CRng}^R$, we define $F(A, \phi)$ as the split short exact sequence

$$
\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\kappa"] & R \ltimes_\phi A \arrow[r, shift left, "\alpha"] & R \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}
$$

where $\kappa(a) = (0, a)$ for $a \in A$, $\alpha(r, a) = r$ for $(r, a) \in R \ltimes_\phi A$ and $\beta(r) = (r, 0)$ for $r \in R$. (We easily see that it is indeed a split short exact sequence.)

Given an arrow $f \colon (A, \phi) \to (B, \psi)$ in CRng^R, we must construct an arrow (f', g) from $F(A, \phi) = (\kappa, \alpha, \beta)$ to $F(B, \psi) = (\lambda, \gamma, \delta)$.

$$
\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, "f'"] & R \ltimes_\phi A \arrow[r, shift left, "\alpha"] \arrow[d, "g"] & R \arrow[l, shift left, "\beta"] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, "\lambda"] & R \ltimes_\psi B \arrow[r, shift left, "\gamma"] & R \arrow[l, shift left, "\delta"] \arrow[r] & 0 \end{tikzcd}
$$

We set $f' = f$ and define $g(r, a) = \big(r, f(a)\big)$ for $(r, a) \in R \ltimes_\phi A$. The only thing that is not clear is the compatibility of g with the multiplication. We compute, for (r, a),

$(r', a') \in R \ltimes_\phi A$,

$$\begin{aligned} g\big((r,a)(r',a')\big) &= g(rr', aa' + r \cdot a' + r' \cdot a) = \big(rr', f(a)f(a') + r \cdot f(a') + r' \cdot f(a)\big) \\ &= \big(r, f(a)\big)\big(r', f(a')\big) = g(r,a)g(r',a'), \end{aligned}$$

as wanted.

We easily see that this construction gives us a functor F and we need to check that it is fully faithful and essentially surjective to conclude that CRng^R and $\mathrm{SSES}_{\mathrm{CRng}}(R)$ are equivalent.

Given an arrow (f, g) from $F(A, \phi) = (\kappa, \alpha, \beta)$ to $F(B, \psi) = (\lambda, \gamma, \delta)$, let us show that f is the only arrow from (A, ϕ) to (B, ψ) such that $F(f) = (f, g)$. The uniqueness of f is clear by construction of F so we just need to check that f is an arrow in CRng^R, i.e. it is equivariant, and that $F(f) = (f, g)$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, "f"] & R \ltimes_\phi A \arrow[r, shift left, "\alpha"] \arrow[d, "g"] & R \arrow[l, shift left, "\beta"] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, "\lambda"] & R \ltimes_\psi B \arrow[r, shift left, "\gamma"] & R \arrow[l, shift left, "\delta"] \arrow[r] & 0 \end{tikzcd}$$

For all $(r, a) \in R \ltimes_\phi A$, we have

$$\begin{aligned} g(r,a) &= g(0,a) + g(r,0) = (g \circ \kappa)(a) + (g \circ \beta)(r) = (\lambda \circ f)(a) + \delta(r) \\ &= \big(0, f(a)\big) + (r, 0) = \big(r, f(a)\big) \end{aligned}$$

so, if we show that f is indeed an arrow in CRng^R, we will directly have that $F(f) = (f, g)$ as wanted. As g is a ring homomorphism, for any $r \in R$ and $a \in A$, we have that

$$g\big((r,0)(0,a)\big) = g(0, r \cdot a) = \big(0, f(r \cdot a)\big)$$

and

$$g(r,0)g(0,a) = (r,0)\big(0, f(a)\big) = \big(0, r \cdot f(a)\big)$$

are equal, which shows that f must be equivariant, i.e. an arrow in CRng^R.

To finish the proof, we need to show that, given a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "\lambda"] & C \arrow[r, shift left, "\gamma"] & R \arrow[l, shift left, "\delta"] \arrow[r] & 0 \end{tikzcd}$$

in CRng, there exists $(A, \phi) \in \mathrm{CRng}^R$ such that $F(A, \phi)$ and $(\lambda, \gamma, \delta)$ are isomorphic. First, we can assume that λ is given by the inclusion of the canonical kernel $\operatorname{Ker}\gamma$ of γ in C and we set $A = B = \operatorname{Ker}\gamma$. We define $\phi \colon R \otimes A \to A$ by $r \cdot a = \phi(r \otimes a) = \delta(r)a$, this product being made in C. We need to check that it is well defined, i.e. $\delta(r)a$ belongs to $A = \operatorname{Ker}\gamma$, which is indeed the case since a is in $A = \operatorname{Ker}\gamma$. We also remark that ϕ is bilinear and compatible with the multiplication (so it is a homomorphism from $R \otimes A$ to A) and verifies $rs \cdot a = r \cdot (s \cdot a)$ and $r \cdot aa' = (r \cdot a)a' = a(r \cdot a')$ for all r, $s \in R$ and all a, $a' \in A$ so it is an action as wanted. Now, to show the essential surjectivity of F, we must construct an isomorphism (f, g) as in the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\kappa"] \arrow[d, dashed, "f"] & R \ltimes_\phi A \arrow[r, shift left, "\alpha"] \arrow[d, dashed, "g"] & R \arrow[l, shift left, "\beta"] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, "\lambda"] & C \arrow[r, shift left, "\gamma"] & R \arrow[l, shift left, "\delta"] \arrow[r] & 0 \end{tikzcd},$$

where κ, α and β are given by $\kappa(a) = (0, a)$, $\alpha(r, a) = r$ and $\beta(r) = (r, 0)$ as above. For f, as we have set $A = B$, we take $f = 1_A$. For g, we define $g(r, a) = \delta(r) + a$ for $(r, a) \in R \ltimes_\phi A$ and we easily see that it is a group homomorphism that makes our diagram commutes so we just need to show that it is also compatible with the multiplication and bijective to finish the proof (like in $\mathrm{SSES}_{\mathrm{Grp}}(C)$, an arrow in $\mathrm{SSES}_{\mathrm{CRng}}(R)$ is an isomorphism if and only if its two components are isomorphisms).

For (r, a), $(r', a') \in R \ltimes_\phi A$, we compute

$$\begin{aligned} g\big((r,a)(r',a')\big) &= g\big(rr', aa' + r \cdot a' + r' \cdot a\big) = g\big(rr', aa' + \delta(r)a' + \delta(r')a\big) \\ &= \delta(rr') + \big(aa' + \delta(r)a' + \delta(r')a\big) \end{aligned}$$

and

$$g(r,a)g(r',a') = \big(\delta(r) + a\big)\big(\delta(r') + a'\big) = \delta(r)\delta(r') + \delta(r)a' + a\delta(r') + aa'$$

so $g\big((r,a)(r',a')\big) = g(r,a)g(r',a')$ as wanted.

For the injectivity, we have $\operatorname{Ker} g = \{(r, a) \in R \ltimes_\phi A \mid \delta(r) + a = 0\}$ so we need to see for which $r \in R$ we have $-\delta(r) \in A = \operatorname{Ker}\gamma$, i.e. $0 = \gamma\big(-\delta(r)\big) = -(\gamma \circ \delta)(r) = -r$. Therefore, we have $\operatorname{Ker} g = \Big\{\big(r, -\delta(r)\big) \in R \ltimes_\phi A \;\Big|\; r = 0\Big\} = 0$, which shows that g is injective.

Finally, given $c \in C$, we set $a = c - (\delta \circ \gamma)(c) \in C$. As $\gamma(a) = \gamma(c) - (\gamma \circ \delta \circ \gamma)(c) = \gamma(c) - \gamma(c) = 0$, a is in fact in $A = \operatorname{Ker}\gamma$. Thus we have $\big(\gamma(c), a\big) \in R \ltimes_\phi A$ that is such that $g\big(\gamma(c), a\big) = \delta\big(\gamma(c)\big) + a = c$, which shows that g is surjective and finishes the proof. $\square$

Corollary 2.2.8. CRng^R *is also equivalent to* $\mathrm{Pt}_R(\mathrm{CRng})$.

Remark 2.2.9. Consider a split short exact sequence in CRng

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\kappa"] & B \arrow[r, shift left, "\alpha"] & R \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

where B is an abelian ring. Then we see that A and R are also abelian as they inject themselves in B. As seen in the previous proof, B is isomorphic to the semidirect product $R \ltimes A$ relative to the action given by $r \cdot a = \beta(r)a$ (where this multiplication is made in B and A is seen as a subring of B). However, since B is abelian, this product is equal to 0 for all $r \in R$ and all $a \in A$ so this action is trivial and the semidirect $R \ltimes A$ is just the direct product $R \times A$. Moreover, as all our objects are abelian, we are in a full subcategory of CRng isomorphic to ${}_R\text{Mod}$ as mentioned in Example 2.2.4. As ${}_R\text{Mod}$ is abelian, the product $R \times A$ can be completed to a biproduct and our short exact sequence is also split on the left.

Chapter 3

The Monad $B\flat(-)$

This chapter is dedicated to the monad $B\flat(-)$ which is the main tool we will use to define the general notion of action in the next chapter. In the first section, we will introduce the theory of monads and define the bifunctor $\flat$ and the last two sections will illustrate these definitions in the categories we have already covered in the previous chapter, Grp and CRng.

3.1 Monads, Algebras for a Monad and the Bifunctor $\flat$

This section is split in two parts, the first one on monads and their algebras and the second one on the bifunctor $\flat$. Theses two subsections will follow a similar structure : we will first define the main concepts then show some of their properties, the main result of the section being the fact that the functor $B\flat(-)$ (i.e. the bifunctor $\flat$ where we have fixed the first component) is a monad.

3.1.1 Monads and their Algebras

The definitions and results of this subsection can also be found in [ML98, Chapter VI] (more precisely in Sections 1 and 2).

A *monad* is a special kind of endofunctor which is interesting since it allows us to construct new categories, their *Eilenberg–Moore categories.* It is exactly this construction applied to the functor $B\flat(-)$ we will use to define the general notion of action in the next chapter.

Definition 3.1.1. A *monad* on a category $\mathcal{C}$ is a triple (T, η, μ) where $T\colon \mathcal{C} \to \mathcal{C}$ is an endofunctor (i.e. a functor from a category to itself) and $\eta\colon 1_{\mathcal{C}} \to T$, $\mu\colon T^2 \to T$ are two natural transformations such that, for all $X \in \mathcal{C}$, $\mu_X \circ T(\mu_X) = \mu_X \circ \mu_{T(X)}$ and $\mu_X \circ T(\eta_X) = \mu_X \circ \eta_{T(X)} = 1_{T(X)}$.

$$\begin{tikzcd} T^3(X) \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{T(X)}"'] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"'] & T(X) \end{tikzcd} \qquad \begin{tikzcd} T(X) \arrow[r, "T(\eta_X)"] \arrow[d, "\eta_{T(X)}"'] \arrow[dr, equal] & T^2(X) \arrow[d, "\mu_X"] \\ T^2(X) \arrow[r, "\mu_X"'] & T(X) \end{tikzcd}$$

For simplicity, we will often abuse notation and write *the monad* T instead of (T, η, μ).

The name *monad* comes from the fact that the definition of a monad is similar to the one of a monoid. Therefore, by analogy with monoids, the natural transformations μ and η are respectively called *multiplication* and *unit*. (More precisely, we can define monads using the categorical notion of monoid in a certain category of functors.)

Definitions 3.1.2. Let (T, η, μ) be a monad on a category $\mathcal{C}$.

- A *T-algebra* is a couple (X, ξ) where X is an object of $\mathcal{C}$ and $\xi \colon T(X) \to X$ is an arrow in $\mathcal{C}$ such that $\xi \circ T(\xi) = \xi \circ \mu_X$ and $\xi \circ \eta_X = 1_X$.

$$\begin{tikzcd} T^2(X) \arrow[r, "T(\xi)"] \arrow[d, "\mu_X"'] & T(X) \arrow[d, "\xi"] \\ T(X) \arrow[r, "\xi"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & T(X) \arrow[d, "\xi"] \\ & X \end{tikzcd}$$

- A *morphism* $f \colon (X, \xi) \to (Y, \zeta)$ *of T-algebras* is an arrow $f \colon X \to Y$ in $\mathcal{C}$ such that $\zeta \circ T(f) = f \circ \xi$.

$$\begin{tikzcd} T(X) \arrow[r, "T(f)"] \arrow[d, "\xi"'] & T(Y) \arrow[d, "\zeta"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

- T-algebras with their morphisms form a category called the *Eilenberg–Moore category of T* and we denote it by $\mathcal{C}^T$. (Composition and identities are given by those of $\mathcal{C}$, by functoriality of T.)

A classical way to construct monads is the following: any adjunction gives rise to a monad as we will show in the following proposition. (Conversely, every monad gives rise to adjunctions that determine it: see [ML98, Sections VI.2 and 5].)

Proposition 3.1.3. *Any adjunction $F \dashv G$, $F \colon \mathcal{C} \to \mathcal{D}$, $G \colon \mathcal{D} \to \mathcal{C}$, with unit $\eta \colon 1_{\mathcal{C}} \to GF$ and counit $\varepsilon \colon FG \to 1_{\mathcal{D}}$ gives rise to a monad (GF, η, μ) where $\mu \colon GFGF \to GF$ is given by $\mu_X = G(\varepsilon_{F(X)})$ for $X \in \mathcal{C}$.*

Proof. The fact that GF is an endofunctor and η is a natural transformation is clear. Let us show that also μ is natural. For $f\colon X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd}
GFGF(X) \arrow[r, "\mu_X = G(\varepsilon_{F(X)})"] \arrow[d, "GFGF(f)"'] & GF(X) \arrow[d, "GF(f)"] \\
GFGF(Y) \arrow[r, "\mu_Y = G(\varepsilon_{F(Y)})"'] & GF(Y)
\end{tikzcd}$$

commutes as wanted by naturality of ε and functoriality of G.

Let X be an object in $\mathcal{C}$. First, we remark that

$$\begin{tikzcd}
FGFGF(X) \arrow[r, "\varepsilon_{FGF(X)}"] \arrow[d, "FG(\varepsilon_{F(X)})"'] & FGF(X) \arrow[d, "\varepsilon_{F(X)}"] \\
FGF(X) \arrow[r, "\varepsilon_{F(X)}"'] & F(X)
\end{tikzcd}$$

commutes by naturality of ε. Using this, we compute

$$\begin{aligned}
\mu_X \circ GF(\mu_X) = G(\varepsilon_{F(X)}) \circ GFG(\varepsilon_{F(X)}) &= G\Big(\varepsilon_{F(X)} \circ FG(\varepsilon_{F(X)})\Big) \\
&= G\Big(\varepsilon_{F(X)} \circ \varepsilon_{FGF(X)}\Big) = G(\varepsilon_{F(X)}) \circ G(\varepsilon_{FGF(X)}) = \mu_X \circ \mu_{GF(X)}.
\end{aligned}$$

Finally, by the counit–unit equations (see (iv) of Proposition 1.1.9), we also have

$$\mu_X \circ GF(\eta_X) = G(\varepsilon_{F(X)}) \circ GF(\eta_X) = G\Big(\varepsilon_{F(X)} \circ F(\eta_X)\Big) = G(1_{F(X)}) = 1_{GF(X)}$$

and

$$\mu_X \circ \eta_{GF(X)} = G(\varepsilon_{F(X)}) \circ \eta_{GF(X)} = 1_{GF(X)},$$

which finishes the proof. □

3.1.2 The Bifunctor $\flat$

The construction of the bifunctor $\flat$ involves coproducts so let us recall some notations and computation rules about them.

Notations 3.1.4. Let X, Y, Z, A, B, C and D be objects in a category $\mathcal{C}$ such that the coproducts $(X + Y, \iota_1, \iota_2)$, $(A + B, i_1, i_2)$ and $(C + D, j_1, j_2)$ exist.

- Given $f\colon X \to Z$ and $g\colon Y \to Z$, the unique arrow induced by f and g by the universal property of $X+Y$ is denoted by $\langle f, g\rangle$.

$$\begin{tikzcd}
X \arrow[r, "\iota_1"] \arrow[dr, "f"'] & X+Y \arrow[d, dashed, "\exists!\langle f,g\rangle"] & Y \arrow[l, "\iota_2"'] \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

- Given $u\colon A \to C$ and $v\colon B \to D$, the unique arrow induced by $j_1 \circ u$ and $j_2 \circ v$ by the universal property of $A+B$ is denoted by $u+v$ (i.e. $u+v = \langle j_1 \circ u, j_2 \circ v\rangle$).

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[d, "u"'] & A+B \arrow[dd, dashed, "\exists! u+v"] & B \arrow[l, "i_2"'] \arrow[d, "v"] \\
C \arrow[dr, "j_1"'] & & D \arrow[dl, "j_2"] \\
& C+D &
\end{tikzcd}$$

Remark 3.1.5. Using the uniqueness of the universal property of the coproduct, we have the following computation rules (whenever they make sense) :

$$h \circ \langle f, g\rangle = \langle h \circ f, h \circ g\rangle,$$
$$\langle f, g\rangle \circ (u+v) = \langle f \circ u, g \circ v\rangle,$$
$$(s+t) \circ (u+v) = (s \circ u) + (t \circ v).$$

Here is the definition of the bifunctor $\flat$ and the functor $B\flat(-)$. The construction of the functor $B\flat(-)$ and its structure of monad were introduced in [BJ98] (it is their T^B at the beginning of their page 44) but we here followed the presentation of [Jan03, Section 1.1].

Definitions 3.1.6. Let $\mathcal{C}$ be a pointed category with binary coproducts and kernels.

- The *bifunctor* $\flat\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is constructed by choosing a coproduct $B+C$ of B and C and a kernel $\kappa_{B,C}\colon B\flat C \to B+C$ of $\langle 1_B, 0\rangle\colon B+C \to B$ for the objects and by taking the uniquely induced arrow given by the universal property of the kernel for the arrows.

$$\begin{tikzcd}
B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, dashed, "f\flat g"] & B+C \arrow[r, "\langle 1_B, 0\rangle"] \arrow[d, "f+g"] & B \arrow[d, "f"] \\
B'\flat C' \arrow[r, "\kappa_{B',C'}"] & B'+C' \arrow[r, "\langle 1_{B'}, 0\rangle"] & B'
\end{tikzcd}$$

- For $B \in \mathcal{C}$ fixed, we define the *functor* $B\flat(-)\colon \mathcal{C} \to \mathcal{C}$ by sending an object C to the object $B\flat C$ and an arrow g to the arrow $B\flat g = 1_B \flat g$.

To show that $B\flat(-)$ is a monad, we will show that it can be recovered from an adjunction, which will give it a structure of monad by Proposition 3.1.3.

Proposition 3.1.7. *Let B be an object in a pointed category $\mathcal{C}$ with binary coproducts and kernels. We define the functors $F\colon \mathcal{C} \to \mathrm{Pt}_B = \mathrm{Pt}_B(\mathcal{C})$ (see Definition 1.3.20) and $K\colon \mathrm{Pt}_B \to \mathcal{C}$ as follows :*

- *for $C \in \mathcal{C}$, $F(C)$ is given by choosing a coproduct $(B + C, \iota_1, \iota_2)$ of B and C then setting $F(C) = (B + C, \langle 1_B, 0\rangle, \iota_1)$,*
- *for $(A, \alpha, \beta) \in \mathrm{Pt}_B$, $K(A, \alpha, \beta)$ is given by a choice of kernel of α,*
- *the behaviour on arrows of these functors is given by taking the uniquely induced arrow by the universal properties of the coproduct and the kernel (for F, it means that $F(f)$ is given by $1_B + f$).*

$$\begin{tikzcd}
B \arrow[r, shift left, "i_1"] \arrow[d, equal] & B+C \arrow[l, shift left, "\langle 1_B,0\rangle"] \arrow[d, dashed, "F(f)"] & C \arrow[l, "i_2"'] \arrow[d, "f"] \\
B \arrow[r, shift left, "j_1"] & B+C' \arrow[l, shift left, "\langle 1_B,0\rangle"] & C' \arrow[l, "j_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
K(A,\alpha,\beta) \arrow[r, "\ker\alpha"] \arrow[d, dashed, "K(f)"] & A \arrow[r, shift left, "\alpha"] \arrow[d, "f"] & B \arrow[l, shift left, "\beta"] \arrow[d, equal] \\
K(A',\alpha',\beta') \arrow[r, "\ker\alpha'"] & A' \arrow[r, shift left, "\alpha'"] & B \arrow[l, shift left, "\beta'"]
\end{tikzcd}$$

Then we have an adjunction $F \dashv K$ whose natural bijection ϕ (see (i) of Proposition 1.1.9) is given by $\phi_{C,(A,\alpha,\beta)}$ sending an arrow $f = \langle g, h\rangle \in \mathrm{Pt}_B\big(F(C), (A, \alpha, \beta)\big)$ to the uniquely induced arrow $\hat{h} \in \mathcal{C}\big(C, K(A, \alpha, \beta)\big)$ given by the universal property of the kernel in the diagram below. Its inverse is given by $\phi^{-1}_{C,(A,\alpha,\beta)}\colon \mathcal{C}\big(C, K(A, \alpha, \beta)\big) \to \mathrm{Pt}_B\big(F(C), (A, \alpha, \beta)\big)\colon f \mapsto \langle \beta, \ker\alpha \circ f\rangle$.

$$\begin{tikzcd}
C \arrow[d, dashed, "\hat{h}=\phi_{C,(A,\alpha,\beta)}(f)"'] \arrow[dr, "f\circ\iota_2=h"] & & \\
K(A,\alpha,\beta) \arrow[r, "\ker\alpha"'] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"]
\end{tikzcd}$$

Proof. First, let us check that F and K are well-defined functors. For F, we see that it is indeed the case because $\langle 1_B, 0\rangle \circ \iota_1 = 1_B$ by definition of $\langle 1_B, 0\rangle$, $F(1_C) = 1_B + 1_C = 1_{F(C)}$ as wanted and the compatibility with composition follows from the third computation rule of Remark 3.1.5. The functoriality of K is immediate (only the compatibility with composition can be less clear but it follows from the uniqueness of the universal property of the kernel).

