

École polytechnique de Louvain

Numerical and experimental modeling of mudflows

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Abstract

Current trends and future projections for natural disasters reinforce the need to develop forecasting tools. Against this backdrop, the current Master's thesis focuses on developing a simulation model for mudflows, destructive events which occur in mountainous regions.

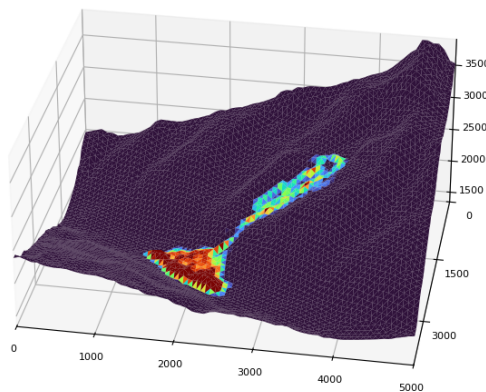
This document outlines the theoretical approach that enabled the construction of an initial mudflow calculation module in the Watlab program. The program, developed by the hydraulic research team at UCLouvain, solves the Shallow Water Equations with a Finite Volume Method. The aim of this thesis was to add an initial calculation function for these very specific and complex flows.

In order to develop such a module, this work first focused on a theoretical synthesis of the behaviors and main mechanisms encountered with mudflows. The model therefore incorporates these concepts or respond to the behaviors that are predicted.

The model is then developed by modifying the friction formulation of the classical Shallow Water Equation system. A single-phase assumption made possible the derivation of the mathematical expressions to be inserted into Watlab. Some formulations were proposed to express Bingham rheology and a Herschel-Bulkley rheology as friction source terms inside the Shallow Water Equations. Also, the numerical difficulty of treating the stopping ability of the material with Finite Volume scheme is exposed along with the solution strategy that has been implemented.

Finally, the model was tested in a controlled laboratory environment using data provided by Politecnico di Milano, and then on a large scale: the devastating flow of rocks, ice, and soil that destroyed the village of Blatten in late spring 2025.

The model has been found to have some weaknesses, but the consistency between observations and actual measurements reinforces its potential if future improvements are expected for the Watlab mudflow calculation module.



Mudflow and numerical simulation – illustration

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Chapter 1

Introduction

Mudflows and debris flow are natural hazards capable of devastating villages, infrastructures and causing many deaths in a short period of time. These catastrophic events generally happen in mountainous areas where steep slopes quickly accelerate a mass of soil mixed with water possessing a high bulk density (Tan et al. 2020). They represent a challenge in terms of theoretical description and modeling but is also a challenge in terms of post catastrophic event mitigation. In addition to having a significant impact force on infrastructures, the deposition volume at the end of the runout makes the cleanup and reconstruction work slow and tedious.

Better understanding the theoretical behaviour of debris flow and mudflows, being able to characterize, quantify and model their runout is essential for places where these events pose a danger to population or properties. The recent glacier collapse (May 2025) in Switzerland (Guardian 2025), where the runout of the flow buried almost entirely the village of Blatten in the Canton Valais, is a tragic example of the importance of prevision in order to warn and evacuate the population on time.

1.1 Mudflows and climate change

Climate change is one of the most important challenges of the 21st century. Mountainous areas are particularly sensitive to the anthropogenic climate change and catastrophic events as debris flows are expected to be impacted by the actual and future changes.

Mudflows or debris flows are highly impacted by climate change regarding their frequency or severity. At a worldwide scale, however, the trend is difficult to identify. A precise contextualization and multiple factors should be taken into account.

Unlike other global warming impacts on mountainous environments as glacier retreat or permafrost degradation, flow mass movements are not directly impacted by temperature change but rather by secondary effects specific to a given context, as rainfall event modification, freezing area retreat or wild-fire making slopes prone to mobilization. Debris flows are mainly activated by heavy or long rainfall events (Scott et al. 2005), (Sidle and Chigira 2004). Thus, in regions where climate change affects the hydrological processes of slopes and the rainfall intensity or frequency, making raining events more brutal, the debris flows risks are expected to increase as in cold mountain areas (Imaizumi et al. 2021). Indeed, according to Fowler and Hennessy (1995), the average temperature rising in these areas leads to a higher capability of the air to store moisture and allows more frequent intense rainfall events.