Let C be an object in $\mathcal{C}$, (A, α, β) be an object in Pt_B and let $\varphi = \phi_{C,(A,\alpha,\beta)}$ and $\psi = \phi^{-1}_{C,(A,\alpha,\beta)}$. As $f = \langle g, h\rangle$ is an arrow from $F(C) = (B + C, \langle 1_B, 0\rangle, \iota_1)$ to (A, α, β) in Pt_B, $\langle 1_B, 0\rangle = \alpha \circ f = \langle \alpha \circ g, \alpha \circ h\rangle$ so $\alpha \circ h = 0$ and we can indeed apply the universal property of the kernel to find $\varphi(f) = \hat{h} \in \mathcal{C}\Big(C, K(A, \alpha, \beta)\Big)$, which shows that ϕ is well defined.

$$\begin{tikzcd}
B + C \arrow[rr, "f = \langle g, h \rangle"] \arrow[dr, shift left, "\langle 1_B, 0 \rangle"] & & A \arrow[dl, shift left, "\alpha"] \\
& B \arrow[ul, shift left, "\iota_1"] \arrow[ur, shift left, "\beta"] &
\end{tikzcd}$$

For ψ, we see that, for $f\colon C \to K(A, \alpha, \beta)$, the diagram

$$\begin{tikzcd}
B + C \arrow[rr, "\langle \beta, \ker \alpha \circ f \rangle"] \arrow[dr, shift left, "\langle 1_B, 0 \rangle"] & & A \arrow[dl, shift left, "\alpha"] \\
& B \arrow[ul, shift left, "\iota_1"] \arrow[ur, shift left, "\beta"] &
\end{tikzcd}$$

commutes so it is also well defined.

Given $f \in \mathcal{C}\Big(C, K(A, \alpha, \beta)\Big)$, we remark that f makes the diagram

$$\begin{tikzcd}
C \arrow[d, dashed] \arrow[dr, "\ker \alpha \circ f = \psi(f) \circ \iota_2"] & & \\
K(A, \alpha, \beta) \arrow[r, "\ker \alpha"'] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"]
\end{tikzcd}$$

commute so $f = (\varphi \circ \psi)(f)$ as wanted. Conversely, given $f = \langle g, h \rangle \in \mathrm{Pt}_B\Big(F(C), (A, \alpha, \beta)\Big)$, as

$$\begin{tikzcd}
B + C \arrow[rr, "f = \langle g, h \rangle"] \arrow[dr, shift left, "\langle 1_B, 0 \rangle"] & & A \arrow[dl, shift left, "\alpha"] \\
& B \arrow[ul, shift left, "\iota_1"] \arrow[ur, shift left, "\beta"] &
\end{tikzcd}$$

must commute, we have $g = f \circ \iota_1 = \beta$ so we compute

$$(\psi \circ \varphi)(f) = \langle \beta, \ker \alpha \circ \varphi(f) \rangle = \langle g, h \rangle = f,$$

which finishes the proof that φ and ψ are each other's inverse.

Now, we must show the naturality of ϕ. Let $f \in \mathrm{Pt}_B\Big(F(C), (A, \alpha, \beta)\Big)$, $\gamma\colon C' \to C$ in $\mathcal{C}$ and $\delta\colon (A, \alpha, \beta) \to (A', \alpha', \beta')$ in Pt_B. We denote the coproducts of B with C and C' by $B \xrightarrow{i_1} B + C \xleftarrow{i_2} C$ and $B \xrightarrow{j_1} B + C' \xleftarrow{j_2} C'$. We must show that

$$K(\delta) \circ \phi_{C,(A,\alpha,\beta)}(f) \circ \gamma = \phi_{C',(A',\alpha',\beta')}\Big(\delta \circ f \circ F(\gamma)\Big).$$

$$\begin{tikzcd}
C & (A,\alpha,\beta) \arrow[d, "\delta"] & \mathrm{Pt}_B\big(F(C),(A,\alpha,\beta)\big) \arrow[r, "\phi_{C,(A,\alpha,\beta)}"] \arrow[d, "\mathrm{Pt}_B(F(\gamma),\delta)=\delta\circ(-)\circ F(\gamma)"'] & \mathcal{C}\big(C,K(A,\alpha,\beta)\big) \arrow[d, "\mathcal{C}(\gamma,K(\delta))=K(\delta)\circ(-)\circ\gamma"] \\
C' \arrow[u, "\gamma"] & (A',\alpha',\beta') & \mathrm{Pt}_B\big(F(C'),(A',\alpha',\beta')\big) \arrow[r, "\phi_{C',(A',\alpha',\beta')}"'] & \mathcal{C}\big(C',K(A',\alpha',\beta')\big)
\end{tikzcd}$$

For that, it suffices to show that $K(\delta)\circ\phi_{C,(A,\alpha,\beta)}(f)\circ\gamma$ makes

$$\begin{tikzcd}
C' \arrow[d, dashed] \arrow[dr, "(\delta\circ f\circ F(\gamma))\circ j_2"] & & \\
K(A',\alpha',\beta') \arrow[r, "\ker\alpha'"'] & A' \arrow[r, shift left, "\alpha'"] & B \arrow[l, shift left, "\beta'"]
\end{tikzcd}$$

commute. We compute (using the definitions of K, ϕ and F)

$$\begin{aligned}
\ker\alpha'\circ\Big(K(\delta)\circ\phi_{C,(A,\alpha,\beta)}(f)\circ\gamma\Big) &= (\delta\circ\ker\alpha)\circ\phi_{C,(A,\alpha,\beta)}(f)\circ\gamma = \delta\circ(f\circ i_2)\circ\gamma \\
&= \delta\circ f\circ\Big(F(\gamma)\circ j_2\Big) = \Big(\delta\circ f\circ F(\gamma)\Big)\circ j_2
\end{aligned}$$

as wanted, which finishes the proof. □

As said earlier, the proposition above combined with Proposition 3.1.3 gives a structure of monad to $B\flat(-)$. Moreover, the following corollary gives us explicit descriptions of the unit and multiplication of this monad, which will be useful for the following.

Corollary 3.1.8. *For an object B in a pointed category $\mathcal{C}$ with binary coproducts and kernels, the triple $\big(B\flat(-),\eta,\mu\big)$ is a monad, where the components of η and μ are given by the universal property of the kernel in the diagrams below (where ι_1, ι_2 are the inclusions in the coproduct $B+C$).*

$$\begin{tikzcd}
C \arrow[d, dashed, "\eta_C"'] \arrow[dr, "\iota_2"] & & \\
B\flat C \arrow[r, "\kappa_{B,C}"'] & B+C \arrow[r, "\langle 1_B,0\rangle"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
B\flat(B\flat C) \arrow[r, "\kappa_{B,B\flat C}"] \arrow[d, dashed, "\mu_C"'] & B+(B\flat C) \arrow[r, "\langle 1_B,0\rangle"] \arrow[d, "\langle\iota_1,\kappa_{B,C}\rangle"'] & B \arrow[d, equal] \\
B\flat C \arrow[r, "\kappa_{B,C}"'] & B+C \arrow[r, "\langle 1_B,0\rangle"'] & B
\end{tikzcd}$$

Proof. Since $B\flat(-)=KF$, where K and F are as in Proposition 3.1.7, by Proposition 3.1.3, it suffices to show that Proposition 3.1.7 gives us the right η and μ. By Proposition 1.1.9, the unit $\tilde{\eta}$ of $B\flat(-)$ is given by $\tilde{\eta}_C=\phi_{C,F(C)}(1_{F(C)})$ and its counit ε by $\varepsilon_{(A,\alpha,\beta)}=\phi^{-1}_{K(A,\alpha,\beta),(A,\alpha,\beta)}(1_{K(A,\alpha,\beta)})$.

We compute $\kappa_{B,C}\circ\tilde{\eta}_C=1_{F(C)}\circ\iota_2=\iota_2$ so $\tilde{\eta}=\eta$ by uniqueness of the universal property of the kernel. For μ, as $F(C)=(B+C,\langle 1_B,0\rangle,\iota_1)$, we compute

$$\kappa_{B,C}\circ K(\varepsilon_{F(C)})=\varepsilon_{F(C)}\circ\kappa_{B,B\flat C}=\Big\langle\iota_1,\ker\langle 1_B,0\rangle\circ 1_{KF(C)}\Big\rangle\circ\kappa_{B,B\flat C}=\langle\iota_1,\kappa_{B,C}\rangle\circ\kappa_{B,B\flat C}$$

so μ is indeed given by $\mu_C = K(\varepsilon_{F(C)})$ as expected. $\square$

3.2 The Monad $B\flat(-)$ and its Algebras in Grp

In this section, we will take a look at the monad $B\flat(-)$ in Grp and show that it captures the concept of action of groups.

In Grp, the coproduct $B+C$ of two groups B and C is given by their free product $B*C$ whose elements can be represented by words on the alphabet $B \amalg C$ (the disjoint union of B and C). Then, $B\flat C$ is the kernel of $\langle 1_B, 0\rangle\colon B*C \to B$, i.e. the subgroup of $B*C$ whose elements are represented by words of the form $b_1c_1\cdots b_nc_n$, $b_i \in B$, $c_i \in C$ for $i \in \{1,\dots,n\}$, such that $b_1\cdots b_n = 1$ in B. Here is a more explicit description of this kernel.

Lemma 3.2.1. *The subgroup $B\flat C$ of $B*C$ is generated by the formal conjugates of elements of C by elements of B, i.e. $B\flat C = \langle bcb^{-1} : b \in B,\, c \in C\rangle$. More precisely, it admits the presentation*

$$\langle bcb^{-1},\, b \in B,\, c \in C \mid b1b^{-1} = 1,\, (bcb^{-1})(bc'b^{-1}) = b(cc')b^{-1} \quad \forall b \in B, \forall c,\, c' \in C\rangle.$$

Proof. The inclusion $\langle bcb^{-1} : b \in B,\, c \in C\rangle \subset B\flat C$ is clear since $\langle 1_B, 0\rangle(bcb^{-1}) = 1$ for all $b \in B$ and $c \in C$ (and $B\flat C$ is a subgroup).

For the other inclusion, let $g = b_1c_1\cdots b_nc_n$, $b_i \in B$ and $c_i \in c$ for $i \in \{1,\dots,n\}$, be an arbitrary element of $B\flat C$. Then, using the fact that $b_1\cdots b_n = 1$, we have

$$\begin{aligned} g &= b_1c_1\dots b_nc_n \\ &= (b_1c_1{b_1}^{-1})\Big((b_1b_2)c_2(b_1b_2)^{-1}\Big)\dots\Big((b_1b_2\dots b_{n-1})c_{n-1}(b_1b_2\dots b_{n-1})^{-1}\Big)(b_1b_2\dots b_n)c_n \\ &= (b_1c_1{b_1}^{-1})\Big((b_1b_2)c_2(b_1b_2)^{-1}\Big)\dots\Big((b_1b_2\dots b_{n-1})c_{n-1}(b_1b_2\dots b_{n-1})^{-1}\Big)(1c_n1^{-1}), \end{aligned} \tag{3.1}$$

which is an element of $\langle bcb^{-1} : b \in B,\, c \in C\rangle$, as expected.

Now, we will prove that $B\flat C$ admits the presentation of the statement. Let G be a group admitting this presentation (unique up to isomorphism) and let us show that $B\flat C$ is isomorphic to G. Since $b1b^{-1} = 1$ and $(bcb^{-1})(bc'b^{-1}) = b(cc')b^{-1}$ for all $b \in B$ and all $c,\, c' \in C$, $\varphi\colon bcb^{-1} \in G \mapsto bcb^{-1} \in B\flat C$ uniquely defines a group homomorphism $\varphi\colon G \to B\flat C$. Conversely, given $g \in B\flat C \leq B*C$, it can be uniquely written in reduced form $g = b_1c_1\cdots b_nc_n$, i.e. with $b_i \in B\setminus\{1\}$ for $i \in \{2,\dots,n\}$, $c_i \in C\setminus\{1\}$ for $i \in \{1,\dots,n-1\}$, $b_1 \in B$ and $c_n \in C$. Then, by (3.1), we can rewrite it as a product of formal conjugates

$$g = (b_1c_1{b_1}^{-1})\Big((b_1b_2)c_2(b_1b_2)^{-1}\Big)\dots\Big((b_1b_2\dots b_{n-1})c_{n-1}(b_1b_2\dots b_{n-1})^{-1}\Big)(1c_n1^{-1}),$$

allowing us to define

$$\psi(g)=(b_1c_1{b_1}^{-1})\Big((b_1b_2)c_2(b_1b_2)^{-1}\Big)\dots\Big((b_1b_2\dots b_{n-1})c_{n-1}(b_1b_2\dots b_{n-1})^{-1}\Big)(1c_n1^{-1})\in G.$$

As this procedure associates to each $g\in B\flat C$ a unique $\psi(g)\in G$ (since the reduced form of an element of $B*C$ is unique and the procedure of (3.1) is deterministic), ψ is a well-defined function from $B\flat C$ to G (it is not yet clear that it is a group homomorphism). Finally, we easily check that $\varphi\circ\psi=1_{B\flat C}$ and we compute that, for a generator bcb^{-1} of G, $b\in B$, $c\in C$,

$$\varphi(bcb^{-1})=bcb^{-1}=bcb^{-1}1=(bcb^{-1})(1\,1\,1^{-1})$$

so

$$(\psi\circ\varphi)(bcb^{-1})=(bcb^{-1})(1\,1\,1^{-1})=(bcb^{-1})1=bcb^{-1},$$

showing that $\psi\circ\varphi=1_G$. Therefore, the arrow φ is an isomorphism between G and $B\flat C$, finishing the proof. (In particular, this also shows that ψ is a group homomorphism.) $\square$

For the behaviour of $B\flat(-)$ on the arrows, given $g\colon C\to C'$ in Grp, $B\flat g=1_B\flat g$ is the (co)restriction to $B\flat C$ and to $B\flat C'$ (induced by the universal property/the functoriality of the kernel) of the arrow 1_B+g from $B*C$ to $B*C'$ induced by $b\in B\mapsto b\in B\leq B*C'$ and $c\in C\mapsto g(c)\in C'\leq B*C'$. In short, this arrow applies g on the letters from C and does nothing on the ones from B. For example, on the generators bcb^{-1}, $b\in B$, $c\in C$, of $B\flat C$, we have $(B\flat g)(bcb^{-1})=bg(c)b^{-1}$.

$$\begin{tikzcd}
B\flat C \arrow[r, hook, "\kappa_{B,C}"] \arrow[d, dashed, "B\flat g"] & B*C \arrow[r, "\langle 1_B,0\rangle"] \arrow[d, "1_B+g"] & B \arrow[d, equal] \\
B\flat C' \arrow[r, hook, "\kappa_{B,C'}"] & B*C' \arrow[r, "\langle 1_B,0\rangle"] & B
\end{tikzcd}$$

Now, let us see what are the unit η and multiplication μ of the monad induced by $B\flat(-)$ in Grp. By Corollary 3.1.8, their components are the induced arrows in the diagrams below.

$$\begin{tikzcd}
C \arrow[d, dashed, "\eta_C"'] \arrow[dr, hook, "\iota_2"] & & \\
B\flat C \arrow[r, hook, "\kappa_{B,C}"'] & B*C \arrow[r, "\langle 1_B,0\rangle"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
B\flat(B\flat C) \arrow[r, hook, "\kappa_{B,B\flat C}"] \arrow[d, dashed, "\mu_C"'] & B*(B\flat C) \arrow[r, "\langle 1_B,0\rangle"] \arrow[d, "\langle\iota_1,\kappa_{B,C}\rangle"'] & B \arrow[d, equal] \\
B\flat C \arrow[r, hook, "\kappa_{B,C}"'] & B*C \arrow[r, "\langle 1_B,0\rangle"'] & B
\end{tikzcd}$$

More explicitly, seeing C as a subgroup of $B*C$, we remark that this subgroup is included in $B\flat C$ and η_C is just the inclusion of C in $B\flat C$. For the multiplication μ, its component μ_C is the (co)restriction to $B\flat(B\flat C)$ and to $B\flat C$ of the arrow f from $B*(B*C)$ to $B*C$ induced by $\iota_1\colon B\to B*C$ and $1_{B*C}\colon B*C\to B*C$. More explicitly, f sends an element of $B*(B*C)=B*B*C$ to the element of $B*C$ represented by the same word but, in $B*C$, we only have one copy of B whereas they are two copies of B in $B*B*C$ so, in short, f "forgets" which copy of B the letters of the word representing our element come from. Alternatively, seeing $B*B*C$ as $(B*B)*C$, we have $f=\nabla_B+1_C$ where $\nabla_B=\langle 1_B,1_B\rangle$ is the codiagonal of B.

$$\begin{tikzcd}
B*B \arrow[r, hook] \arrow[d, "\nabla_B"] & (B*B)*C \arrow[d, dashed, "f=\nabla_B+1_C"] & C \arrow[l, hook'] \arrow[d, equal] \\
B \arrow[r, hook] & B*C & C \arrow[l, hook']
\end{tikzcd}$$

Using the results above, we will now show that the category $\mathrm{Grp}^{B\flat(-)}$ of $B\flat(-)$-algebras is isomorphic to the category Grp^B of B-groups (see Definition 2.1.2).

Proposition 3.2.2. *Given a group B, let us consider the monad $B\flat(-)$ constructed with the canonical coproducts and kernels. Then the categories $\mathrm{Grp}^{B\flat(-)}$ and Grp^B are isomorphic.*

Proof. We will define two functors $F\colon \mathrm{Grp}^{B\flat(-)}\to\mathrm{Grp}^B$, $G\colon\mathrm{Grp}^B\to\mathrm{Grp}^{B\flat(-)}$ and show that they are each other's inverse. Given $(C,\xi)\in\mathrm{Grp}^{B\flat(-)}$, we tentatively define $F(C,\xi)=(C,\phi)$ where $\phi\colon B\to\operatorname{Aut}C$ is defined by $\phi(b)(c)=\xi(bcb^{-1})$ for $b\in B$ and $c\in C$. Conversely, given $(C,\phi)\in\mathrm{Grp}^B$, we tentatively define $G(C,\phi)=(C,\xi)$ where $\xi\colon B\flat C\to C$ is given by $\xi(bcb^{-1})=\phi(b)(c)$ for a generator bcb^{-1} of $B\flat C$, $b\in B$, $c\in C$. For the arrows, as the arrows of both $\mathrm{Grp}^{B\flat(-)}$ and Grp^B are just group homomorphisms satisfying certain conditions, we tentatively say that the functors F and G simply send an arrow to the arrow of the other category corresponding to the same underlying group homomorphism. This makes the functoriality of F and G immediate. Moreover, it is also easy to see that F and G are each other's inverse so we just need to check if they are indeed well defined to finish the proof that they form an isomorphism between $\mathrm{Grp}^{B\flat(-)}$ and Grp^B.

First, given $(C,\xi)\in\mathrm{Grp}^{B\flat(-)}$, we must show that $F(C,\xi)=(C,\phi)$, where $\phi\colon B\to\operatorname{Aut}C$ is defined by $\phi(b)(c)=\xi(bcb^{-1})$ for $b\in B$ and $c\in C$, is a B-group, i.e. that ϕ is indeed a group homomorphism from B to $\operatorname{Aut}C$.

- Let b be an element of B and let us check that $\phi(b)$ is an automorphism of C. As $\xi\colon B\flat C\to C$ is a group homomorphism, for all c_1, $c_2\in C$, we have

  $$\begin{aligned}\phi(b)(c_1c_2)&=\xi(bc_1c_2b^{-1})=\xi\Big((bc_1b^{-1})(bc_2b^{-1})\Big)=\xi(bc_1b^{-1})\xi(bc_2b^{-1})\\&=\phi(b)(c_1)\phi(b)(c_2),\end{aligned}$$

  showing that $\phi(b)$ is a group homomorphism. The fact that it is even an automorphism follows from the identity $\phi(b_1)\circ\phi(b_2)=\phi(b_1b_2)$, b_1, $b_2\in B$, which we will prove in the next item. Indeed, as we easily check that $\phi(1)=1_C$ with the equation $\xi\circ\eta_C=1_C$, this identity implies that $\phi(b^{-1})$ is the inverse of $\phi(b)$.

- As (C, ξ) is a $B\flat(-)$-algebra, $\xi \colon B\flat C \to C$ verifies $\xi \circ B\flat\xi = \xi \circ \mu_C$.

$$\begin{tikzcd} B\flat(B\flat C) \arrow[r, "B\flat\xi"] \arrow[d, "\mu_C"'] & B\flat C \arrow[d, "\xi"] \\ B\flat C \arrow[r, "\xi"'] & C \end{tikzcd}$$

  Applying this equation on a generator $b_1(b_2cb_2{}^{-1})b_1{}^{-1}$ of $B\flat(B\flat C)$ where b_1 is a letter from the first copy of B, b_2 a letter from the second copy of B and c a letter from C, we obtain

$$\begin{aligned} \phi(b_1b_2)(c) &= \xi\Big((b_1b_2)c(b_1b_2)^{-1}\Big) = \xi(b_1b_2cb_2{}^{-1}b_1{}^{-1}) = \xi\Big(\mu_C\Big(b_1(b_2cb_2{}^{-1})b_1{}^{-1}\Big)\Big) \\ &= \xi\Big((B\flat\xi)\Big(b_1(b_2cb_2{}^{-1})b_1{}^{-1}\Big)\Big) = \xi\Big(b_1\xi(b_2cb_2{}^{-1})b_1{}^{-1}\Big) = \phi(b_1)\Big(\phi(b_2)(c)\Big) \\ &= \Big(\phi(b_1) \circ \phi(b_2)\Big)(c), \end{aligned}$$

  showing that $\phi(b_1) \circ \phi(b_2) = \phi(b_1b_2)$ for all b_1, $b_2 \in B$, i.e. that ϕ is a group homomorphism.

Combining these two points, that shows that ϕ is a group homomorphism from B to $\operatorname{Aut} C$ and thus that $F(C, \xi) = (C, \phi)$ is a B-group.

Conversely, given $(C, \phi) \in \mathrm{Grp}^B$, we must show that $G(C, \phi) = (C, \xi)$ is a $B\flat(-)$-algebra, where $\xi \colon B\flat C \to C$ is given by $\xi(bcb^{-1}) = \phi(b)(c)$ for a generator bcb^{-1} of $B\flat C$, $b \in B$, $c \in C$. First, by the presentation of $B\flat C$ from Lemma 3.2.1 and using the fact that ϕ is an action by automorphisms, we easily check that this defines a group homomorphism $\xi \colon B\flat C \to C$ (because $\phi(b)(1) = 1$ and $\phi(b)(c)\phi(b)(c') = \phi(b)(cc')$ for all $b \in B$ and all c, $c' \in C$). Moreover, by definition of $B\flat(-)$-algebra, ξ must verify $\xi \circ B\flat\xi = \xi \circ \mu_C$ and $\xi \circ \eta_C = 1_C$.

$$\begin{tikzcd} B\flat(B\flat C) \arrow[r, "B\flat\xi"] \arrow[d, "\mu_C"'] & B\flat C \arrow[d, "\xi"] \\ B\flat C \arrow[r, "\xi"'] & C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, hook, "\eta_C"] \arrow[dr, equal] & B\flat C \arrow[d, "\xi"] \\ & C \end{tikzcd}$$

For a generator $b_1(b_2cb_2{}^{-1})b_1{}^{-1}$ of $B\flat(B\flat C)$ where b_1 is a letter from the first copy of B, b_2 a letter from the second copy of B and c a letter from C, we compute

$$\begin{aligned} \xi\Big((B\flat\xi)\Big(b_1(b_2cb_2{}^{-1})b_1{}^{-1}\Big)\Big) &= \xi\Big(b_1\xi(b_2cb_2{}^{-1})b_1{}^{-1}\Big) = \phi(b_1)\Big(\phi(b_2)(c)\Big) = \phi(b_1b_2)(c) \\ &= \xi\Big((b_1b_2)c(b_1b_2)^{-1}\Big) = \xi(b_1b_2cb_2{}^{-1}b_1{}^{-1}) \\ &= \xi\Big(\mu_C\Big(b_1(b_2cb_2{}^{-1})b_1{}^{-1}\Big)\Big) \end{aligned}$$

and, for $c \in C$, we have

$$\xi\Big(\eta_C(c)\Big) = \xi(c) = \xi(1c1^{-1}) = \phi(1)(c) = c$$

so $G(C,\phi) = (C,\xi)$ is indeed a $B\flat(-)$-algebra.

Finally, to finish the proof, let us see that a group homomorphism $f\colon C \to C'$ is an arrow between the $B\flat(-)$-algebras (C,ξ), (C',ζ) if and only if it is an arrow between the B-groups (C,ϕ), (C',ψ) where we have the relations $\xi(bcb^{-1}) = \phi(b)(c)$ and $\zeta(bc'b^{-1}) = \psi(b)(c')$, $b \in B$, $c \in C$, $c' \in C'$, between ξ, ζ and ϕ, ψ. Being a morphism of $B\flat(-)$-algebras means that we must have $\zeta \circ B\flat f = f \circ \xi$.

$$\begin{tikzcd}
B\flat C \arrow[r, "B\flat f"] \arrow[d, "\xi"'] & B\flat C' \arrow[d, "\zeta"] \\
C \arrow[r, "f"'] & C'
\end{tikzcd}$$

Applying this equality to a generator bcb^{-1} of $B\flat C$, $b \in B$, $c \in C$, we obtain that f satisfies

$$\psi(b)\Big(f(c)\Big) = \zeta\Big(bf(c)b^{-1}\Big) = \zeta\Big((B\flat f)(bcb^{-1})\Big) = f\Big(\xi(bcb^{-1})\Big) = f\Big(\phi(b)(c)\Big)$$

for all $b \in B$ and all $c \in C$, i.e. f is equivariant for the actions ϕ and ψ, as expected. Similarly, if f is equivariant, we compute, for a generator bcb^{-1} of $B\flat C$, $b \in B$, $c \in C$,

$$\zeta\Big((B\flat f)(bcb^{-1})\Big) = \zeta\Big(bf(c)b^{-1}\Big) = \psi(b)\Big(f(c)\Big) = f\Big(\phi(b)(c)\Big) = f\Big(\xi(bcb^{-1})\Big),$$

showing that the equivariant arrows between B-groups are morphisms between the corresponding $B\flat(-)$-algebras, finishing the proof. $\square$

3.3 The Monad $B\flat(-)$ and its Algebras in CRng

Similarly to what we have done in the previous section for groups, we will show that actions in CRng (see Definition 2.2.2) can be expressed in terms of the monad $B\flat(-)$.

First, the coproduct $B+X$ of two commutative rings B and X is given by the commutative ring whose underlying abelian group is $B \oplus (B \otimes X) \oplus X$ and whose multiplication is given by

$$(b, b'\otimes x', x)(c, c'\otimes y', y) = (bc, bc'\otimes y' + b\otimes y + b'c\otimes x' + b'c'\otimes x'y' + b'\otimes x'y + c\otimes x + c'\otimes xy', xy)$$

for $(b, b'\otimes x', x), (c, c'\otimes y', y) \in B \oplus (B\otimes X) \oplus X$ (and we extend it by bilinearity if we do not have elementary tensors in the middle factor). The inclusions of B and X in $B+X$ are given by $\iota_1\colon b \in B \mapsto (b,0,0) \in B+X$ and $\iota_2\colon x \in X \mapsto (0,0,x) \in B+X$, respectively.

(Since any $(b, b' \otimes x', x) \in B + X$ can be written as $(b,0,0)+(b',0,0)(0,0,x')+(0,0,x)$, one can check that this construction verifies the universal property of the coproduct as wanted.)

Then, we must determine the kernel $B\flat X$ of $\langle 1_B, 0\rangle \colon B + X \to B$. Given an arbitrary $(b, \sum_{i=1}^n b_i \otimes x_i, x) \in B + X$, we have

$$\begin{aligned}\left(b, \sum_{i=1}^n b_i \otimes x_i, x\right) &= (b,0,0) + \sum_{i=1}^n (0, b_i \otimes x_i, 0) + (0,0,x)\\ &= (b,0,0) + \sum_{i=1}^n (b_i,0,0)(0,0,x_i) + (0,0,x)\end{aligned}$$

so

$$\begin{aligned}\langle 1_B, 0\rangle \left(b, \sum_{i=1}^n b_i \otimes x_i, x\right) &= \langle 1_B, 0\rangle \left((b,0,0) + \sum_{i=1}^n (b_i,0,0)(0,0,x_i) + (0,0,x)\right)\\ &= b + \sum_{i=1}^n b_i 0 + 0 = b.\end{aligned}$$

Therefore, $B\flat X$ is isomorphic to the subring of $B+X$ whose elements have a 0 in the first component so we can see $B\flat X$ as the commutative ring whose underlying abelian group is $(B\otimes X)\oplus X$ and whose multiplication is given by that of $B+X$ restricted to the two last components (i.e. it is determined by $(b'\otimes x', x)(c'\otimes y', y) = (b'c'\otimes x'y' + b'\otimes x'y + c'\otimes xy', xy)$ for $(b' \otimes x', x)$, $(c' \otimes y', y) \in (B \otimes X) \oplus X$). Its inclusion $\kappa_{B,X}$ in $B + X$ is given by $\kappa_{B,X}(l, x) = (0, l, x)$ for $(l, x) \in B\flat X$. On the arrows, given $g \colon X \to X'$ in CRng, $B\flat g \colon B\flat X \to B\flat X'$ is determined by $(B\flat g)(b'\otimes x', x) = \big(b'\otimes g(x'), g(x)\big)$ for $(b'\otimes x', x) \in B\flat X$.

For the unit η and multiplication μ, by Corollary 3.1.8, their components are the uniquely induced arrows in the diagrams

$$\begin{tikzcd} X \arrow[d, dashed, "\eta_X"'] \arrow[dr, hook, "\iota_2"] & & \\ B\flat X \arrow[r, "\kappa_{B,X}"'] & B + X \arrow[r, "\langle 1_B, 0\rangle"'] & B \end{tikzcd} \qquad \begin{tikzcd} B\flat(B\flat X) \arrow[r, hook, "\kappa_{B,B\flat X}"] \arrow[d, dashed, "\mu_X"'] & B + (B\flat X) \arrow[r, "\langle 1_B, 0\rangle"] \arrow[d, "\langle \iota_1, \kappa_{B,X}\rangle"'] & B \arrow[d, equal] \\ B\flat X \arrow[r, "\kappa_{B,X}"'] & B + X \arrow[r, "\langle 1_B, 0\rangle"'] & B \end{tikzcd}$$

so we easily see that the components of η are given by $\eta_X \colon x \in X \mapsto (0, x) \in B\flat X$, $X \in$ CRng. For μ_X, $X \in$ CRng, we compute

$$\begin{aligned}&\langle \iota_1, \kappa_{B,X}\rangle\Big(\kappa_{B,B\flat X}\big(b' \otimes (l', x'), (l, x)\big)\Big) = \langle \iota_1, \kappa_{B,X}\rangle\big(0, b' \otimes (l', x'), (l, x)\big)\\ &= \langle \iota_1, \kappa_{B,X}\rangle\Big((b',0,0)\big(0,0,(l',x')\big) + \big(0,0,(l,x)\big)\Big) = (b',0,0)(0,l',x') + (0,l,x)\\ &= (0, b'l' + b' \otimes x' + l, x)\end{aligned}$$

so μ_X is determined by $\mu_X\Big(b'\otimes(l',x'),(l,x)\Big) = (b'l'+b'\otimes x'+l,x)$ for $\Big(b'\otimes(l',x'),(l,x)\Big) \in B\flat(B\flat X)$, $b' \in B$, (l',x'), $(l,x) \in B\flat X$. (Let us recall that l and l' are tensors in $B \otimes X$. In particular, if $l' = \sum_{i=1}^n b_i \otimes x_i$, then $b'l'$ must be understood as $\sum_{i=1}^n b'b_i \otimes x_i$.)

For the end of this section, similarly to what we have done for groups, we will show that the category $\mathrm{CRng}^{B\flat(-)}$ of $B\flat(-)$-algebras is isomorphic to the category CRng^B of B-rings (see Definition 2.2.3).

Proposition 3.3.1. *Given a commutative ring B, let us consider the monad $B\flat(-)$ constructed as above (i.e. with $B\flat X$ being the commutative ring whose underlying abelian group is $(B\otimes X)\oplus X$ described above). Then the categories $\mathrm{CRng}^{B\flat(-)}$ and CRng^B are isomorphic.*

Proof. Similar to the case in Grp, the construction of the functors $F\colon \mathrm{CRng}^{B\flat(-)} \to \mathrm{CRng}^B$, $G\colon \mathrm{CRng}^B \to \mathrm{CRng}^{B\flat(-)}$ forming an isomorphism between $\mathrm{CRng}^{B\flat(-)}$ and CRng^B is relatively straightforward, the most difficult thing to check is the fact that they are well defined. Given $(X,\xi) \in \mathrm{CRng}^{B\flat(-)}$, we tentatively set $F(X,\xi) = (X,\phi)$ where $\phi\colon B\otimes X \to X$ is determined by $\phi(b\otimes x) = \xi(b\otimes x, 0)$ for $b \in B$, $x \in X$ and, given $(X,\phi) \in \mathrm{CRng}^B$, we tentatively set $G(X,\phi) = (X,\xi)$ where $\xi\colon B\flat X \to X$ is determined by $\xi(b'\otimes x', x) = \phi(b'\otimes x') + x$ for $(b'\otimes x', x) \in B\flat X$. For the arrows, as they are again just ring homomorphisms satisfying certain conditions, our two functors F and G tentatively send an arrow to the one of the other category corresponding to the same underlying ring homomorphism. Let us show that F and G are indeed well defined, their functoriality following directly by construction.

Given $(X,\xi) \in \mathrm{CRng}^{B\flat(-)}$, we must verify that $F(X,\xi) = (X,\phi)$, where $\phi\colon B\otimes X \to X$ is determined by $\phi(b\otimes x) = \xi(b\otimes x, 0)$ for $b \in B$, $x \in X$, is a B-ring.

- As $\xi\colon B\flat X \to X$ is a ring homomorphism, we only need to show that the map $\iota\colon l \in B\otimes X \mapsto (l,0) \in B\flat X$ is a ring homomorphism in order to deduce that ϕ is also one, as desired. The underlying abelian group of $B\flat X$ being $(B\otimes X)\oplus X$, the fact that ι is a group homomorphism is clear and we just need to check that it is also compatible with the multiplication, which is clear since it is determined by
$$\iota(b\otimes x)\iota(c\otimes y) = (b\otimes x,0)(c\otimes y,0) = (bc\otimes xy,0) = \iota(bc\otimes xy) = \iota\Big((b\otimes x)(c\otimes y)\Big),$$
b, $c \in B$, x, $y \in X$, the case for arbitrary tensors following by distributivity.
- For b, $c \in B$ and $x \in X$, as ξ verifies $\xi \circ B\flat\xi = \xi \circ \mu_X$,
$$\begin{tikzcd} B\flat(B\flat C) \arrow[r, "B\flat\xi"] \arrow[d, "\mu_C"'] & B\flat C \arrow[d, "\xi"] \\ B\flat C \arrow[r, "\xi"'] & C \end{tikzcd}$$

we have

$$\begin{aligned}\phi\Big(b\otimes\phi(c\otimes x)\Big) &= \xi\Big(b\otimes\xi(c\otimes x,0),0\Big) = \xi\Big(b\otimes\xi(c\otimes x,0),\xi(0,0)\Big)\\ &= \xi\Big((B\flat\xi)\big(b\otimes(c\otimes x,0),(0,0)\big)\Big) = \xi\Big(\mu_X\big(b\otimes(c\otimes x,0),(0,0)\big)\Big)\\ &= \xi(bc\otimes x+b\otimes 0+0,0) = \xi(bc\otimes x,0) = \phi(bc\otimes x),\end{aligned}$$

as wanted.

- Finally, for $b\in B$ and $x,\,y\in X$, as $\xi\circ\eta_X = 1_X$, we compute

$$\begin{aligned}\phi(b\otimes x)y &= \xi(b\otimes x,0)\xi\big(\eta_X(y)\big) = \xi(b\otimes x,0)\xi(0,y) = \xi\big((b\otimes x,0)(0,y)\big)\\ &= \xi(0+b\otimes xy+0,0) = \xi(b\otimes xy,0) = \phi(b\otimes xy)\end{aligned}$$

and, similarly, we have $x\phi(b\otimes y) = \phi(b\otimes xy)$.

Thus ϕ is indeed an action in the sense of Definition 2.2.2, showing that $F(X,\xi) = (X,\phi)$ is a B-ring.

Given $(X,\phi)\in \mathrm{CRng}^B$, we must show that $G(X,\phi) = (X,\xi)$, where $\xi\colon B\flat X\to X$ is determined by $\xi(b'\otimes x',x) = \phi(b'\otimes x')+x$ for $(b'\otimes x',x)\in B\flat X$, is a ring homomorphism and verifies $\xi\circ B\flat\xi = \xi\circ\mu_X$ and $\xi\circ\eta_X = 1_X$.

$$\begin{tikzcd} B\flat(B\flat X) \arrow[r, "B\flat\xi"] \arrow[d, "\mu_X"'] & B\flat X \arrow[d, "\xi"] \\ B\flat X \arrow[r, "\xi"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & B\flat X \arrow[d, "\xi"] \\ & X \end{tikzcd}$$

The fact that ξ is a group homomorphism is clear and, for the multiplication, given $(b'\otimes x',x)$, $(c'\otimes y',y)\in B\flat X$, we compute

$$\begin{aligned}\xi(b'\otimes x',x)\xi(c'\otimes y',y) &= \Big(\phi(b'\otimes x')+x\Big)\Big(\phi(c'\otimes y')+y\Big)\\ &= \phi(b'\otimes x')\phi(c'\otimes y')+\phi(b'\otimes x')y+x\phi(c'\otimes y')+xy\\ &= \phi(b'c'\otimes x'y')+\phi(b'\otimes x'y)+\phi(c'\otimes xy')+xy\\ &= \phi(b'c'\otimes x'y'+b'\otimes x'y+c'\otimes xy')+xy\\ &= \xi(b'c'\otimes x'y'+b'\otimes x'y+c'\otimes xy',xy)\\ &= \xi\Big((b'\otimes x',x)(c'\otimes y',y)\Big).\end{aligned}$$

For the first equation $\xi\circ B\flat\xi = \xi\circ\mu_X$, given $\Big(b'\otimes(c'\otimes x',x),(b\otimes y',y)\Big)\in B\flat(B\flat X)$, we have

$$\begin{aligned}\xi\Big(\mu_X\big(b'\otimes(c'\otimes x',x),(b\otimes y',y)\big)\Big) &= \xi(b'c'\otimes x'+b'\otimes x+b\otimes y',y)\\ &= \phi(b'c'\otimes x'+b'\otimes x+b\otimes y')+y\end{aligned}$$

and

$$\begin{aligned}
\xi\Big((B\flat\xi)\Big(b'\otimes(c'\otimes x',x),(b\otimes y',y)\Big)\Big) &= \xi\Big(b'\otimes\xi(c'\otimes x',x),\xi(b\otimes y',y)\Big)\\
&= \phi\Big(b'\otimes\Big(\phi(c'\otimes x')+x\Big)\Big)+\Big(\phi(b\otimes y')+y\Big)\\
&= \phi\Big(b'\otimes\phi(c'\otimes x')\Big)+\phi(b'\otimes x)+\phi(b\otimes y')+y\\
&= \phi(b'c'\otimes x')+\phi(b'\otimes x)+\phi(b\otimes y')+y,
\end{aligned}$$

these two being equal as wanted by linearity of ϕ. For the other equation, we have $\xi\Big(\eta_X(x)\Big)=\xi(0,x)=\phi(0)+x=x$ for all $x\in X$, as desired.

For the arrows, given an equivariant ring homomorphism $f\colon X\to Y$ between B-rings (X,ϕ), (Y,ψ), we must show that $\zeta\circ B\flat f=f\circ\xi$ where $\xi\colon B\flat X\to X$ and $\zeta\colon B\flat Y\to Y$ are determined by $\xi(b'\otimes x',x)=\phi(b'\otimes x')+x$ and $\zeta(b'\otimes y',y)=\psi(b'\otimes y')+y$ for $b'\in B$, x, $x'\in X$ and y, $y'\in Y$.

$$\begin{tikzcd}
B\flat X \arrow[r, "B\flat f"] \arrow[d, "\xi"'] & B\flat Y \arrow[d, "\zeta"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Using the equivariance of F, we compute

$$\begin{aligned}
\zeta\Big((B\flat f)(b'\otimes x',x)\Big) &= \zeta\Big(b'\otimes f(x'),f(x)\Big)=\psi\Big(b'\otimes f(x')\Big)+f(x)\\
&= f\Big(\phi(b'\otimes x')\Big)+f(x)=f\Big(\phi(b'\otimes x')+x\Big)=f\Big(\xi(b'\otimes x',x)\Big),
\end{aligned}$$

as expected. Conversely, given a ring homomorphism $f\colon X\to Y$ between $B\flat(-)$-algebras (X,ξ), (Y,ζ), we must show that it is equivariant between the B-rings (X,ϕ), (Y,ψ) where $\phi\colon B\otimes X\to X$ and $\psi\colon B\otimes Y\to Y$ are determined by $\phi(b\otimes x)=\xi(b\otimes x,0)$ and $\psi(b\otimes y)=\zeta(b\otimes y,0)$, $b\in B$, $x\in X$ and $y\in Y$. As f makes

$$\begin{tikzcd}
B\flat X \arrow[r, "B\flat f"] \arrow[d, "\xi"'] & B\flat Y \arrow[d, "\zeta"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

commute, we obtain, for $b\otimes x\in B\otimes X$,

$$\begin{aligned}
\psi\Big(b\otimes f(x)\Big)=\zeta\Big(b\otimes f(x),0\Big)&=\zeta\Big(b\otimes f(x),f(0)\Big)=\zeta\Big((B\flat f)(b\otimes x,0)\Big)\\
&= f\Big(\xi(b\otimes x,0)\Big)=f\Big(\phi(b\otimes x)\Big),
\end{aligned}$$

showing the equivariance of f.

To finish the proof, we must show that F and G are each other's inverse. On the arrows, it is immediate and we also easily see that $F \circ G = 1_{\mathrm{CRng}^B}$ for the objects. For $G \circ F$, we must show that, given $(X, \xi) \in \mathrm{CRng}^{B\flat(-)}$, the formula $\phi(b' \otimes x') + x$, where $\phi(b \otimes x) = \xi(b \otimes x, 0)$ for $b \otimes x \in B \otimes X$, gives us $\xi(b' \otimes x', x)$ for any $(b' \otimes x', x) \in B\flat X$. Therefore, we must have $\xi(b' \otimes x', x) = \xi(b' \otimes x', 0) + x$, which is indeed the case because ξ is a group homomorphism and $x = \xi(0, x)$ by the equation $\xi \circ \eta_X = 1_X$ (as η_X is given by $\eta_X(x) = (0, x)$, $x \in X$). $\square$

Chapter 4

Internal Actions in Semi-Abelian Categories

We now have everything we need to define actions in a more general context. In the first section, we will present this context of semi-abelian categories and state some of its properties we will need afterwards. The second section is dedicated to the definition of actions in this context with the main result being that we keep the equivalence with the split short exact sequences as in Grp and CRng (see Propositions 2.1.4 and 2.2.7).