On the other hand, the increase of temperature also leads to a lower seasonal freezing duration of the soils. Frozen grounds lower the infiltration capacity of the soil, promote runoff and erosion, leading to higher risks of mudflows (for those triggered by erosion). Imaizumi et al. (2021) have studied ground temperature around the Osawa failure on Mt. Fuji, Japan, in order to evaluate the influence of frozen ground with the rainfall intensity-duration (ID) threshold leading to debris flow. They showed that the ID threshold is lower when the ground is frozen, meaning the shorter the duration of the frozen ground, the lower the debris flows initiation risks. Reduction of the seasonal frozen ground duration due to climate change can thus lead to a decrease of debris flows initiation in some regions.

Regarding sediments feeding, the balance between positive and negative effects are also difficult to evaluate. On one side, the glacier retreat and permafrost degradation bring unstable sediments to the slopes and increases the available materials. On the other side, the retreat of freezing conditions also leads to a frost-weathering reduction (processes of freezing and thawing of water trapped in cracks of rocks

which leads to loose materials production) and a sediment production decrease (Hirschberg et al. 2021). Wildfires are also known to increase the risks of mudflows and debris flows. Post-wildfire debris flows are a great concern on steep slopes prone to wildfires. These events could more than double in likelihood considering modest climate change scenarios in some countries (Kean and Staley 2021).

These few examples briefly illustrate the uncertainty surrounding the impacts of climate change on flow mass movements. The topics of mountain disasters and climate change are widely studied and documented. It will definitely be a key point to consider in the future years. Here, increasing or decreasing factors are intertwined, making it complex and uncertain to predict how the trend will evolve in the future years. Nevertheless, the tendency should be studied for specific context in specific regions in order to apprehend in a better way its trajectory, adopting a local view risk mitigation instead of a worldwide approach is crucial.

1.2 Objectives and organization of the document

This work is part of the process of understanding the challenges posed by mudflows and searching for simulation tools to better assess risk mitigation. It first aims to bring a summary around the existing knowledge of the subject in order to better apprehend the global mechanisms behind it. Then, a model is exposed, from global theory to precise building of a calculation model. The model is a new axis of improvement in the program Watlab by adding a module to simulate mudflows. Watlab¹ is a program developed at UCLouvain by the hydraulic research team. Watlab resolves the shallow-water equations with finite-volumes method using a *C++* computing core and a *python* user interface. The thesis aims both to provide a theoretical overview of debris flows and mudflows, and to present an initial basis for the module integrated into Watlab that will be used to study mudflows, in the hope that this work will provide the keys to further advances should this module be improved in the future.

The document is divided into 5 main parts. This first chapter introduced the context, objectives of the current work. A second part concerns debris flows and their theoretical descriptions. This section aims to provide a brief theoretical overview of the main characteristics governing the events in order to understand their key behavior and lay the foundations for more in-depth work. Then mathematical models and numerical techniques generally used to model such flows are presented. In the context of this work, the Shallow Water Equations are chosen to approach the numerical modeling, using single phase assumption. This assumption is exposed in the fourth chapter and the rheology ruling it is discussed. The next part focuses on the experimental campaign carried out in the Gap2 lab of the University of Politecnico di Milano (Lecco campus, Italy). The campaign was carried out by Professor Laura Longoni, PhD doctoral researcher Lorenzo Panzeri and Master's Candidate Yixin Guo (Guo 2025). It is introduced here to build a first database for the numerical characterization and code validation purposes. Then the adaptation to the Watlab program is discussed. This sixth part is the heart of the work done and presents both the mathematical model and the numerical approach used in the implementation of the new module. Finally, the last two parts concern the application of the code to real test cases and the validation. First, the model is compared to the experimental campaign carried out in the Gap2 lab of the University of Politecnico di Milano. Then, the model is applied to a real case study that occurred in Switzerland in the early summer of 2025. A conclusion will summarize the work accomplished and outline future perspectives, in the hope that it will enable future progress.