4.1 Semi-Abelian Categories

In this section, we will present semi-abelian categories, which are the context where we will prove the main result of this thesis. Semi-abelian categories were introduced in the early 2000s to unify a number of older and isolated approaches towards such a definition with the concept of Barr-exactness, which is at the very heart of topos theory and the theory of abelian categories alike. The "missing ingredient" was Bourn's concept of protomodularity, which allows generalising some of the core reasoning involving diagram chasing which makes use of the additive structure in an abelian category to a non-additive context. However, the context of semi-abelian categories is a vast subject so we will not cover it in detail here and only state the definitions and results (without proofs) we will need later. We redirect the interested reader to [JMT02] and [BB04], as well as [BCF$^+$21, Chapter 4] for regular categories and relations. The following definitions mainly come from [JMT02, Sections 2.1 and 2.5].

In short, a *semi-abelian category* is a pointed category with binary coproducts that is exact and protomodular. We already covered protomodularity in Section 1.3 so we will define here what is an *exact category*. This definition uses the concepts of *regular category* ([BCF$^+$21] do not give the same definition as ours (see their Definition 1.10) but they show that these two definitions are equivalent in their Theorem 1.14) and *relation* so let us begin with these two notions.

We first define *regular categories* whose main characteristic is the existence of *regular epimorphism/monomorphism factorisations.*

Definitions 4.1.1.

- A *regular epimorphism/monomorphism factorisation of an arrow* $f\colon X \to Y$ is a factorisation $X \xrightarrow{e} I \xrightarrow{m} Y$ of f where e is a regular epimorphism and m is a monomorphism.

- A category $\mathcal{C}$ is *regular* if it is finitely complete and has pullback-stable regular epimorphism/monomorphism factorisations, i.e. every arrow f in $\mathcal{C}$ has a regular epimorphism/monomorphism factorisation and the pullback of this factorisation gives us a regular epimorphism/monomorphism factorisation of the pullback of f. In other words, for each arrow $f\colon X \to Y$ in $\mathcal{C}$ with regular epimorphism/monomorphism factorisation $X \xrightarrow{e} I \xrightarrow{m} Y$ and for $p\colon Y' \to Y$, the regular epimorphism/monomorphism factorisation of the pullback p^*f of f along p is given as in the diagram below where both squares are pullbacks (which implies that the outer rectangle is a pullback as well so it is indeed a factorisation of p^*f).

$$\begin{tikzcd}
X' \arrow[r, two heads] \arrow[d] \arrow[rr, bend left, "p^*f"] & I' \arrow[r, tail] \arrow[d] & Y' \arrow[d, "p"] \\
X \arrow[r, two heads, "e"'] \arrow[rr, bend right, "f"'] & I \arrow[r, tail, "m"'] & Y
\end{tikzcd}$$

Let us now define *relations* and some attributes they can have. The article [JMT02] sees relations via their images by a Hom-functor but we will instead present the approach of [BCF+21, Chapter 4, Section 2].

Definitions 4.1.2.

- An *(internal) relation from X to Y* is a couple of arrows $(r_1\colon R \to X, r_2\colon R \to Y)$ that are jointly monic. If the product $X \times Y$ exists, we can see such a relation as a monomorphism $r\colon R \to X \times Y$. If $X = Y$, we say that we have a *relation on X*.

$$\begin{tikzcd}
 & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\
X & & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
R \arrow[d, tail, "r"] \\
X \times Y
\end{tikzcd}$$

- A relation $(r_1\colon R \to X, r_2\colon R \to X)$ on X is *reflexive* if there exists an arrow $\Delta\colon X \to R$ that splits both r_1 and r_2, i.e. such that $r_1 \circ \Delta = r_2 \circ \Delta = 1_X$.

$$\begin{tikzcd}
R \arrow[r, shift left=2, "r_1"] \arrow[r, shift right=2, "r_2"'] & X \arrow[l, "\Delta" description]
\end{tikzcd}$$

- A relation $(r_1\colon R \to X, r_2\colon R \to X)$ on X is *symmetric* if there exists an arrow $\sigma\colon R \to R$ such that $r_1 \circ \sigma = r_2$ and $r_2 \circ \sigma = r_1$.

$$\begin{tikzcd} R \arrow[rr, shift left, "\sigma"] \arrow[dr, "r_1"'] & & R \arrow[ll, shift left, "\sigma"] \arrow[dl, "r_2"] \\ & X & \end{tikzcd}$$

- If the pullback

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_2"'] & X \end{tikzcd}$$

exists, we say that the relation $(r_1\colon R \to X, r_2\colon R \to X)$ on X is *transitive* if there exists an arrow $\tau\colon P \to R$ such that $r_1 \circ \tau = r_1 \circ p_1$ and $r_2 \circ \tau = r_2 \circ p_2$.

$$\begin{tikzcd} P \arrow[r, "\tau"] \arrow[d, "p_1"'] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_1"'] & X \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\tau"] \arrow[d, "p_2"'] & R \arrow[d, "r_2"] \\ R \arrow[r, "r_2"'] & X \end{tikzcd}$$

- A relation on X is an *equivalence relation* if it is reflexive, symmetric and transitive.

Remark 4.1.3. As mentioned just before Lemma 1.2.14, this lemma tells us that every kernel pair is a reflexive relation in the sense we defined just above. In fact, every kernel pair is even an equivalence relation (see [BCF⁺21, Chapter 4, Lemma 2.2]).

We can now define exact and semi-abelian categories.

Definitions 4.1.4.

- A category $\mathcal{C}$ is *(Barr-)exact* if it is regular and every equivalence relation in $\mathcal{C}$ is *effective*, i.e. it is the kernel pair of some arrow $f\colon X \to Y$.
- A category $\mathcal{C}$ is *semi-abelian* if it is pointed, exact, protomodular and has binary coproducts.

Examples 4.1.5. Abelian categories, Grp, CRng and $\mathrm{Set}_*^{\mathrm{op}}$ are semi-abelian. We already know that they are pointed protomodular by Examples 1.3.6 and we also remark that they have binary coproducts. For their exactness, see [BB04, Example 5.1.8] for $\mathrm{Set}_*^{\mathrm{op}}$, [Bor94, Theorem 2.6.11] for abelian categories and [Bor94, Theorem 3.5.4] for Grp and CRng.

For the end of this section, here are the two results about semi-abelian categories we will need later.

Proposition 4.1.6. *Semi-abelian categories are finitely cocomplete.*

Proof. See [JMT02, Corollary p.377]. □

Lemma 4.1.7. *In a semi-abelian category $\mathcal{C}$, if we have a commutative diagram*

$$\begin{tikzcd} A \arrow[r, two heads, "q"] \arrow[d, tail, "w"'] & B \arrow[d, tail, "v"] \\ C \arrow[r, two heads, "p"'] & D \end{tikzcd}$$

with normal epimorphisms p, q, a monomorphism v and a normal monomorphism w, then v is a normal monomorphism.

Proof. See [JMT02, Proposition p.382] where (SA*6) corresponds to the statement of the lemma. □

4.2 Internal Actions and Semidirect Products in Semi-Abelian Categories

We have seen in Sections 3.2 and 3.3 that, in Grp and CRng, the categories of B-objects are isomorphic to the Eilenberg–Moore category $\mathcal{C}^{B\flat(-)}$ (see Propositions 3.2.2 and 3.3.1), which motivates us to use this category as the general definition for the category of B-objects.

Definition 4.2.1. Let $\mathcal{C}$ be a semi-abelian category. For $B \in \mathcal{C}$, we define the *category* $\mathcal{C}^{B\flat(-)}$ *of B-objects* as the Eilenberg–Moore category of the monad $B\flat(-)$ (see Corollary 3.1.8). Given a B-object (C, ξ), the arrow ξ is called an *(internal) action of B on C*.

As a reminder, to show the equivalence between the B-objects and the split short exact sequences in Grp and CRng, we used the *semidirect product*. We will do the same here so let us define this notion in the semi-abelian context.

Let B be an object of a semi-abelian category $\mathcal{C}$. Given a B-object (C, ξ), we consider the two parallel arrows $\kappa_{B,C}$ and $\iota_2 \circ \xi \colon B\flat C \to B + C$ where $\iota_2 \colon C \to B + C$ denotes the inclusion of C in the coproduct $B + C$. Since $\mathcal{C}$ is finitely cocomplete by Proposition 4.1.6, we can take a coequaliser $q \colon B + C \to Q$ of these two arrows and it is this object Q that we will use as the definition of the *semidirect product of* (C, ξ).

Definition 4.2.2. Let (C,ξ) be a B-object. A *semidirect product of C by B (relative to the action ξ* is the object Q obtained by taking a coequaliser $q\colon B+C \to Q$ of $\kappa_{B,C}$ and $\iota_2 \circ \xi$ where $\iota_2\colon C \to B+C$ is the inclusion of C in the coproduct $B+C$. We write $B \ltimes_\xi C := Q$.

$$\begin{tikzcd} B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, "\xi"'] & B+C \arrow[r, two heads, "q"] & B \ltimes_\xi C \\ C \arrow[ur, "\iota_2"'] & & \end{tikzcd}$$

A categorical definition of the semidirect product already appears in [BJ98, Section 3.2] but with a different approach (for example, they work with points instead of split short exact sequences but, as we have shown in Proposition 1.3.23, they are equivalent in the pointed protomodular context). Here, we tried to give a definition not too difficult to understand and such that it is relatively easy to see that it is a generalisation of the semidirect products we already know, as we will show for Grp in the following example.

Example 4.2.3. Let us check that the semidirect in Grp (see Reminder 2.1.3) verifies this definition. Let us recall that the action $\phi\colon B \to \operatorname{Aut} C$ induced by a B-object (C,ξ) is given by $\phi(b)(c) = \xi(bcb^{-1})$, $b \in B$, $c \in C$. We will show that

$$q\colon B * C \to C \rtimes_\phi B\colon \begin{cases} b \mapsto (1,b) \\ c \mapsto (c,1) \end{cases}$$

is a coequaliser of $\kappa_{B,C}$ and $\iota_2 \circ \xi$ (we remark that it is indeed a group homomorphism since it is the arrow induced by the inclusions of B and C in $C \rtimes_\phi B$ by the universal property of the coproduct $B * C$).

$$\begin{tikzcd} B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, "\xi"'] & B*C \arrow[r, two heads, "q"] & C \rtimes_\phi B \\ C \arrow[ur, "\iota_2"'] & & \end{tikzcd}$$

First, let us evaluate $q \circ \kappa_{B,C}$ and $q \circ (\iota_2 \circ \xi)$ on a generator bcb^{-1} of $B\flat C$ in order to show that q coequalises $\kappa_{B,C}$ and $\iota_2 \circ \xi$. We compute

$$q\Big(\kappa_{B,C}(bcb^{-1})\Big) = q(bcb^{-1}) = (1,b)(c,1)(1,b)^{-1} = \Big(\phi(b)(c), b\Big)(1,b^{-1}) = \Big(\phi(b)(c), 1\Big)$$

and

$$q\Big((\iota_2 \circ \xi)(bcb^{-1})\Big) = \Big(\xi(bcb^{-1}), 1\Big)$$

which are equal as desired by construction of ϕ.

Now, given $f\colon B * C \to D$ such that $f \circ \kappa_{B,C} = f \circ (\iota_2 \circ \xi)$, we must show that we have a unique group homomorphism $\varphi\colon C \rtimes_\phi B \to D$ such that $\varphi \circ q = f$.

$$\begin{tikzcd}
B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, "\xi"'] & B * C \arrow[r, two heads, "q"] \arrow[drr, "f"'] & C \rtimes_\phi B \arrow[dr, phantom, ""] \arrow[d, dashed, "\exists!\varphi\,?"] \\
C \arrow[ur, "\iota_2"'] & & D
\end{tikzcd}$$

We set $\varphi\colon C \rtimes_\phi B\colon (c, b) \mapsto f(cb)$. We easily see that we have $\varphi \circ q = f$ as wanted (by checking for the generators of $B * C$). Let us check that φ is a group homomorphism. Given (c, b), $(c', b') \in C \rtimes_\phi B$, we compute

$$\begin{aligned}
\varphi(c, b)\varphi(c', b') = f(cb)f(c'b') = f(cbc'b') &= f\Big(c(bc'b^{-1})bb'\Big) = f(c)f\Big(\kappa_{B,C}(bc'b^{-1})\Big)f(bb') \\
&= f(c)f\Big((\iota_2 \circ \xi)(bc'b^{-1})\Big)f(bb') = f(c)f\Big(\phi(b)(c')\Big)f(bb') = f\Big(c\phi(b)(c')bb'\Big) \\
&= \varphi\Big(c\phi(b)(c'), bb'\Big) = \varphi\Big((c, b)(c', b')\Big)
\end{aligned}$$

so φ is a group homomorphism as desired. Finally, its uniqueness follows from the fact that q is an epimorphism since it is surjective.

The fact that the semidirect product is a coequaliser lets us see it as a quotient of $B * C$. More precisely, $C \rtimes_\phi B$ is the quotient of $B * C$ by the normal closure of $\{(bcb^{-1})\phi(b)(c)^{-1} : b \in B,\, c \in C\}$. In this quotient, we have $bc = \phi(b)(c)b$ for $b \in B$, $c \in C$ so we can rewrite its words by gathering the letters from C on the left and those from B on the right. If we do this, we easily see that the multiplication of this quotient corresponds to the one of the semidirect product from Reminder 2.1.3 as expected. More formally, using this fact, we can show that the "natural" map from this quotient to the semidirect product is an isomorphism (similarly to the proof of Lemma 3.2.1).

Now, we will show that, for an object B of a semi-abelian category $\mathcal{C}$, the categories $\mathcal{C}^{B\flat(-)}$ and $\mathrm{SSES}_{\mathcal{C}}(B)$ are equivalent as announced in the introduction of the chapter. Like for the semidirect products, this result was first obtained in [BJ98] but we chose to follow a different approach.

Let us describe this equivalence. Given a B-object (C, ξ), we associate it to the split short exact sequence forming the second line of the diagram below where the arrow α is given by the universal property of the coequaliser q of $\kappa_{B,C}$ and $\iota_2 \circ \xi$ (ι_1 and ι_2 being the inclusions in the coproduct $B + C$).

$$\begin{tikzcd}
0 \arrow[r] & B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, "\xi"'] & B + C \arrow[r, shift left, "\langle 1_B, 0\rangle"] \arrow[d, "q"] & B \arrow[l, shift left, "\iota_1"] \arrow[d, equal] & \\
0 \arrow[r] & C \arrow[r, "\kappa = q\circ\iota_2"'] \arrow[ur, "\iota_2"] & B \ltimes_\xi C \arrow[r, shift left, dashed, "\alpha"] & B \arrow[l, shift left, "\beta = q\circ\iota_1"] \arrow[r] & 0
\end{tikzcd}$$

Conversely, to a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd},$$

we associate the action $\xi\colon B\flat C \to C$ induced by the universal property of the kernel κ in the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, dashed, "\xi"'] & B+C \arrow[r, "\langle 1_B{,}0\rangle"] \arrow[d, "\langle\beta{,}\kappa\rangle"'] & B \arrow[d, equal] \\ 0 \arrow[r] & C \arrow[r, "\kappa"'] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

Theorem 4.2.4. *Let $\mathcal{C}$ be a semi-abelian category and B be an object of $\mathcal{C}$. Then the constructions above give us an equivalence between the category $\mathcal{C}^{B\flat(-)}$ of B-objects and the category $\mathrm{SSES}_{\mathcal{C}}(B)$ of split short exact sequences over B.*

To show this equivalence, we will again use Proposition 1.1.7 and construct a weak equivalence $F\colon \mathrm{SSES}_{\mathcal{C}}(B) \to \mathcal{C}^{B\flat(-)}$. Since the proof of this theorem is quite long, we will split it in the following subsections: the first one is dedicated to the construction of this weak equivalence F and we will show that this functor F is essentially surjective and fully faithful in the other two.

4.2.1 Construction of the Functor F

Given a split short exact sequence over B

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd},$$

we consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, dashed, "\xi"'] & B+C \arrow[r, "\langle 1_B{,}0\rangle"] \arrow[d, "\langle\beta{,}\kappa\rangle"'] & B \arrow[d, equal] \\ 0 \arrow[r] & C \arrow[r, "\kappa"'] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd} \tag{4.1}$$

where $\xi\colon B\flat C \to C$ is the uniquely induced arrow given by the universal property of the kernel κ (we easily check that the right square with α commutes), and set $F(\kappa,\alpha,\beta) = (C,\xi)$. We must check that (C,ξ) is indeed a B-object, i.e. that $\xi\circ\eta_C = 1_C$ and $\xi\circ B\flat\xi = \xi\circ\mu_C$. As a reminder, η_C and μ_C are given by the induced arrows in the diagrams below.

$$\begin{tikzcd}
C \arrow[d, dashed, "\eta_C"'] \arrow[dr, "\iota_2"] & & \\
B\flat C \arrow[r, "\kappa_{B,C}"'] & B+C \arrow[r, "\langle 1_B,0\rangle"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
B\flat(B\flat C) \arrow[r, "\kappa_{B,B\flat C}"] \arrow[d, dashed, "\mu_C"'] & B+(B\flat C) \arrow[r, "\langle 1_B,0\rangle"] \arrow[d, "\langle \iota_1,\kappa_{B,C}\rangle"'] & B \arrow[d, equal] \\
B\flat C \arrow[r, "\kappa_{B,C}"'] & B+C \arrow[r, "\langle 1_B,0\rangle"'] & B
\end{tikzcd} \tag{4.2}$$

- For the first equation, using the uniqueness of the universal property of the kernel κ (or the fact that κ is monic) on the diagram

$$\begin{tikzcd}
& C \arrow[d, dashed, "\eta_C"] \arrow[dr, "\iota_2"] \arrow[dd, dashed, bend right, "1_C"'] & & & \\
& B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, dashed, "\xi"] & B+C \arrow[r, "\langle 1_B,0\rangle"] \arrow[d, "\langle \beta,\kappa\rangle"'] & B \arrow[d, equal] & \\
0 \arrow[r] & C \arrow[r, "\kappa"'] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0
\end{tikzcd}$$

(obtained by combining the diagram (4.1) and the first diagram of (4.2)), we have directly the equality $\xi \circ \eta_C = 1_C$ since both $\xi \circ \eta_C$ and 1_C are equal to $\langle \beta, \kappa\rangle \circ \iota_2 = \kappa$ when postcomposed with κ.

- For the second equation, let us recall that $B\flat\xi$ is the image of $1_B + \xi$ by the kernel functor.

$$\begin{tikzcd}
B\flat(B\flat C) \arrow[r, "\kappa_{B,B\flat C}"] \arrow[d, dashed, "B\flat\xi"] & B+(B\flat C) \arrow[r, "\langle 1_B,0\rangle"] \arrow[d, "1_B+\xi"] & B \arrow[d, equal] \\
B\flat C \arrow[r, "\kappa_{B,C}"] & B+C \arrow[r, "\langle 1_B,0\rangle"] & B
\end{tikzcd}$$

Combining this diagram with the diagrams (4.1) and (4.2) (the second one), we obtain

$$\begin{tikzcd}
0 \arrow[r] & B\flat(B\flat C) \arrow[r, "\kappa_{B,B\flat C}"] \arrow[d, dashed, shift right, "\mu_C"'] \arrow[d, dashed, shift left, "B\flat\xi"] & B+(B\flat C) \arrow[r, "\langle 1_B,0\rangle"] \arrow[d, shift right, "\langle \iota_1,\kappa_{B,C}\rangle"'] \arrow[d, shift left, "1_B+\xi"] & B \arrow[d, equal] & \\
0 \arrow[r] & B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, dashed, "\xi"'] & B+C \arrow[r, "\langle 1_B,0\rangle"] \arrow[d, "\langle \beta,\kappa\rangle"'] & B \arrow[d, equal] & \\
0 \arrow[r] & C \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0
\end{tikzcd}.$$

Using the computation rules of Remark 3.1.5, we notice that both $\langle \beta, \kappa\rangle \circ \langle \iota_1, \kappa_{B,C}\rangle$ and $\langle \beta, \kappa\rangle \circ (1_B + \xi)$ are equal to $\langle \beta, \kappa \circ \xi\rangle$. Hence both $\xi \circ B\flat\xi$ and $\xi \circ \mu_C$ give us to $\langle \beta, \kappa \circ \xi\rangle \circ \kappa_{B,B\flat C}$ when postcomposed with κ, showing that $\xi \circ B\flat\xi = \xi \circ \mu_C$ as expected since κ is monic.

These two things shows that F is well defined on the objects.