¹<https://sites.uclouvain.be/hydraulics-group/watlab/index.html>

Chapter 2

Technical description of debris flows and mudflows

Mudflows and debris flows terms are used to refer to a global mechanism involving a sliding mass of mixture composed by rocks, wetted soils and sometimes other kind of debris. Where mudflows are generally composed by a high percentage of fine particles (silt and clay), debris flows contain a significant proportion of coarse particles as rocks or even trees carried within the flow. Some researchers associate mudflows (as well as debris torrent, mudslides mudspates or lahars) as a subcategory of debris flows (Johnson 1984), this classification is kept here. The nomenclature refers to the origin of the flow composition. The present document will treat debris flow in a more descriptive and general way which will include mudflows in its definition in order to avoid confusion. However, numerical modeling and the experimental campaign will focus primarily on mudflows, whose finer composition makes them more likely to be homogeneous and to obey the hypotheses that will be presented.

Debris flows are complex catastrophic events that are dangerous, difficult to predict and to model. They are fluid-solid mixtures composed by enough water to travel long distances and reach relatively high velocities but also possess solid particles concentration high enough to increase the bulk densities usually more than twice the one of water. Large flows can exceed the speed of 10 m/s and release a potential energy of $10^{16} J$ with a volume of $10^9 m^3$ (Richard M. Iverson 1997). They travel long distances when they are channelized in gullies or rivers, and can spread over large areas when they reach flatter terrains. Later in this document, the experimental campaign will illustrate both the channelized and the deposition flow behavior (see section 5).

The typical behaviour of debris flows can be divided into three main phases. In a first phase, the process starts from a resting mass of materials on a slope. For some reasons, the mass can fail to sustain itself and is entrained to a sediment-mixture flow in a liquid-like behavior. The second phase is the flow dynamic. The flow can travel long distances downslope, eroding the ground and devastating everything on its path. In a third phase, the flow can stop when the speed or the slope decrease enough. The flow transforms back into a solid material and compose the solid deposition (Richard M. Iverson 1997). The two last phases cause enormous damage, whether during powerful runoff or deposition, which makes reconstruction difficult.

The duality of solid-liquid behavior brings many challenges in terms of theoretical and numerical approaches. Both interactions between solid particles and the fluid phase at a microscopic scale and the solid-liquid viscous behavior at a macroscopic scale are complex. Coussot and Meunier (1996) said: *With debris flow material, the complexity could scarcely be greater since the interactions at a mesoscopic scale are various and complex. Clay particles immersed in water give rise to colloidal interactions, larger grains interact with one another via solid frictions or collisions, clay particles and grains interact via unknown processes and the solid fraction is very high.* The theoretical aspects about those fluid are interesting and necessary to provide a clever model. Hence, this section aims to bring a theoretical review of the main characteristics of debris flows. It will classify the different flows encountered in mountain disasters, the main physical characteristics of debris flow composition will then be discussed. In order to characterize the flow behavior, the three main phases of a debris flow will be presented: their triggering mechanisms (from solid-static state to liquid-like flow), the flow dynamic behavior (how the flow behaves) with finally the deposition process (the return to rest state). Figure 2.3 shows the three phases and their recognizable characteristics; hillslope scar at the initiation location or where the runoff has washed away the soil, the runoff gully often channelizing the flow and the deposition (or accumulation).



Figure 2.1: Mudflows considered as a subcategory of debris flow, regarding their more homogeneous composition (Inspired Pencil 2023), (Patrick 2022).

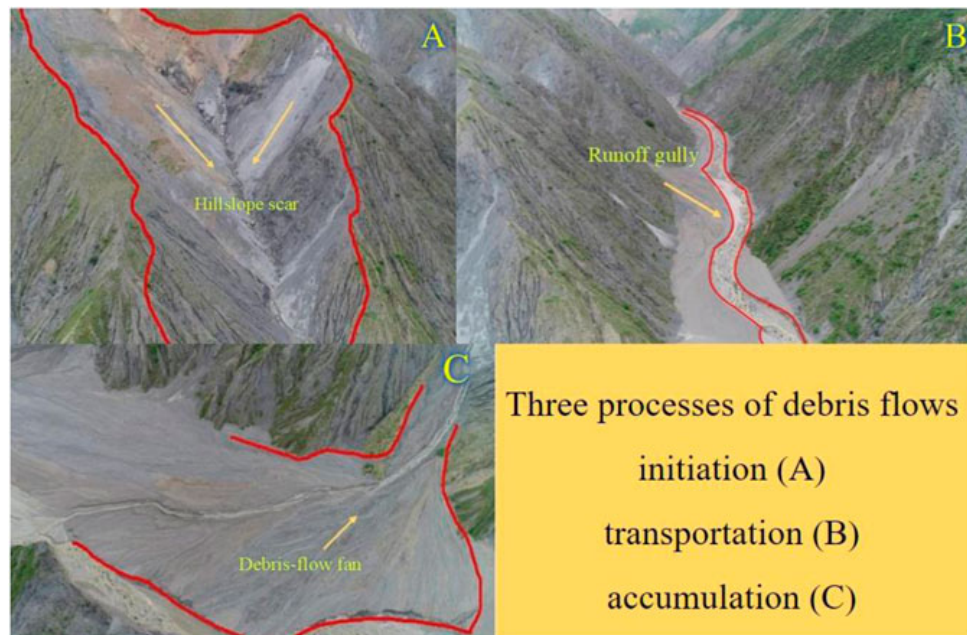


Figure 2.2: Schematic representation of the three main phases of a debris flow: initiation, flow dynamics (or transport) and deposition (or accumulation). From (Zhang et al. 2021).

2.1 Debris flow classification

Debris flows are part of the wider family of geophysical mass movements. They are usually classified between solid landslides and hyperconcentrated flows with high bed load concentration. The classification is based on the solid fraction and the type of material involved, from muddy debris flows to granular debris flows (Coussot and Meunier 1996).

Debris flows is a term covering different types of flows during which a mixture of water, fine and coarse solid particles highly concentrated flows down a slope. The solid particles can range from clay and silt to boulders and trees or even cars and huge debris. Simple classification during the flow only with velocity, depth or width is not entirely relevant as those parameters vary a lot during the disaster (Coussot and Meunier 1996). The solid composition and the materials involved are more stable parameters and easier to measure after an event. As mentioned, debris flows can be encapsulated between hyperconcentrated flows and landslides, positionned at both ends of the mountain flow mass movements spectrum.

Debris flows and hyperconcentrated flows

Debris flow are usually highly transient and unsteady, and can cause surges and waves during their runout. For example, the 1996 Jiangjia Ravine mudflows in China generated 126 successive surges during 5 hours (Li et al. 1983). In comparison with stream and hyperconcentrated flows, debris flows have a tendency to sharp variation of velocity and depth where stream and hyperconcentrated flows have smoother transitions. They also transport the biggest solid particles as part of the flow in a viscous and sort of one phase behavior for a first approximation (Coussot and Meunier 1996). At the opposite, stream flows transports their biggest particles in a slower pushed and rolled mechanism compared to the speed of the surrounding water and suspended particles. Solid coarse or other important materials can even travel faster than the fluid phase due to sieving processes (see section 2.3.2). The stop of a debris flow results in a solid deposition with characteristics close to the flowing state. The deposition is usually poorly sorted (except for the front part highly concentrated of coarse particles, also see section 2.3.2). Other deposits from stream flows would be better sorted and small solids would be washed away by the water phase (the drainage process of debris flow is slow compared to stream flows). The solid fraction is also clearly higher; 1 to 25% for stream flows, compared to 50 to 90% for debris flows.



Figure 2.3: Comparison of solid landslide movement, debris flow and hyperconcentrated flow respectively. (Natural Resources Canada 2025), (Agence France-Presse (AFP) 2017), (Ergenzinger and Jong 2017).

Debris flows and landslides

Landslides are at the opposite end of the spectrum. Landslides movements can generate debris flows under certain conditions, see Section 2.3.1. In general, if no debris flow is generated, landslides behave as a solid mass moving downslope with negligible water effects on the movement (Pierson and Costa 1987). Landslides are usually slower in their active phase until a brutal failure for which the speed can be comparable to debris flows. The deposition characteristics are close to the its original state with a clear stratification of the material and a better sorting of the grains where debris flow, at the opposite, are highly mixed during the movement and grain rearrangement is important.

Coussot and Meunier (1996) classifies the different types of steep slopes mass movements with two mains criteria: solid fraction type (from cohesive clays to non-cohesive cobbles) and solid volume/weight fraction. The classification is conceptual, qualitative and don't take into account a change of composition during the process, as the add of water but gives a good representation of the influence of the type of material and the solid concentration on the behavior of the mass movement. Figure 2.4 illustrates the classification proposed by Coussot and Meunier (1996).