For the arrows, given an arrow (f, g) in $\mathrm{SSES}_{\mathcal{C}}(B)$ as in

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "\kappa"] \arrow[d, "f"] & A \arrow[r, shift left, "\alpha"] \arrow[d, "g"] & B \arrow[l, shift left, "\beta"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C' \arrow[r, "\lambda"] & A' \arrow[r, shift left, "\gamma"] & B \arrow[l, shift left, "\delta"] \arrow[r] & 0
\end{tikzcd},$$

we set $F(f,g) = f\colon (C,\xi) \to (C',\zeta)$ where $\xi\colon B\flat C \to C$ and $\zeta\colon B\flat C' \to C'$ are the induced arrows given by the universal properties of κ and λ in the diagram below.

$$\begin{tikzcd}
B\flat C \arrow[rr, "\kappa_{B,C}"] \arrow[dr, dashed, "\xi"] \arrow[dd, "B\flat f"'] & & B+C \arrow[rr, "\langle 1_B, 0\rangle"] \arrow[dr, "\langle \beta, \kappa\rangle"] \arrow[dd, "1_B+f", near start] & & B \arrow[dr, equal] \arrow[dd, equal] & \\
& C \arrow[rr, "\kappa", near start] \arrow[dd, "f", near start] & & A \arrow[rr, shift left, "\alpha"] \arrow[dd, "g", near start] & & B \arrow[ll, shift left, "\beta"] \arrow[dd, equal] \\
B\flat C' \arrow[rr, "\kappa_{B,C'}", near end] \arrow[dr, dashed, "\zeta"'] & & B+C' \arrow[rr, "\langle 1_B, 0\rangle", near end] \arrow[dr, "\langle \delta, \lambda\rangle"'] & & B \arrow[dr, equal] & \\
& C' \arrow[rr, "\lambda"'] & & A' \arrow[rr, shift left, "\gamma"] & & B \arrow[ll, shift left, "\delta"]
\end{tikzcd}$$

Let us check that f is indeed a morphism of $B\flat(-)$-algebras, i.e. that $\zeta \circ B\flat f = f \circ \xi$. By construction of ξ and ζ, the upper and lower faces of the diagram above commute and, by definition of arrows in $\mathrm{SSES}_{\mathcal{C}}(B)$, the front faces commute too. By definition of $B\flat f$ and the computation rules of Remark 3.1.5, we also see that the back faces commute. For the middle face, using the computation rules and the commutativity of the front faces, we compute

$$\langle \delta, \lambda\rangle \circ (1_B + f) = \langle \delta \circ 1_B, \lambda \circ f\rangle = \langle \delta, g \circ \kappa\rangle = \langle g \circ \beta, g \circ \kappa\rangle = g \circ \langle \beta, \kappa\rangle$$

so it is also a commutative square. Finally, now that we know that all the squares commute except the left one, we compute

$$\begin{aligned}
\lambda \circ (f \circ \xi) = (g \circ \kappa) \circ \xi = g \circ (\langle \beta, \kappa\rangle \circ \kappa_{B,C}) &= \big(\langle \delta, \lambda\rangle \circ (1_B + f)\big) \circ \kappa_{B,C} \\
&= \langle \delta, \lambda\rangle \circ (\kappa_{B,C'} \circ B\flat f) = \lambda \circ (\zeta \circ B\flat f)
\end{aligned}$$

so $f \circ \xi = \zeta \circ B\flat f$ as desired since λ is a monomorphism.

The compatibility of F with composition and identities is immediate so it is indeed a functor.

4.2.2 F is Essentially Surjective

Let (C,ξ) be a B-object. By construction of F, we must find an object A and arrows κ, α, β in $\mathcal{C}$ in order to complete the following diagram with a split short exact sequence on the second line (making the two squares commute).

$$\begin{tikzcd}
0 \arrow[r] & B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, "\xi"'] & B+C \arrow[r, "\langle 1_B,0\rangle"] \arrow[d, dashed, "\langle\beta,\kappa\rangle"'] & B \arrow[d, equal] & \\
0 \arrow[r] & C \arrow[r, dashed, "\kappa"'] & A \arrow[r, dashed, shift left, "\alpha"] & B \arrow[l, dashed, shift left, "\beta"] \arrow[r] & 0
\end{tikzcd} \tag{4.3}$$

Let $\iota_1 \colon B \to B+C$ and $\iota_2 \colon C \to B+C$ be the inclusions in the coproduct $B+C$. As explained before, we take a semidirect product $q \colon B+C \to B \ltimes_\xi C$ of C by B. (Let us remind that it means that q is a coequaliser of $\kappa_{B,C}$ and $\iota_2 \circ \xi$.)

$$\begin{tikzcd}
0 \arrow[r] & B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, "\xi"'] & B+C \arrow[r, shift left, "\langle 1_B,0\rangle"] \arrow[d, "q"] & B \arrow[l, shift left, "\iota_1"] \arrow[d, equal] & \\
0 \arrow[r] & C \arrow[r, "\kappa=q\circ\iota_2"'] \arrow[ur, "\iota_2"] & B \ltimes_\xi C \arrow[r, dashed, shift left, "\alpha"] & B \arrow[l, shift left, "\beta=q\circ\iota_1"] \arrow[r] & 0
\end{tikzcd}$$

We set $\kappa = q \circ \iota_2$ and $\beta = q \circ \iota_1$. Since $\langle 1_B, 0\rangle \circ (\iota_2 \circ \xi) = 0 \circ \xi = 0 = \langle 1_B, 0\rangle \circ \kappa_{B,C}$, by the universal property of the coequaliser q, we have a unique arrow $\alpha \colon B \ltimes_\xi C \to B$ such that $\alpha \circ q = \langle 1_B, 0\rangle$. Now that we have a candidate, let us check that it is indeed a split short exact sequence that completes the diagram (4.3). By construction, it is easy to see that the squares commute (the upper triangle on the left does not commute but q coequalises $\kappa_{B,C}$ and $\iota_2 \circ \xi$ so the left square commutes as wanted) and that q is equal to the induced arrow $\langle \beta, \kappa\rangle$ (it follows directly from our choices of κ and β) so we only need to check that we have a split short exact sequence.

- For the splitting, we have $\alpha \circ \beta = (\alpha \circ q) \circ \iota_1 = \langle 1_B, 0\rangle \circ \iota_1 = 1_B$.
- Let us show that α is a cokernel of κ. First, we compute $\alpha \circ \kappa = (\alpha \circ q) \circ \iota_2 = \langle 1_B, 0\rangle \circ \iota_2 = 0$. Then, given $f \colon B \ltimes_\xi C \to D$ such that $f \circ \kappa = 0$, we must show that we have a unique arrow $\varphi \colon B \to D$ such that $\varphi \circ \alpha = f$.

$$\begin{tikzcd}
C \arrow[r, "\kappa"] & B \ltimes_\xi C \arrow[r, "\alpha"] \arrow[dr, "f"'] & B \arrow[d, dashed, "\exists!\varphi\,?"] \\
& & D
\end{tikzcd}$$

Let us set $\varphi = f\circ\beta$. We compute $(\varphi\circ\alpha)\circ\kappa = 0 = f\circ\kappa$ and $(\varphi\circ\alpha)\circ\beta = \varphi\circ 1_B = f\circ\beta$. Moreover, since we have $\kappa = q\circ\iota_2$ and $\beta = q\circ\iota_1$ with q an epimorphism and ι_2, ι_1 jointly epic (by the uniqueness of the universal property of the coproduct), κ and β are jointly epic by Lemma 1.3.4 so $\varphi\circ\alpha = f$ as wanted. Finally, the uniqueness of φ follows since α is epic (because it is split by β), showing that α is a cokernel of κ.

To conclude that we have a split short exact sequence, we must still show that κ is a kernel of α. In order to do this, we will show that κ is monic and use Lemma 4.1.7 to deduce that κ is a normal monomorphism, implying that it is a kernel of its cokernel α as wanted by Lemma 1.2.1.

First, let us show that ξ is a coequaliser of μ_C and $B\flat\xi$. We already know that $\xi \circ \mu_C = \xi \circ B\flat\xi$ since (C, ξ) is a $B\flat(-)$-algebra and, again by definition of $B\flat(-)$-algebra, we have $\xi \circ \eta_C = 1_C$. Moreover, using the commutativity of the diagram below and the fact that $\iota_2 = \kappa_{B,C} \circ \eta_C$ by construction of the monad $\big(B\flat(-), \eta, \mu\big)$ (see Corollary 3.1.8), we compute

$$\kappa_{B,C} \circ (B\flat\xi \circ \eta_{B\flat C}) = \Big((1_B + \xi) \circ \kappa_{B,B\flat C}\Big) \circ \eta_{B\flat C} = (1_B + \xi) \circ \widetilde{\iota_2} = \iota_2 \circ \xi = \kappa_{B,C} \circ (\eta_C \circ \xi),$$

where $\widetilde{\iota_2} \colon B\flat C \to B + (B\flat C)$ is the inclusion of $B\flat C$ in the coproduct $B + (B\flat C)$, and

$$\kappa_{B,C} \circ (\mu_C \circ \eta_{B\flat C}) = \Big(\langle \iota_1, \kappa_{B,C} \rangle \circ \kappa_{B,B\flat C}\Big) \circ \eta_{B\flat C} = \langle \iota_1, \kappa_{B,C} \rangle \circ \widetilde{\iota_2} = \kappa_{B,C}$$

with $\kappa_{B,C}$ a monomorphism so $B\flat\xi \circ \eta_{B\flat C} = \eta_C \circ \xi$ and $\mu_C \circ \eta_{B\flat C} = 1_{B\flat C}$. This shows that η_C and $\eta_{B\flat C}$ split the fork $(\mu_C, B\flat\xi, \xi)$ (see Definitions 1.2.17) so ξ is a coequaliser of μ_C and $B\flat\xi$ by Lemma 1.2.18.

$$\begin{tikzcd}
B\flat C \arrow[d, "\eta_{B\flat C}"'] \arrow[dr, "\widetilde{\iota_2}"] & \\
B\flat(B\flat C) \arrow[r, "\kappa_{B,B\flat C}"] \arrow[d, shift right, "\mu_C"'] \arrow[d, shift left, "B\flat\xi"] & B + (B\flat C) \arrow[d, shift right, "\langle \iota_1, \kappa_{B,C} \rangle"'] \arrow[d, shift left, "1_B+\xi"] \\
B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, "\xi"'] & B + C \arrow[d, "\langle \beta, \kappa \rangle"] \\
C \arrow[r, "\kappa"] & B \ltimes_\xi C
\end{tikzcd}$$

We also have $q = \langle \beta, \kappa \rangle = \operatorname{coeq}(\langle \iota_1, \kappa_{B,C} \rangle, 1_B + \xi)$. Indeed, let us recall that q was defined as a coequaliser of $\kappa_{B,C}$ and $\iota_2 \circ \xi$ so, by the computation rules of Remark 3.1.5,

$$q \circ \langle \iota_1, \kappa_{B,C} \rangle = \langle q \circ \iota_1, q \circ \kappa_{B,C} \rangle = \langle q \circ \iota_1, q \circ (\iota_2 \circ \xi) \rangle = q \circ \langle \iota_1, \iota_2 \circ \xi \rangle = q \circ (1_B + \xi).$$

Moreover, if we have an arrow $f \colon B + C \to D$ such that $f \circ \langle \iota_1, \kappa_{B,C} \rangle = f \circ (1_B + \xi)$, we also have $f \circ \kappa_{B,C} = f \circ (\iota_2 \circ \xi)$ by precomposing the previous equality with $\widetilde{\iota_2}$ (since $1_B + \xi = \langle \iota_1, \iota_2 \circ \xi \rangle$) so we have a unique factorisation $\varphi \colon D \to B \ltimes_\xi C$ such that

$\varphi \circ f = q$ by the universal property of $q = \mathrm{coeq}(\kappa_{B,C}, \iota_2 \circ \xi)$, showing that $q = \langle \beta, \kappa \rangle$ is also a coequaliser of $\langle \iota_1, \kappa_{B,C} \rangle$ and $1_B + \xi$.

$$\begin{tikzcd}
B + (B \flat C) \arrow[r, shift left, "\langle \iota_1, \kappa_{B,C} \rangle"] \arrow[r, shift right, "1_B + \xi"'] & B + C \arrow[r, "q"] \arrow[dr, "f"'] & B \ltimes_\xi C \arrow[d, dashed, "\varphi"] \\
& & D
\end{tikzcd}$$

Now, we will use Lemma 1.2.15 to show that the two squares

$$\begin{tikzcd}
B\flat(B\flat C) \arrow[r, "B\flat\xi"] \arrow[d, "\mu_C"'] & B\flat C \arrow[d, "\xi"] \\
B\flat C \arrow[r, "\xi"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
B + (B\flat C) \arrow[r, "1_B + \xi"] \arrow[d, "\langle \iota_1, \kappa_{B,C} \rangle"'] & B + C \arrow[d, "\langle \beta, \kappa \rangle"] \\
B + C \arrow[r, "\langle \beta, \kappa \rangle"'] & B \ltimes_\xi C
\end{tikzcd}$$

are pushouts in order to apply Corollary 1.2.6. For the right one, $1_B + \xi$ is split by $1_B + \eta_C$ by functoriality of $1_B + (-)$ and, by the computation rules of Remark 3.1.5, we also have

$$\langle \iota_1, \kappa_{B,C} \rangle \circ (1_B + \eta_C) = \langle \iota_1 \circ 1_B, \kappa_{B,C} \circ \eta_C \rangle = \langle \iota_1, \iota_2 \rangle = 1_{B+C}$$

so this square is a pushout by Lemma 1.2.15.

$$\begin{tikzcd}
B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, "B\flat\eta_C"'] & B + C \arrow[d, "1_B + \eta_C"] \\
B\flat(B\flat C) \arrow[r, "\kappa_{B,B\flat C}"] \arrow[d, "\mu_C"'] & B + (B\flat C) \arrow[d, "\langle \iota_1, \kappa_{B,C} \rangle"] \\
B\flat C \arrow[r, "\kappa_{B,C}"] & B + C
\end{tikzcd}$$

For the other one, $B\flat\xi$ is split by $B\flat\eta_C$ by functoriality of $B\flat(-)$ and, by the commutativity of the diagram above and the fact that $\kappa_{B,C} \circ \eta_C = \iota_2$ by construction of η, we compute

$$\begin{aligned}
\kappa_{B,C} \circ (\mu_C \circ B\flat\eta_C) &= (\langle \iota_1, \kappa_{B,C} \rangle \circ \kappa_{B,B\flat C}) \circ B\flat\eta_C = \langle \iota_1, \kappa_{B,C} \rangle \circ \big((1_B + \eta_C) \circ \kappa_{B,C} \big) \\
&= \langle \iota_1, \kappa_{B,C} \circ \eta_C \rangle \circ \kappa_{B,C} = \langle \iota_1, \iota_2 \rangle \circ \kappa_{B,C} = 1_{B+C} \circ \kappa_{B,C} \\
&= \kappa_{B,C} \circ 1_{B\flat C}.
\end{aligned}$$

Thus we also have $\mu_C \circ B\flat\eta_C = 1_{B\flat C}$ since $\kappa_{B,C}$ is monic and we can again apply Lemma 1.2.15 to deduce that we have another pushout as wanted. Therefore, by Corollary 1.2.6, we have that $\xi = \mathrm{coker}(B\flat\xi \circ \ker \mu_C)$ and $\langle \beta, \kappa \rangle = \mathrm{coker}\big((1_B + \xi) \circ \ker \langle \iota_1, \kappa_{B,C} \rangle \big)$ since

μ_C and $\langle \iota_1, \kappa_{B,C} \rangle$ are cokernels of their kernels by Lemma 1.3.7.

$$\begin{tikzcd}
B\flat(B\flat C) \arrow[r, "\kappa_{B,B\flat C}"] \arrow[d, "\mu_C"'] \arrow[dr, phantom, "(*)"] & B+(B\flat C) \arrow[r, "\langle 1_B,0\rangle"] \arrow[d, "\langle \iota_1,\kappa_{B,C}\rangle"] & B \arrow[d, equal] \\
B\flat C \arrow[r, "\kappa_{B,C}"'] & B+C \arrow[r, "\langle 1_B,0\rangle"'] & B
\end{tikzcd}$$

Moreover, by Lemma 1.2.7, the square $(*)$ is a pullback so the kernels of μ_C and $\langle \iota_1, \kappa_{B,C} \rangle$ are such that $\ker\langle \iota_1, \kappa_{B,C} \rangle = \kappa_{B,B\flat C} \circ \ker \mu_C$ by Lemma 1.2.4.

Now, let us consider the regular epimorphism/monomorphism factorisations

$$\operatorname{Ker} \mu_C \overset{a'}{\twoheadrightarrow} I \overset{a}{\rightarrowtail} B\flat C \qquad \text{and} \qquad \operatorname{Ker}\langle \iota_1, \kappa_{B,C} \rangle \overset{b'}{\twoheadrightarrow} J \overset{b}{\rightarrowtail} B+C$$

of $B\flat\xi \circ \ker \mu_C$ and $(1_B+\xi) \circ \ker\langle \iota_1, \kappa_{B,C} \rangle$, respectively, given by definition of a regular category (see Definitions 4.1.1). By Lemma 1.2.9, ξ and $\langle \beta, \kappa \rangle$ are cokernels of a and b, respectively.

$$\begin{tikzcd}
 & \operatorname{Ker} \mu_C \arrow[r, equal] \arrow[d, tail, "\ker \mu_C"'] \arrow[dl, bend right, two heads, "a'"'] & \operatorname{Ker}\langle \iota_1, \kappa_{B,C} \rangle \arrow[d, tail, "\ker\langle \iota_1,\kappa_{B,C}\rangle"] \arrow[dr, bend left, two heads, "b'"] & \\
I \arrow[dr, bend right, tail, "a"'] & B\flat(B\flat C) \arrow[r, "\kappa_{B,B\flat C}"] \arrow[d, shift right, "\mu_C"'] \arrow[d, shift left, "B\flat\xi"] & B+(B\flat C) \arrow[d, shift right, "\langle \iota_1,\kappa_{B,C}\rangle"'] \arrow[d, shift left, "1_B+\xi"] & J \arrow[dl, bend left, tail, "b"] \\
 & B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, two heads, "\xi"'] & B+C \arrow[d, two heads, "\langle \beta,\kappa\rangle"] & \\
 & C \arrow[r, "\kappa"] & B \ltimes_\xi C &
\end{tikzcd} \tag{4.4}$$

Moreover, applying Lemma 4.1.7 on the squares

$$\begin{tikzcd}
\operatorname{Ker} \mu_C \arrow[r, two heads, "a'"] \arrow[d, tail, "\ker \mu_C"'] & I \arrow[d, tail, "a"] \\
B\flat(B\flat C) \arrow[r, two heads, "B\flat\xi"'] & B\flat C
\end{tikzcd} \qquad \begin{tikzcd}
\operatorname{Ker}\langle \iota_1, \kappa_{B,C} \rangle \arrow[r, two heads, "b'"] \arrow[d, tail, "\ker\langle \iota_1,\kappa_{B,C}\rangle"'] & J \arrow[d, tail, "b"] \\
B+(B\flat C) \arrow[r, two heads, "1_B+\xi"'] & B+C
\end{tikzcd}$$

where $B\flat\xi$, $1_B+\xi$ are normal epimorphisms by Lemma 1.3.7 since they are split and a', b' are also normal epimorphisms by Lemma 1.3.11, we have that a and b are normal monomorphisms.

Let us take a pullback $(p_1 \colon P \to C, p_2 \colon P \to B+C)$ of κ and $\langle \beta, \kappa \rangle$. Then, by its universal property, we have a unique arrow $\varphi \colon B\flat C \to P$ that makes the diagram

$$\begin{tikzcd}
B\flat C \arrow[drr, bend left, "\kappa_{B,C}"] \arrow[dr, dashed, "\varphi"] \arrow[ddr, bend right, "\xi"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] & B+C \arrow[d, "\langle\beta,\kappa\rangle"] \\
& C \arrow[r, "\kappa"'] & B \ltimes_\xi C
\end{tikzcd}$$

commute. We will show that this induced arrow φ is an isomorphism so κ will be a monomorphism as a reflection of the monomorphism $\kappa_{B,C}$ by a pullback (see Lemma 1.3.13). By Lemma 1.2.4, a kernel of p_1 (which exists since we are in a finitely complete category) is given by an arrow $\ker p_1 \colon J \to P$ such that $p_2 \circ \ker p_1 = b$ since b is a kernel of its cokernel $\langle\beta, \kappa\rangle$ by Lemma 1.2.1. Consider the following square.

$$\begin{tikzcd}
\operatorname{Ker}\mu_C = \operatorname{Ker}\langle\iota_1, \kappa_{B,C}\rangle \arrow[r, two heads, "a'"] \arrow[d, "b'"'] & I \arrow[d, "\varphi\circ a"] \\
J \arrow[r, tail, "\ker p_1"'] & P
\end{tikzcd}$$

Using the definition of φ and various equalities involved in the diagram (4.4), we compute

$$p_1 \circ \Big((\varphi \circ a) \circ a'\Big) = \xi \circ (B\flat\xi \circ \ker \mu_C) = (\xi \circ \mu_C) \circ \ker \mu_C = 0 = p_1 \circ (\ker p_1 \circ b')$$

and

$$\begin{aligned} p_2 \circ \Big((\varphi \circ a) \circ a'\Big) &= \kappa_{B,C} \circ (B\flat\xi \circ \ker \mu_C) = \Big((1_B + \xi) \circ \kappa_{B,B\flat C}\Big) \circ \ker \mu_C \\ &= (1_B + \xi) \circ \ker\langle\iota_1, \kappa_{B,C}\rangle = b \circ b' = p_2 \circ (\ker p_1 \circ b') \end{aligned}$$

so the square above commutes since p_1 and p_2 are jointly monic as the projections of a pullback. Since a' is strongly epic by Lemma 1.2.19, this square gives us a factorisation $j \colon I \to J$ such that $j \circ a' = b'$ and $\ker p_1 \circ j = \varphi \circ a$. Moreover, by Lemma 1.2.22, this first equality implies that j is extremally epic since b' is a regular epimorphism so it is an extremal epimorphism by Lemma 1.2.21. Using the developments above, we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "a"] \arrow[d, two heads, "j"] & B\flat C \arrow[r, shift left, "\xi"] \arrow[d, "\varphi"] & C \arrow[l, shift left, "\eta_C"] \arrow[d, equal] \\
0 \arrow[r] & J \arrow[r, "\ker p_1"] & P \arrow[r, shift left, "p_1"] & C \arrow[l, shift left, "\varphi\circ\eta_C"]
\end{tikzcd}$$

on which we can apply Corollary 1.3.16 to deduce that φ is an extremal epimorphism. Moreover, since $p_2 \circ \varphi = \kappa_{B,C}$ is monic, φ is also monic so it is an isomorphism by Lemma 1.2.23, implying that κ is a monomorphism by Lemma 1.3.13. To finish the proof that

κ is a kernel of α, we use Lemma 4.1.7 on the square

$$\begin{tikzcd}
B\flat C \arrow[r, tail, "\kappa_{B,C}"] \arrow[d, two heads, "\xi"'] & B+C \arrow[d, two heads, "\langle\beta,\kappa\rangle"] \\
C \arrow[r, tail, "\kappa"'] & B \ltimes_\xi C
\end{tikzcd}$$

to deduce that κ is a normal monomorphism so it is a kernel of its cokernel α as desired by Lemma 1.2.1.