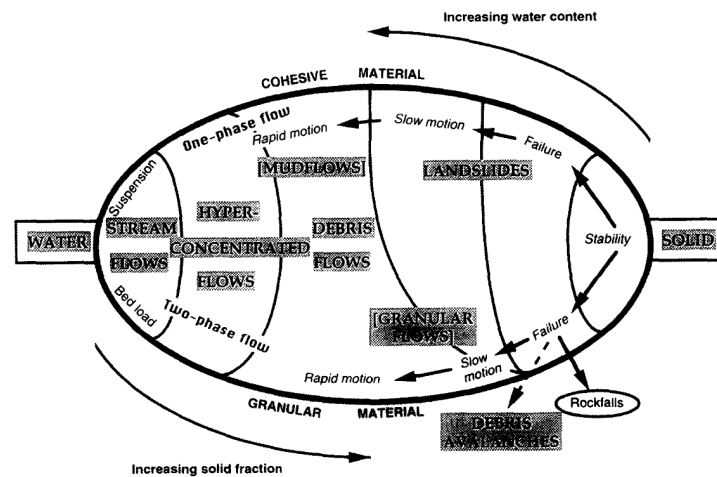


Figure 2.4: *Classification of mass movements on steep slopes as a function of solid fraction and material type.* (Coussot and Meunier 1996).

Inside the debris flow category, different types of flows can be distinguished based on their solid composition. Coussot and Meunier (1996) precise the category into muddy debris flows and granular debris flows. Muddy debris flows are composed by a high fine particles concentration (clay fraction above 10%) with a significant cohesion between grains. Granular debris flows would be composed by less fine particles in order to be dominated by frictional and collisional interactions between grains. Furthermore, Coussot (1992) defined other qualitative limits between debris flows, landslides and hyperconcentrated flows which can be crossed with the previous classification. The limits are based on the solid volume fraction and the clay fraction. Transition from landslide to debris flow would happen for a critical solid fraction (and material type) where the mass movement would induce a fracture after sufficiently large deformation. The transition from debris flow to hyperconcentrated flow would happen for a critical solid fraction (and material type) where settling of particle becomes significant over a given period of time. Of course, the boundaries between the classification is not completely fixed as debris flow can be initiated from a landslide itself if a water source is encounter to saturate the material or high local speed variation breaks the solid-sliding motion of landslide to a flow-like aspect. The same can happen for hyperconcentrated flow, where extremely high erosive capability can sometimes lead to a debris flow. Coussot (1992) made a comparison between these mass movement concerning no transition or crossing between the categories. Figure 2.5 illustrates the classification proposed by (Coussot 1992).

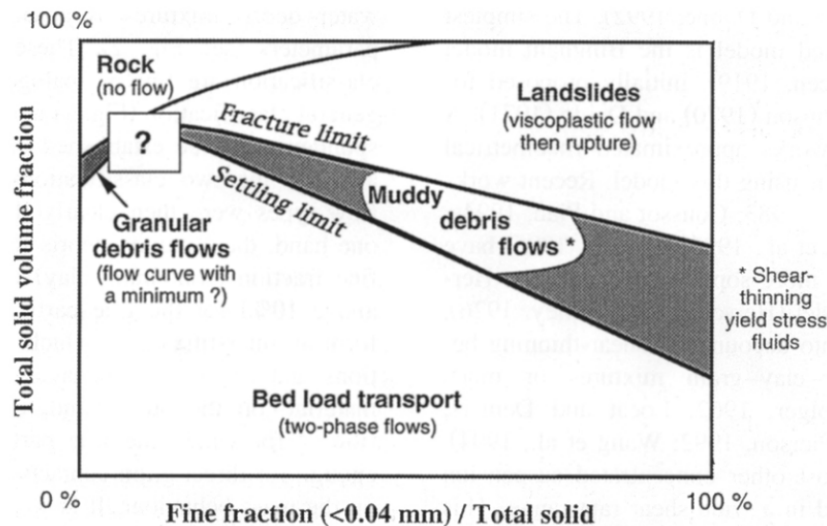


Figure 2.5: *Conceptual rheological classification of mass movements as a function of fine content and solid fraction. The exact limits between the different parts of the diagram should be determined for each material and may slightly vary with flow characteristics.* From (Coussot 1992).