4.2.3 F is Fully Faithful

We must show that, given an arrow $f\colon F(\kappa,\alpha,\beta) = (C,\xi) \to F(\lambda,\gamma,\delta) = (C',\zeta)$ where (κ,α,β) and (λ,γ,δ) are as in the diagram below, there is a unique arrow $(f',g)\colon(\kappa,\alpha,\beta)\to(\lambda,\gamma,\delta)$ such that $F(f',g)=f$.

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "\kappa"] \arrow[d, dashed, "f'"] & A \arrow[r, shift left, "\alpha"] \arrow[d, dashed, "g"] & B \arrow[l, shift left, "\beta"] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C' \arrow[r, "\lambda"] & A' \arrow[r, shift left, "\gamma"] & B \arrow[l, shift left, "\delta"] \arrow[r] & 0
\end{tikzcd}$$

Since $F(f',g)=f'$, we must set $f'=f$. Similarly to what we have done previously, we have the diagram

$$\begin{tikzcd}
B\flat C \arrow[rr, "\kappa_{B,C}"] \arrow[dr, "\xi"] \arrow[dd, "B\flat f"'] & & B+C \arrow[rr, "\langle 1_B,0\rangle"] \arrow[dr, "\langle\beta,\kappa\rangle"] \arrow[dd, "1_B+f"] & & B \arrow[dr, equal] \arrow[dd, equal] & \\
& C \arrow[rr, "\kappa"] \arrow[dd, "f"] & & A \arrow[rr, shift left, "\alpha"] \arrow[dd, dashed, "g"] & & B \arrow[ll, shift left, "\beta"] \arrow[dd, equal] \\
B\flat C' \arrow[rr, "\kappa_{B,C'}"] \arrow[dr, "\zeta"] & & B+C' \arrow[rr, "\langle 1_B,0\rangle"] \arrow[dr, "\langle\delta,\lambda\rangle"] & & B \arrow[dr, equal] & \\
& C' \arrow[rr, "\lambda"'] & & A' \arrow[rr, shift left, "\gamma"] & & B \arrow[ll, shift left, "\delta"]
\end{tikzcd}$$

and, this time, we already know that everything commutes (with the dashed arrow g removed) and we must show that there is a unique $g\colon A\to A'$ that makes the two front faces commute (the front right face having to commute in the two directions).

Let us take a coequaliser $q\colon B+C\to B\ltimes_\xi C$ of $\kappa_{B,C}$ and $\iota_2\circ\xi$ to construct a split short exact sequence like we have done to show the essential surjectivity of F in the previous subsection.

$$\begin{tikzcd}
0 \arrow[r] & B\flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, "\xi"'] & B+C \arrow[r, shift left, "\langle 1_B,0\rangle"] \arrow[d, "q"] & B \arrow[l, shift left, "\iota_1"] \arrow[d, equal] & \\
0 \arrow[r] & C \arrow[r, "\overline{\kappa}=q\circ\iota_2"'] \arrow[ur, "\iota_2"] & B \ltimes_\xi C \arrow[r, shift left, "\overline{\alpha}"] & B \arrow[l, shift left, "\overline{\beta}=q\circ\iota_1"] \arrow[r] & 0
\end{tikzcd}$$

Let us recall that $\overline{\alpha}$ is the unique arrow such that $\overline{\alpha}\circ q = \langle 1_B, 0\rangle$ given by the universal property of q. Since we have

$$\langle\beta,\kappa\rangle\circ\kappa_{B,C} = \kappa\circ\xi = \langle\beta,\kappa\rangle\circ(\iota_2\circ\xi),$$

by the universal property of q, we have a unique arrow $\varphi\colon B\ltimes_\xi C \to A$ such that $\varphi\circ q = \langle\beta,\kappa\rangle$.

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[rr, "\overline{\kappa}=q\circ\iota_2"] \arrow[dd, equal] \arrow[dr, "\iota_2"] & & B\ltimes_\xi C \arrow[rr, shift left, "\overline{\alpha}"] \arrow[dd, dashed, "\varphi"] & & B \arrow[ll, shift left, "\overline{\beta}=q\circ\iota_1"] \arrow[r] \arrow[dl, equal] \arrow[dd, equal] & 0 \\
B\flat C \arrow[ur, "\xi"] \arrow[dr, "\xi"'] \arrow[rr, "\kappa_{B,C}"] & & B+C \arrow[ur, "q"] \arrow[rr, shift left, "\langle 1_B,0\rangle"] \arrow[dr, "\langle\beta,\kappa\rangle"] & & B \arrow[ll, shift left, "\iota_1"] \arrow[dr, equal] & & \\
0 \arrow[r] & C \arrow[rr, "\kappa"'] & & A \arrow[rr, shift left, "\alpha"] & & B \arrow[ll, shift left, "\beta"] \arrow[r] & 0
\end{tikzcd}$$

Using the fact that q is epic, we check that this factorisation φ makes the two front squares commute (the right one having to commute in the two directions). Therefore, by the Split Short Five Lemma (Theorem 1.3.14), φ is an isomorphism such that $\varphi\circ q = \langle\beta,\kappa\rangle$ so $\langle\beta,\kappa\rangle\colon B+C\to A$ is also a coequaliser of $\kappa_{B,C}$ and $\iota_2\circ\xi$.

Let $\iota_1'\colon B\to B+C'$ and $\iota_2'\colon C'\to B+C'$ denote the inclusions of B and C', respectively, in the coproduct $B+C'$. We compute

$$\begin{aligned}\big(\langle\delta,\lambda\rangle\circ(1_B+f)\big)\circ(\iota_2\circ\xi) &= \langle\delta,\lambda\rangle\circ(\iota_2'\circ f)\circ\xi = \lambda\circ f\circ\xi = \lambda\circ(\zeta\circ B\flat f)\\ &= (\langle\delta,\lambda\rangle\circ\kappa_{B,C'})\circ B\flat f = \big(\langle\delta,\lambda\rangle\circ(1_B+f)\big)\circ\kappa_{B,C}\end{aligned}$$

so, by the universal property of the coequaliser $\langle\beta,\kappa\rangle$ (we just showed that it is a coequaliser of $\kappa_{B,C}$ and $\iota_2\circ\xi$), we have an arrow $g\colon A\to A'$ such that $g\circ\langle\beta,\kappa\rangle =$

$\langle \delta, \lambda \rangle \circ (1_B + f)$.

$$
\begin{tikzcd}
B\flat C \arrow[rr, "\kappa_{B,C}"] \arrow[dr, "\xi"] \arrow[dd, "B\flat f"'] & & B+C \arrow[rr, shift left, "\langle 1_B{,}0_{C,B}\rangle"] \arrow[dr, "\langle \beta{,}\kappa\rangle"] \arrow[dd, "1_B+f" near end] & & B \arrow[ll, shift left, "\iota_1"] \arrow[dr, equal] \arrow[dd, equal] & \\
& C \arrow[ur, "\iota_2"] \arrow[rr, "\kappa" near start] \arrow[dd, "f" near start] & & A \arrow[rr, shift left, "\alpha"] \arrow[dd, dashed, "g" near start] & & B \arrow[ll, shift left, "\beta"] \arrow[dd, equal] \\
B\flat C' \arrow[rr, "\kappa_{B,C'}" near end] \arrow[dr, "\zeta"'] & & B+C' \arrow[rr, shift left, "\langle 1_B{,}0_{C',B}\rangle" near end] \arrow[dr, "\langle \delta{,}\lambda\rangle"] & & B \arrow[ll, shift left, "\iota'_1" near start] \arrow[dr, equal] & \\
& C' \arrow[ur, "\iota'_2"'] \arrow[rr, "\lambda"'] & & A' \arrow[rr, shift left, "\gamma"] & & B \arrow[ll, shift left, "\delta"]
\end{tikzcd}
$$

Let us check that this arrow g makes the front faces commute as wanted. We compute

- $g \circ \kappa = g \circ (\langle \beta, \kappa \rangle \circ \iota_2) = \big(\langle \delta, \lambda \rangle \circ (1_B + f)\big) \circ \iota_2 = \langle \delta, \lambda \rangle \circ (\iota'_2 \circ f) = \lambda \circ f$,
- $g \circ \beta = g \circ (\langle \beta, \kappa \rangle \circ \iota_1) = \big(\langle \delta, \lambda \rangle \circ (1_B + f)\big) \circ \iota_1 = \langle \delta, \lambda \rangle \circ (\iota'_1 \circ 1_B) = \delta$,
- $(\gamma \circ g) \circ \langle \beta, \kappa \rangle = \gamma \circ \big(\langle \delta, \lambda \rangle \circ (1_B + f)\big) = \langle 1_B, 0_{C',B} \rangle \circ (1_B + f) = \langle 1_B, 0_{C,B} \rangle = \alpha \circ \langle \beta, \kappa \rangle$ so $\gamma \circ g = \alpha$ since $\langle \beta, \kappa \rangle$ is epic as it is a regular epimorphism (as a coequaliser of $\kappa_{B,C}$ and $\iota_2 \circ \xi$).

Moreover, g is indeed the unique arrow making the front faces commute since κ and β are jointly epic by Lemma 1.3.3 combined with the definition of protomodularity (Definition 1.3.5). Therefore, this shows that F is fully faithful and finishes the proof of Theorem 4.2.4.

Chapter 5

Examples of Internal Actions

To end this thesis, let us look at some examples to apply the theory we developed in the previous chapters. First, we will look at actions in Examples 4.1.5 we did not already cover, $\mathrm{Set}_*^{\mathrm{op}}$ and abelian categories. Then we will present how we can use the equivalence of Theorem 4.2.4 to generalise two kinds of actions well-known in some particular contexts (like Grp, for example), *trivial actions* and *conjugation actions*. In the fifth subsection, we will see how we can change the object that is acting and how this so-called *change of base* behaves. Finally, we will take a very quick look at actions of monoids, which is an example where we cannot apply our theory since Mon is not semi-abelian (we have seen in Example 1.3.8 that it is not even protomodular).

5.1 Internal Actions in $\mathrm{Set}_*^{\mathrm{op}}$

We will construct the monad $B\flat(-)$ and its algebras from the point of view of its opposite category, Set_*. In order to avoid heavy notations, we will omit the op's and the base points when it is not necessary.

Given $(B, *_B), (C, *_C) \in \mathrm{Set}_*$, the cokernel $\kappa_{B,C} \colon B \times C \to B\flat C$ of $(1_B, 0) \colon B \to B \times C$ is given by the canonical projection from $B \times C$ to $B\flat C$ where $B\flat C$ is the quotient of $B \times C$ in which we identify all the couples of $B \times C$ of the form $(b, *_C)$. Given an arrow $g \colon C \to C'$ in Set_*, its image $B\flat g \colon B\flat C \to B\flat C'$ is the arrow between the quotients $B\flat C$ and $B\flat C'$ induced by the map $1_B \times g \colon (b, c) \in B \times C \mapsto \big(b, g(c)\big)$.

$$\begin{tikzcd}
B \arrow[r, "{(1_B,0)}"] \arrow[d, equal] & B \times C \arrow[r, "\kappa_{B,C}"] \arrow[d, "1_B \times g"] & B\flat C \arrow[d, dashed, "B\flat g"] \\
B \arrow[r, "{(1_B,0)}"] & B \times C' \arrow[r, "\kappa_{B,C'}"] & B\flat C'
\end{tikzcd}$$

For $C \in \mathrm{Set}_*$, the components of the natural transformations $\eta \colon B\flat(-) \to 1_{\mathrm{Set}_*}$ and $\mu \colon B\flat(-) \to \big(B\flat(-)\big)^2$ are given as follows (see Corollary 3.1.8):

- $\eta_C \colon [(b,c)] \in B\flat C \mapsto c \in C$,
- $\mu_C \colon [(b,c)] \in B\flat C \mapsto \Big[\Big(b, [(b,c)]\Big)\Big] \in B\flat(B\flat C)$.

$$\begin{tikzcd} B \arrow[r, "(1_B{,}0)"] & B \times C \arrow[r, "\kappa_{B,C}"] \arrow[dr, "\pi_2"'] & B\flat C \arrow[d, dashed, "\eta_C"] \\ & & C \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "(1_B{,}0)"] \arrow[d, equal] & B \times C \arrow[r, "\kappa_{B,C}"] \arrow[d, "(\pi_1{,}\kappa_{B,C})"] & B\flat C \arrow[d, dashed, "\mu_C"] \\ B \arrow[r, "(1_B{,}0)"] & B \times (B\flat C) \arrow[r, "\kappa_{B,B\flat C}"] & B\flat(B\flat C) \end{tikzcd}$$

Now, let us look at the algebras of $B\flat(-)$. Such an algebra is an object $(C, *_C) \in \mathrm{Set}_*$ together with an arrow $\xi \colon C \to B\flat C$ satisfying the equations $B\flat\xi \circ \xi = \mu_C \circ \xi$ and $\eta_C \circ \xi = 1_C$.

$$\begin{tikzcd} C \arrow[r, "\xi"] \arrow[d, "\xi"'] & B\flat C \arrow[d, "B\flat\xi"] \\ B\flat C \arrow[r, "\mu_C"'] & B\flat(B\flat C) \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\xi"] \arrow[dr, equal] & B\flat C \arrow[d, "\eta_C"] \\ & C \end{tikzcd}$$

The second equation means that, if we write $\xi(c) = [(b, \tilde{c})]$, we must have $\tilde{c} = c$. Therefore, we can write $\xi \colon c \in C \mapsto \Big[\Big(\phi(c), c\Big)\Big] \in B\flat C$ for a certain map $\phi \colon C \to B$. Moreover, since all the couples of $B \times C$ of the form $(b, *_C)$ are identified in the quotient $B\flat C$ and all other couples are the unique representative of their class, we have a unique way of choosing ϕ such that it is a pointed map from $(C, *_C)$ to $(B, *_B)$. Now, let us look at the first equation. We compute, for $c \in C$,

$$\begin{aligned} (B\flat\xi)\Big(\xi(c)\Big) = (B\flat\xi)\Big(\Big[\Big(\phi(c), c\Big)\Big]\Big) = \Big[\Big(\phi(c), \Big[\Big(\phi(c), c\Big)\Big]\Big)\Big] = \mu_C\Big(\Big[\Big(\phi(c), c\Big)\Big]\Big) \\ = \mu_C\Big(\xi(c)\Big) \end{aligned}$$

so this equation does not tell us anything new.

A morphism $f \colon (C, \xi) \to (C', \zeta)$ of $B\flat(-)$-algebras is an arrow $f \colon C \to C'$ between the underlying objects making the diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\xi"'] & C' \arrow[d, "\zeta"] \\ B\flat C \arrow[r, "B\flat f"'] & B\flat C' \end{tikzcd}$$

commute. Let $\phi \colon C \to B$ and $\psi \colon C' \to B$ be the unique pointed maps such that

$$\xi \colon c \in C \mapsto \Big[\Big(\phi(c), c\Big)\Big] \in B\flat C \quad \text{and} \quad \zeta \colon c' \in C' \mapsto \Big[\Big(\psi(c'), c'\Big)\Big] \in B\flat C'.$$

Then the commutativity of the diagram above means that, for all $c \in C$,

$$\zeta\Big(f(c)\Big) = \Big[\Big(\psi\Big(f(c)\Big), f(c)\Big)\Big] \quad \text{and} \quad (B\flat f)\Big(\xi(c)\Big) = \Big[\Big(\phi(c), f(c)\Big)\Big]$$

must be equal. Since it is always true for the second component, this equation tells us that we must have $\psi \circ f = \phi$ on all C except the kernel of f (i.e. except for $c \in C$ such that $f(c)$ is the base point $*_{C'}$ of C') for which we have no condition since all couples of the form $(b, *_{C'})$ are identified in the quotient $B\flat C'$. Therefore, a morphism $f\colon (C,\xi) \to (C',\zeta)$ of $B\flat(-)$-algebras is an arrow $f\colon C \to C'$ in Set_* such that, for all $c \in C \setminus \operatorname{Ker} f$, $\psi\big(f(c)\big) = \phi(c)$.

Now, let us look at the equivalence between these actions and the split short exact sequences given by Theorem 4.2.4.

Consider a B-object (C,ξ) and let $\phi\colon C \to B$ be the unique pointed map such that $\xi\colon c \in C \mapsto \Big[\big(\phi(c), c\big)\Big] \in B\flat C$. Its semidirect product is the equaliser of $\kappa_{B,C}$ and $\xi \circ \pi_2$, i.e. we have

$$\begin{aligned} B \ltimes_\xi C &= \Big\{(b,c) \in B \times C \mid \kappa_{B,C}(b,c) = \xi\big(\pi_2(b,c)\big)\Big\} = \{(b,c) \in B \times C \mid [(b,c)] = \xi(c)\} \\ &= \Big\{(b,c) \in B \times C \mid [(b,c)] = \Big[\big(\phi(c), c\big)\Big]\Big\} \\ &= \{(b,c) \in B \times C \mid b = \phi(c)\} \cup \{(b,c) \in B \times C \mid c = *_C\}. \end{aligned}$$

Therefore, $B \ltimes_\xi C$ is the union the graph of ϕ (with the factors of $B \times C$ swapped) and the axis $\{(b,c) \in B \times C \mid c = *_C\}$. Then the split short exact sequence corresponding to (C,ξ) is

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, shift left, "\alpha"] & B \ltimes_\xi C \arrow[l, shift left, "\beta"] \arrow[r, "\kappa"] & C \arrow[r] & 0 \end{tikzcd}$$

where α, β and κ are given by $\alpha(b) = (b, *_C)$, $\beta(b,c) = b$ and $\kappa(b,c) = c$.

Conversely, given a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, shift left, "\alpha"] & A \arrow[l, shift left, "\beta"] \arrow[r, "\kappa"] & C \arrow[r] & 0 \end{tikzcd}$$

in $\mathrm{Set}_*^{\mathrm{op}}$, it induces an action ξ of B on C as in the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, shift left, "\alpha"] \arrow[d, equal] & A \arrow[l, shift left, "\beta"] \arrow[r, "\kappa"] \arrow[d, "{(\beta,\kappa)}"] & C \arrow[r] \arrow[d, dashed, "\xi"] & 0 \\ & B \arrow[r, "{(1_B,0)}"'] & B \times C \arrow[r, "\kappa_{B,C}"'] & B\flat C \arrow[r] & 0 \end{tikzcd}$$

Explicitly, since κ is a normal epimorphism, it has a unique splitting $\delta\colon C \to A$ and this splitting is such that $\delta\big(\kappa(a)\big) = a$ for all $a \in A$ that is not sent to the base point of C by κ (i.e. for $a \notin \operatorname{Ker}\kappa$). Then the action ξ is given by $\xi(c) = \Big[\Big(\beta\big(\delta(c)\big), c\Big)\Big]$ for $c \in C$. In terms of pointed maps from C to B, this action corresponds to the pointed map $\phi = \beta \circ \delta$.

5.2 Internal Actions in Abelian Categories

In an abelian category $\mathcal{C}$, every coproduct is a biproduct and we have that, for $B, C \in \mathcal{C}$, a kernel of $\langle 1_B, 0\rangle\colon B \oplus C \to B$, which is in fact the projection $\pi_1\colon B \oplus C \to B$ of $B \oplus C$ on B, is given by the inclusion $\iota_2\colon C \to B \oplus C$ so we can choose $B\flat C = C$. Then we also see that, given an arrow $g\colon C \to C'$ in $\mathcal{C}$, its image by the functor $B\flat(-)$ is g itself so the functor $B\flat(-)$ is simply the identity functor.

$$\begin{tikzcd}
B\flat C = C \arrow[r, "\kappa_{B,C}=\iota_2"] \arrow[d, dashed, "B\flat g = g"] & B \oplus C \arrow[r, "\langle 1_B, 0\rangle"] \arrow[d, "1_B + g"] & B \arrow[d, equal] \\
B\flat C' = C \arrow[r, "\kappa_{B,C'}=\iota_2'"] & B \oplus C' \arrow[r, "\langle 1_B, 0\rangle"] & B
\end{tikzcd}$$

Since we may choose kernels in such a way that $B\flat(-)$ is the identity functor, it allows us to easily describe its algebras. By definition, such an algebra is a couple (C, ξ) where C is an object of $\mathcal{C}$ and $\xi\colon B\flat C \to C$ is an arrow such that $\xi \circ B\flat\xi = \xi \circ \mu_C$ and $\xi \circ \eta_C = 1_C$. By Corollary 3.1.8, η_C is the uniquely induced arrow in the diagram

$$\begin{tikzcd}
C \arrow[d, dashed, "\eta_C"'] \arrow[dr, "\iota_2"] & & \\
B\flat C = C \arrow[r, "\kappa_{B,C}"'] & B \oplus C \arrow[r, "\langle 1_B, 0\rangle"'] & B
\end{tikzcd}$$

but the arrow $\kappa_{B,C}\colon B\flat C \to B \oplus C$ is also the inclusion ι_2 so we deduce that η_C is the identity 1_C. Then the equation $\xi \circ \eta_C = 1_C$ tells us that ξ must be a splitting of $\eta_C = 1_C$ so we find $\xi = 1_C$. This shows that every action in an abelian category is trivial. Moreover, $\xi = 1_C$ also verifies $\xi \circ B\flat\xi = \xi \circ \mu_C$ since we can check that μ_C is given by 1_C so these trivial actions always exist.

For the morphisms, we remark that, since the actions are trivial and $B\flat(-)$ is the identity functor, the equation they must verify is always satisfied so these morphisms are simply arrows in $\mathcal{C}$ with no other condition.

$$\begin{tikzcd}
B\flat C = C \arrow[r, "B\flat f = f"] \arrow[d, "\xi = 1_C"'] & B\flat C' \arrow[d, "\zeta = 1_{C'}"] \\
C \arrow[r, "f"'] & C'
\end{tikzcd}$$

These two things show us that the category $\mathcal{C}^{B\flat(-)}$ of B-objects is isomorphic to the underlying category $\mathcal{C}$. Let us remark that we only use the fact that $B\flat(-)$ is the identity functor to show that the category $\mathcal{C}^{B\flat(-)}$ of its algebras is isomorphic to $\mathcal{C}$, the assumption that $\mathcal{C}$ is abelian is only needed to show that $B\flat(-) = 1_{\mathcal{C}}$, not in the rest of the argument. Moreover, to show that we have the identity functor, we only used the fact that our coproducts are biproducts. To summarise, the fact that every action is trivial in an abelian category follows from the combination of the three following independent properties :

- any abelian category has binary biproducts,
- the existence of binary biproducts (with B) implies that $B\flat(-)$ is the identity functor and, more precisely, is part of a monad whose unit and multiplication are $1_{1_{\mathcal{C}}}$,
- the Eilenberg–Moore category $\mathcal{C}^{1_{\mathcal{C}}}$ of a monad of the form $(1_{\mathcal{C}}, 1_{1_{\mathcal{C}}}, 1_{1_{\mathcal{C}}})$ is isomorphic to the category $\mathcal{C}$ itself.

(In fact, this reasoning holds in an additive category (see [Bor94, Section 1.2]) : see [BJ98, Example 3.3].)

In terms of split short exact sequences, the fact that every action is trivial corresponds to the fact that every split short exact sequence in an abelian category is of the form

$$0 \longrightarrow C \underset{\iota_2}{\overset{\pi_2}{\leftrightarrows}} B \oplus C \underset{\iota_1}{\overset{\pi_1}{\rightleftarrows}} B \longrightarrow 0$$

where ι_1, ι_2 and π_1, π_2 are the inclusions and projections of the biproduct $B \oplus C$, respectively.

5.3 Trivial Actions

In the previous example, we said that every action in an abelian category is trivial but, even if we can intuitively understand why these actions can be considered as trivial, we did not give a clear definition of what we meant by calling an action trivial. Therefore, in this subsection, we will try to give a formal definition of this notion.

5.3.1 Definition

Since the form of the monad $B\flat(-)$ can widely vary based on the category we consider, we will use the equivalence of categories of Theorem 4.2.4 to define the general notion of trivial action in terms of split short exact sequences.

In the previous section, we have seen that trivial actions in an abelian category correspond to split short exact sequences of the form

$$0 \longrightarrow C \underset{\iota_2}{\overset{\pi_2}{\leftrightarrows}} B \oplus C \underset{\iota_1}{\overset{\pi_1}{\rightleftarrows}} B \longrightarrow 0$$

(where ι_1, ι_2 and π_1, π_2 are the inclusions and projections of the biproduct $B \oplus C$, respectively) but, in an arbitrary semi-abelian category, biproducts do not necessarily exist (in fact, while looking at the abelian context, we have seen that their existence implies that every action is trivial) so we must determine if we must use products or coproducts for the generalisation we are looking for. For that, let us look at trivial actions in Grp. A trivial action in Grp is a group homomorphism of the form $\phi\colon B \to \operatorname{Aut} C\colon b \mapsto 1_C$ and, in terms of split short exact sequences, by Proposition 2.1.4, it corresponds to the split short exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "{(0,1_C)}"] & B \times C \arrow[r, shift left, "\pi_1"] & B \arrow[l, shift left, "{(1_B,0)}"] \arrow[r] & 0
\end{tikzcd}
$$

where π_1 is the projection of $B \times C$ on B. Thus, with this example in Grp in mind, we use these split short exact sequences of the form

$$
\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "{(0,1_C)}"] & B \times C \arrow[r, shift left, "\pi_1"] & B \arrow[l, shift left, "{(1_B,0)}"] \arrow[r] & 0
\end{tikzcd} ,
$$

π_1 being the projection of $B \times C$ on B, as the general definition of a *trivial action* via the equivalence of categories of Theorem 4.2.4. (Let us note that split short exact sequences of this form always exist since we only need a binary product, which exists since semi-abelian categories are finitely complete. Therefore, for all B, $C \in \mathcal{C}$, there is always a trivial action of B on C.)

Furthermore, since we have a product in the middle of these split short exact sequences, we can give an explicit formula for trivial actions as follows: such a trivial action is induced as in a diagram of the form

$$
\begin{tikzcd}
0 \arrow[r] & B \flat C \arrow[r, "\kappa_{B,C}"] \arrow[d, dashed, "\xi"'] & B + C \arrow[r, "{\langle 1_B,0\rangle}"] \arrow[d, "{\langle (1_B,0),(0,1_C)\rangle}"'] & B \arrow[d, equal] & \\
0 \arrow[r] & C \arrow[r, "{(0,1_C)}"'] & B \times C \arrow[r, shift left, "\pi_1"] & B \arrow[l, shift left, "{(1_B,0)}"] \arrow[r] & 0
\end{tikzcd}
$$

so, using the identity $\langle (f_{11}, f_{21}), (f_{12}, f_{22}) \rangle = (\langle f_{11}, f_{12} \rangle, \langle f_{21}, f_{22} \rangle)$ (whenever it makes sense) and the commutativity of the left square, we compute

$$
\begin{aligned}
\xi &= \pi_2 \circ (0, 1_C) \circ \xi = \pi_2 \circ \Big(\langle (1_B, 0), (0, 1_C) \rangle \circ \kappa_{B,C} \Big) = \pi_2 \circ (\langle 1_B, 0 \rangle, \langle 0, 1_C \rangle) \circ \kappa_{B,C} \\
&= \langle 0, 1_C \rangle \circ \kappa_{B,C}
\end{aligned}
$$

and we check that it indeed corresponds to trivial actions in abelian categories, Grp and the two following examples.

5.3.2 CRng

Now, let us apply this new definition in CRng. By Proposition 2.2.7, a split short sequence as above gives us an action $\phi\colon B \otimes C \to C$ of B on C defined by the formula $\phi(b \otimes c) = (1_B, 0)(b)c = 0$ where the multiplication $(1_B, 0)(b)c$ and the last equality are made via the identification $C \xrightarrow{\cong} \operatorname{Ker} \pi_1 = \{0\} \times C \leq B \times C\colon c \mapsto (0, c)$. Therefore, a trivial action of B on C in CRng is simply a zero arrow $0\colon B \otimes C \to C$. Let us remark that, even if we also obtain a zero arrow $0\colon B \to \operatorname{Aut} C\colon b \mapsto 1_C$ for trivial actions in Grp, the associated set-theoretical maps from the Cartesian product $B \times C$ to C are quite different : in Grp, we have $(b, c) \in B \times C \mapsto c \in C$ whereas this map is $(b, c) \in B \times C \mapsto 0 \in C$ in CRng.

5.3.3 $\mathrm{Set}_*^{\mathrm{op}}$

Let us now examine trivial actions in $\mathrm{Set}_*^{\mathrm{op}}$. By definition, a trivial action in $\mathrm{Set}_*^{\mathrm{op}}$ corresponds to a split short exact sequence in Set_* of the form

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, shift left, "\iota_1"] & B + C \arrow[l, shift left, "\langle 1_B, 0\rangle"] \arrow[r, "\langle 0, 1_C\rangle"] & C \arrow[r] & 0 \end{tikzcd}$$

where ι_1 denotes the inclusion of B in $B + C$. Since the unique pointed splitting of $\langle 0, 1_C\rangle$ is the inclusion ι_2 of C in $B + C$, by the developments we made in Section 5.1, the action corresponding to this split short exact sequence is the pointed map $\phi\colon C \to B\colon c \mapsto \langle 1_B, 0\rangle\big(\iota_2(c)\big) = *_B$ (where $*_B$ is the base point of B), which is again a zero arrow, as we could have expected.

5.3.4 Actions of Trivial Objects

We remark that, in Grp, CRng and $\mathrm{Set}_*^{\mathrm{op}}$, every action of a trivial object is trivial and, not surprisingly, it is a general fact as we will see in this subsection. In terms of split short exact sequences, if we have a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

with $B = 0$, we immediately have that α and β are zero arrows and, since $\alpha = 0$, we also deduce that $\kappa = \ker \alpha$ is an isomorphism. Since the only choice (the choice of κ) we have is unique up to isomorphism, this shows that, if $B = 0$, all the split short exact sequences of the form

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

are isomorphic to each other and, in particular, are isomorphic to

$$0 \longrightarrow C \xrightarrow{(0,1_C)} B \times C \underset{(1_B,0)}{\overset{\pi_1}{\rightleftarrows}} B \longrightarrow 0$$

as wanted. From the point of view of $B\flat(-)$-algebras, for each $C \in \mathcal{C}$, the biproduct $B \oplus C$ exists since B is trivial so, by the observation at the end of Section 5.2, every action of B is trivial and, more precisely, the category $\mathcal{C}^{B\flat(-)}$ of B-objects is isomorphic to the underlying category $\mathcal{C}$.

5.3.5 Actions on Trivial Objects

In the previous subsection, we have seen that an action of B on C is trivial if $B = 0$. Let us now look at what happens when it is C that is trivial. Since an action is an arrow from $B\flat C$ to C and $C = 0$ is terminal, we have a unique action $\phi\colon B\flat C \to C$ and this action is trivial since trivial actions always exist (the definition given in Subsection 5.3.1 only need a binary product, which exists since $\mathcal{C}$ is finitely complete). In terms of split short exact sequences, if we have a split short exact sequence

$$0 \longrightarrow C \xrightarrow{\kappa} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longrightarrow 0$$

with $C = 0$, κ is trivial thus $\alpha = \operatorname{coker} \kappa$ is an isomorphism and so is β as a splitting of an isomorphism. Therefore, all split short exact sequences of the form above are isomorphic and thus corresponds to a trivial action (since such an action always exists).

Finally, let us show that $(0, \xi)$ is a zero object in $\mathcal{C}^{B\flat(-)}$ (where ξ is the unique action on 0, which is trivial). Since a morphism of B-objects is an arrow in $\mathcal{C}$, the uniqueness of the arrows from and to 0 is immediate (since 0 is a zero object in $\mathcal{C}$) so we just need to check that these arrows are morphisms of B-objects. For that, let us first look at $B\flat 0$. Since $\langle 1_B, 0\rangle\colon B + 0 \to B$ is an isomorphism, we find that $B\flat 0 = 0$ as the kernel of an isomorphism. Therefore, we easily see that the following diagrams always commute (both members of the equations they express are trivial arrows), which shows that the unique arrows from and to 0 are morphisms of B-objects, finishing the proof that $(0, \xi)$ is a zero object in $\mathcal{C}^{B\flat(-)}$.

$$\begin{tikzcd}
B\flat 0 = 0 \arrow[r, "B\flat f"] \arrow[d, "\xi"'] & B\flat C \arrow[d, "\zeta"] \\
0 \arrow[r, "f"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
B\flat C \arrow[r, "B\flat f"] \arrow[d, "\zeta"'] & B\flat 0 = 0 \arrow[d, "\xi"] \\
C \arrow[r, "f"'] & 0
\end{tikzcd}$$

5.4 Conjugation Actions

If we have a normal subgroup N of a group X, we know that X acts on N by conjugation, i.e. the formula $\phi(x)(n) = xnx^{-1}$ for $x \in X$ and $n \in N$ defines an action $\phi\colon X \to \operatorname{Aut} N$. Let us see how we can generalise this notion in a semi-abelian context.

5.4.1 Definition

First, let us define a normal subobject. A *subobject* of an object X is a monomorphism $i\colon N \to X$ (up to some equivalence relation but we omit it since we will not use it in the following) and such a subobject i is *normal* if it is a normal monomorphism.

Now, let $i\colon N \to X$ be a normal subobject of an object X in a semi-abelian category $\mathcal{C}$ and let us explain how we define the conjugation action of X on N. We consider the cokernel $q\colon X \to X/N$ of i and the kernel pair $\big(\mathrm{Eq}(q), p_1, p_2\big)$ of this cokernel q. First, by Lemma 1.2.4, the kernel object of p_1 is the same as the one of q, i.e. it is N since i is the kernel of its cokernel q by Lemma 1.2.1.

$$\begin{tikzcd}
N \arrow[r, tail, "\ker p_1"] \arrow[dr, tail, "i"'] & \mathrm{Eq}(q) \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "q"] \\
& X \arrow[r, "q"'] & X/N
\end{tikzcd} \tag{5.1}$$

Moreover, p_1 is a split epimorphism by Lemma 1.2.14 so, by Corollary 1.3.10, we have a split short exact sequence

$$0 \longrightarrow N \xrightarrow{\ker p_1} \mathrm{Eq}(q) \underset{\Delta}{\overset{p_1}{\rightleftarrows}} X \longrightarrow 0\ ,$$

where the chosen splitting Δ is the unique common splitting with p_2 given by Lemma 1.2.14. It is this split short exact sequence we use as the definition of the *conjugation action of X on N*, passing through the equivalence of categories of Theorem 4.2.4.

5.4.2 Grp

Let us check that this definition is indeed a generalisation of the conjugation actions we know in Grp. Let us consider a normal subgroup N of a group X. We denote $i\colon N \to X$ the inclusion of N in X and $q\colon X \to X/N$ the canonical projection of the quotient X/N (which is indeed a cokernel of i). Then the kernel pair of q is the group

$$\mathrm{Eq}(q) = \{(x_1, x_2) \in X \times X \mid q(x_1) = q(x_2)\} \leq X \times X$$

with the projections p_1, $p_2\colon \mathrm{Eq}(q) \to X$ given by the projections of the product $X \times X$ restricted to the subgroup $\mathrm{Eq}(q) \leq X \times X$. Now, we consider the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{N} \arrow[r, "\kappa"] & \mathrm{Eq}(q) \arrow[r, shift left, "p_1"] & X \arrow[l, shift left, "\Delta"] \arrow[r] & 0 \end{tikzcd}$$

where κ is the inclusion of the canonical kernel $\widetilde{N}$ of p_1 in $\mathrm{Eq}(q)$ and Δ is given by $\Delta(x) = (x, x)$ for $x \in X$. The first object $\widetilde{N}$ of this split short exact sequence is not exactly N but we have

$$\begin{aligned}\widetilde{N} &= \{x \in \mathrm{Eq}(q) \mid p_1(x) = 1\} = \{(x_1, x_2) \in X \times X \mid x_1 = 1 \text{ and } q(x_1) = q(x_2)\}\\ &= \{(x_1, x_2) \in X \times X \mid x_1 = 1 \text{ and } 1N = x_2N\}\\ &= \{(x_1, x_2) \in X \times X \mid x_1 = 1 \text{ and } x_2 \in N\} = \{1\} \times N\end{aligned}$$

which is isomorphic to N via the isomorphism $\varphi\colon N \to \widetilde{N}\colon n \mapsto (1, n)$ so we will be able to transport our action from $\widetilde{N}$ to N. First, let us look at the action corresponding to the split short exact sequence above. By the construction from Proposition 2.1.4, this action of X on $\widetilde{N}$ is given by

$$x \cdot (1, n) = \Delta(x)(1, n)\Delta(x)^{-1} = (x, x)(1, n)(x^{-1}, x^{-1}) = (1, xnx^{-1})$$

for $x \in X$ and $(1, n) \in \widetilde{N}$. Finally, using the isomorphism $\varphi\colon N \xrightarrow{\cong} \widetilde{N}$ above, we obtain the action of X on N given by

$$x \cdot n = \varphi^{-1}\Big(x \cdot \varphi(n)\Big) = \varphi^{-1}\Big(x \cdot (1, n)\Big) = \varphi^{-1}(1, xnx^{-1}) = xnx^{-1}$$

for $x \in X$ and $n \in N$, which is the conjugation action of X on N as expected.

To go a little further, let us note that, if X is an abelian group, we know that every subgroup N of X is normal and that the conjugation action of X on N is trivial. In terms of split short exact sequences, it corresponds to the fact that the map $X \times N \to \mathrm{Eq}(q)\colon (x, n) \mapsto (x, xn)$ is an isomorphism of groups (the fact that X is abelian is used to show that it is indeed a group homomorphism) and this isomorphism induces an isomorphism between the split short exact sequences associated to the trivial and conjugation actions. Let us remark that, instead of the assumption that the group X is abelian, we can just ask that N is a normal subgroup of X contained in the centre $Z(X) = \{x \in X \mid xy = yx \quad \forall y \in X\}$ of X to find that this conjugation action is trivial. Let us also note that, since we can rewrite the equation $xy = yx$ as $xyx^{-1} = y$ or $yxy^{-1} = x$, we can see $Z(X)$ as the subset of elements of X that act trivially by conjugation or as the largest normal subgroup of X on which X acts trivially by conjugation.

5.4.3 CRng

The conjugation actions in CRng follows more or less the same idea as the ones in Grp. We obtain the same split short exact sequence (except that the additive group of our rings is denoted additively instead of multiplicatively as in the example in Grp) so, by Proposition 2.2.7, we have an action of X on $\widetilde{N}$ given by

$$x \cdot (0, n) = \Delta(x)(0, n) = (x, x)(0, n) = (0, xn)$$

for $x \in X$ and $(0, n) \in \widetilde{N}$. Then, by the isomorphism between N and $\widetilde{N}$, we find that the conjugation action of X on N is given by $x \cdot n = xn$, $x \in X$, $n \in N$. Let us remark that, similarly to what happens in Grp, a normal subobject $N \triangleleft X$, i.e. an ideal $N \triangleleft X$, is exactly a subobject that is closed under the conjugation action, i.e. such that $x \cdot n \in N$ for all $x \in X$ and all $n \in N$.

Furthermore, like in Grp, if X is abelian (see Example 2.2.4), every subring N of X is an ideal and we see that the conjugation action of X on N is trivial. In this case, the isomorphism between the split short exact sequences is induced by the isomorphism $X \times N \xrightarrow{\cong} \mathrm{Eq}(q)\colon (x, n) \mapsto (x, x + n)$ where the fact that X is abelian is again used to show that this map is indeed an arrow in the category we consider. Let us note that, similarly to the case in Grp, the assumption that X is abelian is stronger than what we need and we can just ask that the ideal $N \triangleleft X$ is contained in the centre $Z(X) = \{x \in X \mid xy = 0 \quad \forall y \in X\}$ of X, which is again the ideal of elements of X that act trivially by conjugation or the largest ideal of X on which X acts trivially by conjugation.

5.4.4 $\mathrm{Set}_*^{\mathrm{op}}$

Let us end these examples with $\mathrm{Set}_*^{\mathrm{op}}$. A normal subobject in $\mathrm{Set}_*^{\mathrm{op}}$ is a normal epimorphism $i\colon (X, *_X) \to (N, *_N)$ in Set_* so it has a unique splitting $j\colon N \to X$. Then we must look at the pushout

$$\begin{tikzcd}
X/N \arrow[r, "q"] \arrow[d, "q"'] & X \arrow[d, "p_2"] \\
X \arrow[r, "p_1"'] & \mathrm{Coeq}(q)
\end{tikzcd}$$

where q is a kernel of i. More explicitly, we have $X/N = \{x \in X \mid i(x) = *_N\}$ with q being its inclusion in X and $\mathrm{Coeq}(q) = (X + X)/\sim$ where $X + X$ is the disjoint union of two copies of X with the two base points identified and $x_1 \sim x_2$ if x_1 and x_2 are equal or represent the same element of $X/N \subset X$. The inclusions p_1 and p_2 are given by the inclusions of X in $X + X$ postcomposed with the canonical projection $X + X \twoheadrightarrow (X + X)/\sim$. Then a cokernel of p_1 is given by the pointed map from $\mathrm{Coeq}(q)$ to N defined as follows:

- if $x \in \mathrm{Coeq}(q)$ comes from the first copy of X, it is sent to the base point,
- if $x \in \mathrm{Coeq}(q)$ is one of the remaining elements of the second copy, i.e. an element of $X \setminus (X/N)$ from this copy since its elements in X/N are identified with their analogues in the first copy, it is sent to $i(x) \in N$.

The unique splitting $\delta\colon N \to \mathrm{Coeq}(q)$ of this cokernel is given by the pointed map that sends $n \in N$ to the class of $j(n)$ from the second copy of X. For the common splitting $\Delta\colon \mathrm{Coeq}(q) \to X$ of p_1 and p_2, it sends an element of $\mathrm{Coeq}(q)$ to the element of X it represents. Combining all these things, we obtain the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, shift left, "p_1"] & \mathrm{Coeq}(q) \arrow[l, shift left, "\Delta"] \arrow[r, "\operatorname{coker} p_1"] & N \arrow[r] & 0 \end{tikzcd}$$

which gives us the action $\phi\colon N \to X\colon n \mapsto \Delta\big(\delta(n)\big) = j(n)$ (see Subsection 5.1). Therefore, we find that the conjugation action we were looking for is simply the unique splitting j of $i\colon X \to N$.

5.4.5 Conjugation Actions and Normal Subobjects

To end this section, let us briefly look at the link between normal subobjects and conjugation actions. In Grp and CRng, we have seen that normal subobjects are exactly the subobjects that are stable by the conjugation of X on itself. We can reformulate this by saying that they are the subobjects that admit a conjugation action of X on them and, in fact, the existence of such an action is equivalent to the normality in a more general context, the semi-abelian one.