2.2 Composition and material characteristics

Composition and material type are essential to understand the behavior of mass movements. The next section will develop the theoretical main mechanics of debris flow composition and characteristics. The relationships presented here will be used as a tool box for the construction of the mathematical model integrated into Watlab. The mathematical model draws on the physical elements of the composition and attempts to reproduce them as accurately as possible.

In order to evaluate the composition of debris flows, measurements can be carried out in situ on samples harvested after the catastrophic event settled or in laboratory on small scale reproductions. Several parameters are usually considered to characterize the composition: the bulk densities (i.e total and dry bulk density), the solid composition (i.e grain size distribution) and mixture properties (e.g viscosity, hydraulic permeability, compression stiffness). From those parameters, it is possible to reconstruct the main characteristic of the flow.

In situ, if the sample is not extracted right after the deposition, the bulk density is reconstructed by adding the appropriate amount of water to samples from fields after the events. It adds uncertainties and needs estimations. In laboratory, bulk density can be directly measured from samples harvested on the deposition volume at the end of the flow or from normal basal stress data to characterize the evolution of the values during the flow (Richard M. Iverson 1997). From a simple static equilibrium, the bulk density can be estimated based on: $\sigma = \rho g h \cos \theta$. Where σ is the basal normal stress, ρ the bulk density, g the gravity acceleration, h the flow height and θ being the slope angle. This type of measurement gives a precise value along time, considering spacial and time variations of the bulk density, for which debris flow are sensible. Furthermore, it is a non intrusive technique that does not disturb the flow and does not imply having to collect samples by plunging buckets in a powerful flow. The disadvantage of this technique is that it is only valid for steady uniform flows.

The solid material composition, for real case debris flow, can be determined by studying the grain size distribution of the deposition. However, those size distributions are usually biased by the lack of consideration of the biggest particles (cobble, boulders, trees or other large objects) that are difficult to sample (Jon J. Major and Voight 1986) and they should be extracted as soon as possible after the event. The drainage of the fluid phase influences the sediment size distribution, especially for the fine

particles (silt and clay) that can be washed away. Nonetheless, the major solid composition is usually sand, gravel and larger cobbles which can be sampled. The fine particles represent around 10% of the flow mass (Takahashi 1978). In total, the solid fraction can represent 0.5 to 0.8 of the total volume of the flow (Richard M. Iverson 1997). As the grain size distribution is usually diverse, the flow can reach higher densities than homogeneous mixtures by a better grain arrangement.

To precise more the physical properties, solid and liquid phases characteristics are distinguished. The rest of this section follows Richard M. Iverson (1997) notations definitions and reasoning. The solid phase (i.e relatively large grains) is denoted by index s and the fluid phase (i.e pore fluid between grains) by f .

- The mass density of the solid grains is ρ_s and the characteristic diameter is represented by δ (diameter of a sphere equivalent in volume to the grain). The friction angle of the grains is ϕ_g and the restitution coefficient e represents the elastic state of the grains. $e = 1$ defines a perfectly elastic grain and $e = 0$ a perfectly inelastic grain. The volume fraction occupied by solid grains is v_s .
- The mass density of the fluid component phase is ρ_f , its dynamic viscosity μ and the fraction of the volume occupied by the fluid is v_f .

To assamble back the two phases and reconstruct the debris flow mixture, some definition and assumptions are made. At classical shear stress conditions ($< 10^4 Pa$), both phases are assumed to be incompressible. By definition, the volume fraction is such as $v_s + v_f = 1$. We consider the mixture to be fully saturated, meaning that no air is trapped in the pore space between grains. And the bulk density obeys the relation: $\rho = \rho_s v_s + \rho_f v_f$. Finally, k is the hydraulic permeability of the mixture. It defines the ease with which the fluid can flow through the pore space and thus the relative motion of the two phases. The hydraulic conductivity K is linked to the permeability by the relation: $K = \frac{k \rho_f g}{\mu}$. The hydraulic permeability is independent from the fluid and is associated to the geometry of the pores where K considers the fluid properties.