First, let us define what we mean by a *conjugation action* (in Subsection 5.4.1, we used the normality of N so we cannot use it as a general definition). Each object X gives us a normal subobject $1_X\colon X \to X$ so we can apply the construction of Subsection 5.4.1 to have a conjugation action $\xi_X\colon X\flat X \to X$ of X on itself. Then, for a subobject $i\colon N \to X$ of X, we say that *X acts by conjugation on N* if we have a factorisation $\xi_N\colon X\flat N \to N$ in the diagram below.

$$\begin{tikzcd} X\flat N \arrow[r, dashed, "\xi_N"] \arrow[d, "X\flat i"'] & N \arrow[d, tail, "i"] \\ X\flat X \arrow[r, "\xi_X"'] & X \end{tikzcd}$$

Such a factorisation ξ_N is unique since i is a monomorphism and we call it the *conjugation action of X on N*.

Before showing the equivalence between the normality of a subobject and the existence of a conjugation action, let us look more in detail at the conjugation action ξ_X of X on itself. This action ξ_X corresponds to the split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "{(0,1_X)}"] & X\times X \arrow[r, shift left, "\pi_1"] & X \arrow[l, shift left, "{(1_X,1_X)}"] \arrow[r] & 0 \end{tikzcd}$$

where π_1 is the projection of the product $X\times X$ on its first factor since we easily check that the kernel pair of the cokernel of 1_X, which is $0\colon X\to 0$, is the product $X\times X$ with its projections, that the common splitting of these projections is the diagonal $(1_X,1_X)$ of $X\times X$ and that $(0,1_X)\colon X\to X\times X$ is a kernel of π_1. Let us note that we have a short exact sequence of the form $0 \longrightarrow X \xrightarrow{(0,1_X)} X\times X \xrightarrow{\pi_1} X \longrightarrow 0$ like for trivial actions (see Subsection 5.3) but a conjugation action of an object on itself is not necessarily trivial since the splittings of π_1 we consider in these two situations are not the same in general. Moreover, again similarly to trivial actions, we can show that this conjugation action is given explicitly by $\xi_X=\langle 1_X,1_X\rangle\circ\kappa_{X,X}$.

Now, let us show that the definition of conjugation action above corresponds to the construction of Subsection 5.4.1 if the subobject $i\colon N\to X$ is normal. Putting together the diagrams used to construct the conjugation actions ξ_N and ξ_X in the sense of Subsection 5.4.1, we obtain the diagram below where ι_1, ι'_1 are the inclusions of the first factor in $X+N$ and $X+X$, respectively (the diagrams of the constructions of ξ_N and ξ_X are the top and bottom faces, respectively).

$$\begin{tikzcd}
X\flat N \arrow[rr, "\kappa_{X,N}"] \arrow[dr, "\xi_N"] \arrow[dd, "X\flat i"'] & & X+N \arrow[rr, shift left, "{\langle 1_X,0_{N,X}\rangle}"] \arrow[dr, "{\langle \Delta,\ker p_1\rangle}"] \arrow[dd, "1_X+i"] & & X \arrow[ll, shift left, "\iota_1"] \arrow[dr, equal] \arrow[dd, equal] & \\
& N \arrow[rr, "\ker p_1"] \arrow[dd, "i"] & & \mathrm{Eq}(q) \arrow[rr, shift left, "p_1"] \arrow[dd, "{(p_1,p_2)}"] & & X \arrow[ll, shift left, "\Delta"] \arrow[dd, equal] \\
X\flat X \arrow[rr, "\kappa_{X,X}"] \arrow[dr, "\xi_X"'] & & X+X \arrow[rr, shift left, "{\langle 1_X,0_{X,X}\rangle}"] \arrow[dr, "{\langle(1_X,1_X),(0,1_X)\rangle}"'] & & X \arrow[ll, shift left, "\iota'_1"] \arrow[dr, equal] & \\
& X \arrow[rr, "{(0,1_X)}"'] & & X\times X \arrow[rr, shift left, "\pi_1"] & & X \arrow[ll, shift left, "{(1_X,1_X)}"]
\end{tikzcd}$$

In order to show that ξ_N is a conjugation action as we defined in the general case, we must show that the left face commute. By some computations, we see that all the other faces commute (for the left front face and the middle one, let us remind that $p_2\circ\ker p_1=i$: see (5.1)) so the left face commute as expected since $(0,1_X)$ is monic.

Finally, let us show that, if we have a subobject $i\colon N\to X$ of X such that the conjugation action ξ_N of X on N exists, then i is normal. For that, we will apply Lemma 4.1.7 on the square below.

$$\begin{tikzcd}
X\flat N \arrow[r, tail, "\kappa_{X,N}"] \arrow[d, two heads, "\xi_N"'] & X+N \arrow[d, two heads, "\langle 1_X, i\rangle"] \\
N \arrow[r, tail, "i"'] & X
\end{tikzcd}$$

The arrow $\langle 1_X, i\rangle$ is a normal epimorphism by Lemmas 1.2.10 and 1.3.11 since it is split by the inclusion ι_1 of X in $X+N$ and $\kappa_{X,B}$ is a normal monomorphism by definition. Therefore, we just need to show that ξ_N is a normal epimorphism and that this square commutes in order to apply Lemma 4.1.7, which will let us conclude that i is a normal monomorphism as desired.

- To show that ξ_N is a normal epimorphism, we will show that it is a split epimorphism split by η_N (let us note that it is not a consequence of the fact that (N, ξ_N) is a X-object since we have not yet proved that ξ_N is an action), which implies that it is a normal epimorphism, again by Lemmas 1.2.10 and 1.3.11. For that, consider the diagram

$$\begin{tikzcd}
N \arrow[r, "\eta_N"] \arrow[d, "i"'] & X\flat N \arrow[r, "\xi_N"] \arrow[d, "X\flat i"] & N \arrow[d, "i"] \\
X \arrow[r, "\eta_X"'] & X\flat X \arrow[r, "\xi_X"'] & X
\end{tikzcd}$$

  where ξ_X denotes the conjugation action of X on itself. This diagram is commutative by naturality of η and definition of ξ_N so we have $i \circ (\xi_N \circ \eta_N) = (\xi_X \circ \eta_X) \circ i = 1_X \circ i = i \circ 1_N$ since ξ_X is an action. Finally, since i is monic, we conclude that $\xi_N \circ \eta_N = 1_N$ (i.e. that ξ_N is split by η_N) as announced.

- For the commutativity of the square, let us consider the cube below.

$$\begin{tikzcd}
X\flat N \arrow[rr, "X\flat i"] \arrow[dr, "\kappa_{X,N}"] \arrow[dd, "\xi_N"'] & & X\flat X \arrow[dr, "\kappa_{X,X}"] \arrow[dd, "\xi_X"] & \\
& X+N \arrow[rr, "1_X+i"] \arrow[dd, "\langle 1_X, i\rangle"] & & X+X \arrow[dd, "\langle (0,1_X),(0,1_X)\rangle"] \\
N \arrow[rr, "i"] \arrow[dr, "i"'] & & X \arrow[dr, "(0,1_X)"] & \\
& X \arrow[rr, "(0,1_X)"'] & & X \times X
\end{tikzcd}$$

  To finish the proof, we must show that the left face commutes. In order to do that, let us first show that all the other faces commute. It is immediate for the bottom one and the top and back ones commute by definition of $X\flat i$ and ξ_N, respectively. For the right one, using the identity $\langle (f_{11}, f_{21}), (f_{12}, f_{22})\rangle = (\langle f_{11}, f_{12}\rangle, \langle f_{21}, f_{22}\rangle)$

(whenever it makes sense), we compute

$$\langle (0,1_X),(0,1_X)\rangle \circ \kappa_{X,X} = (\langle 0,0\rangle, \langle 1_X,1_X\rangle)\circ \kappa_{X,X} = (0,\langle 1_X,1_X\rangle \circ \kappa_{X,X})$$

and

$$(0,1_X)\circ \xi_X = (0,1_X)\circ (\langle 1_X,1_X\rangle \circ \kappa_{X,X}) = (0,\langle 1_X,1_X\rangle \circ \kappa_{X,X})$$

since we have seen before that $\xi_X = \langle 1_X, 1_X\rangle \circ \kappa_{X,X}$. Therefore, these two composites are equal as wanted, showing the commutativity of the right face. Finally, for the front face, we have

$$\langle (0,1_X),(0,1_X)\rangle \circ (1_X + i) = \langle (0,1_X),(0,1_X)\circ i\rangle$$

which is indeed equal to $(0,1_X)\circ \langle 1_X, i\rangle$. Now that we know that all the faces except the left one commute, we can also deduce that the left one commutes as desired since $(0,1_X)$ is monic (it is a kernel of the projection π_1 of $X\times X$ on its first factor).

5.5 Change of Base

In this section, we will show that, given an arrow $f\colon B' \to B$ in $\mathcal{C}$, we obtain the *change-of-base functor* $f^*\colon \mathcal{C}^{B\flat(-)} \to \mathcal{C}^{B'\flat(-)}$ that sends a B-object (C,ξ) to the B'-object $(C, \xi\circ f\flat 1_C)$ and an arrow in $\mathcal{C}^{B\flat(-)}$ to the arrow in $\mathcal{C}^{B'\flat(-)}$ corresponding to the same underlying arrow in $\mathcal{C}$ (let us recall that morphisms of algebras for a monad are arrows in the underlying category making a certain diagram commute : see Definition 3.1.2).

Before checking that the constructions above define a functor as wanted, let us see what is the corresponding functor from the point of view of the split short exact sequences. This corresponding functor sends a split short exact sequence of $\mathrm{SSES}(B)$ to the split short exact sequence of $\mathrm{SSES}(B')$ obtained as the first line of the diagram below by taking first a pullback of α along f then a kernel of α' (a kernel of α' is indeed given as in the diagram by Lemma 1.2.4). Since this functor is a pullback functor, we sometimes use the terminology *pullback of an action* to talk about this change of base.

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "\kappa'"] \arrow[d, equal] & A' \arrow[r, shift left, "\alpha'"] \arrow[d] & B' \arrow[l, shift left, "\beta'"] \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & C \arrow[r, "\kappa"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0
\end{tikzcd}$$

Let us briefly explain why this pullback functor is indeed the functor corresponding to our change-of-base functor. Using the equivalence of categories of Theorem 4.2.4, we can construct the following diagram between the split short exact sequences corresponding to the actions ξ and $\xi \circ f\flat 1_C$ where the middle arrow is induced by the universal property of $B' \ltimes C$ (which is defined as a coequaliser, as a reminder).

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d, equal] & B' \ltimes C \arrow[r, shift left] \arrow[d] & B' \arrow[r] \arrow[l, shift left] \arrow[d, "f"] & 0 \\
0 \arrow[r] & C \arrow[r] & B \ltimes C \arrow[r, shift left] & B \arrow[r] \arrow[l, shift left] & 0
\end{tikzcd}$$

Then one can check that this diagram commutes so Corollary 1.3.17 tells us that the right square with the horizontal arrows pointing to the right is a pullback, showing that the functor corresponding to the change-of-base functor passing through the equivalence of Theorem 4.2.4 is a pullback functor, as expected.

Now, let us check that the assignments of the first paragraph of the section form a well-defined functor. First, let us look at the multiplication μ' and unit η' of the monad $B'\flat(-)$. By Corollary 3.1.8, their components are the induced arrows in the diagrams

$$\begin{tikzcd}
C \arrow[d, dashed, "\eta'_C"'] \arrow[dr, "\iota'_2"] & & \\
B'\flat C \arrow[r, "\kappa_{B',C}"'] & B' + C \arrow[r, "\langle 1_{B'},0\rangle"'] & B'
\end{tikzcd}
\qquad
\begin{tikzcd}
B'\flat(B'\flat C) \arrow[r, "\kappa_{B',B'\flat C}"] \arrow[d, dashed, "\mu'_C"'] & B' + (B'\flat C) \arrow[r, "\langle 1_{B'},0\rangle"] \arrow[d, "\langle \iota'_1, \kappa_{B',C}\rangle"'] & B' \arrow[d, equal] \\
B'\flat C \arrow[r, "\kappa_{B',C}"'] & B' + C \arrow[r, "\langle 1_{B'},0\rangle"'] & B'
\end{tikzcd}$$

where ι'_1, ι'_2 denote the inclusions in the coproduct $B' + C$. We have similar diagrams for $B\flat(-)$ and, using the arrow f to link them, we obtain the diagrams below, where ι_1, ι_2 denote the inclusions in the coproduct $B + C$.

$$\begin{tikzcd}
C \arrow[dr, equal] \arrow[dd, "\eta'_C"'] \arrow[ddrr, bend left, "\iota'_2"] & & & & & \\
& C \arrow[dd, "\eta_C"'] \arrow[ddrrr, bend left, "\iota_2"] & & & & \\
B'\flat C \arrow[rr, "\kappa_{B',C}"] \arrow[dr, "f\flat 1_C"'] & & B' + C \arrow[rr, "\langle 1_{B'},0\rangle"] \arrow[dr, "f+1_C"] & & B' \arrow[dr, "f"] & \\
& B\flat C \arrow[rr, "\kappa_{B,C}"'] & & B + C \arrow[rr, "\langle 1_B,0\rangle"'] & & B
\end{tikzcd}$$

$$
\begin{tikzcd}
B'\flat(B'\flat C) \arrow[rr, "\kappa_{B',B'\flat C}"] \arrow[dr, "f\flat(f\flat 1_C)"] \arrow[dd, "\mu'_C"'] & & B'+(B'\flat C) \arrow[rr, "\langle 1_{B'},0\rangle"] \arrow[dr, "f+(f\flat 1_C)"] \arrow[dd, "\langle \iota'_1,\kappa_{B',C}\rangle"] & & B' \arrow[dr, "f"] \arrow[dd, equal] & \\
& B\flat(B\flat C) \arrow[rr, "\kappa_{B,B\flat C}"] \arrow[dd, "\mu_C"] & & B+(B\flat C) \arrow[rr, "\langle 1_B,0\rangle"] \arrow[dd, "\langle \iota_1,\kappa_{B,C}\rangle"] & & B \arrow[dd, equal] \\
B'\flat C \arrow[rr, "\kappa_{B',C}"] \arrow[dr, "f\flat 1_C"'] & & B'+C \arrow[rr, "\langle 1_{B'},0\rangle"] \arrow[dr, "f+1_C"] & & B' \arrow[dr, "f"] & \\
& B\flat C \arrow[rr, "\kappa_{B,C}"'] & & B+C \arrow[rr, "\langle 1_B,0\rangle"'] & & B
\end{tikzcd}
$$

We want to show that the two left faces commute to have relations between the multiplication and unit of $B'\flat(-)$ and those of $B\flat(-)$ so we will show that all the other faces (including the top one of the first diagram representing the equation $(f+1_C)\circ\iota'_2=\iota_2$) commute. Let us check that for the middle face of the second diagram, the other ones being more or less straightforward. Once again thanks to the computation rules of Remark 3.1.5, we compute

$$\langle \iota_1,\kappa_{B,C}\rangle \circ \Big(f+(f\flat 1_C)\Big) = \langle \iota_1\circ f, \kappa_{B,C}\circ f\flat 1_C\rangle$$

and

$$(f+1_C)\circ\langle \iota'_1,\kappa_{B',C}\rangle = \langle (f+1_C)\circ\iota'_1,(f+1_C)\circ\kappa_{B',C}\rangle = \langle \iota_1\circ f,(f+1_C)\circ\kappa_{B',C}\rangle.$$

Since $(f+1_C)\circ\kappa_{B',C}=\kappa_{B,C}\circ f\flat 1_C$ by the commutativity of the bottom left face, these two expressions are equal so the middle face commutes as wanted. Now that we know that all the faces except the left ones commute, we deduce that these two left faces also commute since $\kappa_{B,C}$ is a monomorphism. Therefore, we obtain the relations $f\flat 1_C\circ\eta'_C=\eta_C$ and $f\flat 1_C\circ\mu'_C=\mu_C\circ f\flat(f\flat 1_C)$ linking the multiplication and unit of $B'\flat(-)$ with those of $B\flat(-)$.

We now have all the necessary tools to check that the assignments described at the beginning of the section form a well-defined functor f^*.

- We must check that, given a B-object (C,ξ), its image $(C,\xi\circ f\flat 1_C)$ is a B'-object, i.e. that

$$(\xi\circ f\flat 1_C)\circ\Big(B\flat(\xi\circ f\flat 1_C)\Big)=(\xi\circ f\flat 1_C)\circ\mu'_C \quad\text{and}\quad (\xi\circ f\flat 1_C)\circ\eta'_C=1_C.$$

For the first equation, we consider the diagram

$$\begin{tikzcd}
B'\flat(B'\flat C) \arrow[r, "B'\flat(f\flat 1_C)"] \arrow[rr, bend left, "B'\flat(\xi\circ f\flat 1_C)"] \arrow[dd, "\mu'_C"'] \arrow[dr, dashed, "f\flat(f\flat 1_C)"] & B'\flat(B\flat C) \arrow[r, "B'\flat\xi"] \arrow[d, dashed, "f\flat 1_{B\flat C}"] & B'\flat C \arrow[d, "f\flat 1_C"] \\
& B\flat(B\flat C) \arrow[r, dashed, "B\flat\xi"] \arrow[d, dashed, "\mu_C"'] & B\flat C \arrow[d, "\xi"] \\
B'\flat C \arrow[r, "f\flat 1_C"'] & B\flat C \arrow[r, "\xi"'] & C
\end{tikzcd}$$

which commutes as wanted by the equality $f\flat 1_C \circ \mu'_C = \mu_C \circ f\flat(f\flat 1_C)$, the bifunctoriality of $\flat$ and the fact that (C, ξ) is a B-object. For the second equation, it follows directly from the equality $f\flat 1_C \circ \eta'_C = \eta_C$ and the fact that (C, ξ) is a B-object.

- Let $g\colon C \to C'$ be a morphism between two B-objects (C, ξ) and (C', ζ), i.e. such that $\zeta \circ B\flat g = g \circ \xi$. We must check that the same underlying arrow g is also a morphism between the B'-objects $(C, \xi \circ f\flat 1_C)$ and $(C', \zeta \circ f\flat 1_{C'})$, i.e. that $(\zeta \circ f\flat 1_{C'}) \circ B'\flat g = g \circ (\xi \circ f\flat 1_C)$. It is indeed the case, as the diagram below shows (the top square commutes by bifunctoriality of $\flat$ and the bottom one since g is a morphism of B-objects).

$$\begin{tikzcd}
B'\flat C \arrow[r, "B'\flat g"] \arrow[d, "f\flat 1_C"'] & B'\flat C' \arrow[d, "f\flat 1_{C'}"] \\
B\flat C \arrow[r, "B\flat g"] \arrow[d, "\xi"'] & B\flat C' \arrow[d, "\zeta"] \\
C \arrow[r, "g"'] & C'
\end{tikzcd}$$

- The functoriality of f^* is immediate since it sends an arrow in $\mathcal{C}^{B\flat(-)}$ to the arrow of $\mathcal{C}^{B'\flat(-)}$ corresponding to the same underlying arrow in $\mathcal{C}$.

Before looking at some examples, let us note that, for every $f\colon B'' \to B'$ and $g\colon B' \to B$, we have $(g \circ f)^* = f^* \circ g^*$ (it follows directly from the functoriality of $\flat$ in its first component) and that we also have the same functoriality property for the pullback functor in terms of split short exact sequences (by a composition property of pullbacks).

Finally, let us finish this section by mentioning how this change of base is given in Grp, CRng and $\mathrm{Set}_*^{\mathrm{op}}$.

- For an action $\phi\colon B \to \operatorname{Aut} C$ in Grp, $f^*\phi$ is given by $f^*\phi = \phi \circ f$.
- For $\phi\colon B \otimes C \to C$ in CRng, we have $(f^*\phi)(b' \otimes c) = \phi\Big(f(b') \otimes c\Big)$, $b' \in B'$, $c \in C$.

- For $\phi\colon C \to B$ in Set^*, our new action is $f^*\phi = f \circ \phi$.

5.6 Internal Actions of Monoids

An *action of a monoid B on a monoid C* is a monoid homomorphism ϕ from B to the monoid $\operatorname{End} C$ of endomorphisms of C. However, as we said in the introduction of the chapter, Mon is not a semi-abelian category so we cannot use the theory of internal actions presented in the previous chapter. In fact, there is also an equivalence between actions and split short exact sequences in Mon but not *all* split short exact sequences, only a subcategory of them, the *Schreier split short exact sequences*. See [BMFMS13] for further information.

Conclusion

The goal of this thesis was to explore a categorical notion of internal actions. In the first place, we looked at two algebraic examples of actions — groups and commutative rings — to highlight their similarities, which motivated a generalisation. After that, we presented this generalisation and showed the main result of this work, the equivalence between internal actions and split short exact sequences in a semi-abelian context. Finally, using the fact that split short exact sequences have a “more transparent” categorical structure, we used this equivalence to show how we can generalise some particular types of actions like trivial actions and conjugation actions.

We have seen, in the last chapter, some applications of the theory of internal actions we presented, but these are just the beginning of what this theory lets us accomplish. First, gathering known actions like group actions and ring actions in a unifying framework could allow us to use our knowledge on actions in certain categories to deduce properties of actions in other categories. Moreover, having another point of view is never a bad thing and could highlight some properties of actions that would have been more difficult to notice with the traditional definition, for example. Finally, such a theory of internal actions is also interesting from a purely categorical point of view, through its former links with commutator theory, for instance. Internal actions also allow us to characterise certain properties of categories, like algebraic coherence or representability of actions.

However, the theory we presented also has its limitations. In the last section of the last chapter, we briefly saw that a notion of actions also exists in Mon but is not covered by what we developed, since Mon is not semi-abelian. So a theory of internal actions in a non-semi-abelian context could be an interesting thing to study. Another example of actions not covered by our theory is the actions we mentioned in the introduction of the document, where the object that acts and the one that is acted on do not belong in the same category. Therefore, it also seems interesting to develop a theory of *external actions* and see what are the relations between such a theory and the one of internal actions.

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