As grain size distribution is usually wide, the threshold between the effective solid and fluid component is delicate to highlight. Richard M. Iverson (1997) rises two questions: (1) *What effectively constitutes the fluid fraction, when a debris flow may contain solids of any size, including colloidal and clay particles carried in solution and suspension?* (2) *If the fluid fraction includes fine solid particles, can it be characterized by the simple properties ρ_f and μ ?* The paper gives some answers by introducing time and length scales. They define a length criterion δ' as the characteristic length of particles for which $\delta \leq \delta'$ are considered as part of the effective fluid phase. This length is totally arbitrary and the modeling is directly impacted by the choice of δ' . Richard M. Iverson (1997) suggests that δ' should be defined in the following way: *If the duration t_D of a debris flow is long in comparison with the timescale for settling of a grain of diameter δ in static, pure water with viscosity μ , the grain must be considered part of the solid fraction.* In order to find the characteristic length δ' , the settling velocity of a grain in a fluid is given by Stokes law ² (Equation 2.1) where ρ_s , ρ_w , μ_w are the bulk density of the grain (generally around $2650 kg/m^3$ but depends on the material) and of water ($\sim 1000 kg/m^3$) and the dynamic viscosity of water ($\sim 10^{-3} Pa \cdot s$) respectively. The timescale for settling is defined as $t_D = \frac{h/2}{v_s}$ ³ where h is the flow thickness and v_s the settling velocity:

$$v_{set} = \frac{(\rho_s - \rho_w)g\delta'^2}{18\mu_w} \quad (2.1)$$

Developing both Stokes and the time t_D with isolating δ' gives:

²The Stokes' formulation comes from a force balance on a spherical grain settling in a fluid with several hypothesis. The grain falling in the fluid should induce a steady state and laminar flow around it (low Reynolds number associate to the grain fall: $Re_s = \delta' \rho_f v_{set} / \mu_w < 1$). The fluid should be incompressible and Newtonian with a bulk density ρ_w and a dynamic viscosity μ_w . The balance of force implies the weight of the grain ($\rho_s V g$), the buoyancy force ($\rho_w V g$) and the drag force ($6\pi\mu_w v_{set} \delta'/2$), $V = \pi\delta'^3/6$ being the volume occupied by the considered grain. At equilibrium (i.e no acceleration, the grain reached the final speed), the sum of those forces is zero and by isolating v_{set} , Stokes law (Equation 2.1) is obtained. The development shows that the formulation is valid for strict hypothesis and should be used with caution. Nonetheless, it gives a first estimation of the settling velocity and help to define the critical diameter δ' separating the solid and fluid phase, as proposed by (Richard M. Iverson 1997).

³ $h/2$ is the mean distance that the grain should travel to settle.

$$\delta' = \sqrt{\frac{h/2 \cdot 18\mu_w}{(\rho_s - \rho_w)g \cdot t_D}} \quad (2.2)$$

For instance, a flow thickness of $h = 2m$ and lasting for $t_D = 240s$, the critical diameter would be around $\delta' = 0.068mm$. This value is close to the boundary between silt and sand ($\sim 0.0625mm$). It means that all particles bigger than the critical diameter should be considered as solid and would be transporter by pushing process during the flow duration. Whereas smaller particles are part of the effective fluid and are suspended during the flow. However, the turbulence created by the flow and the interactions between grains also keep larger grains in suspension. Regarding the fluid phase, it is now possible to decompose it into fine particles and pure water. The bulk density of the fluid can be expressed as in Equation (2.3):

$$\rho_f = \rho_s v_{fines} + \rho_w v_w = \rho_s v_{fines} + \rho_w (1 - v_{fines}) \quad (2.3)$$

Where v_{fines} is the volume fraction of fine particles (defined by the threshold δ') and v_w the volume fraction of pure water. The mass density of the fine particles is assumed to be the same as the solid part of the flow. In practice, measuring ρ_f in situ is difficult. It represents the fluid flowing out of the deposition material once the debris flow has settled. To be able to evaluate this quantity, it is possible to use Equation 2.3, the dry bulk density and the grain size distribution from sample harvested on site. One can evaluate the volume fraction occupied by the fines as follows:

$$m_{dry} = \rho_s V_s + \rho_{fines} V_{fines} = \rho_s v_s V + \rho_s v_{fines} V_f \quad (2.4)$$

As mentioned, $\rho_{fines} = \rho_s$ and $V_f = V - V_s$ is the volume occupied by the fluid phase. Isolating v_{fines} gives:

$$\rho_{dry} = \rho_s v_s + \rho_s v_{fines} (1 - v_s) \Leftrightarrow v_{fines} = \frac{\rho_{dry} - \rho_s v_s}{\rho_s (1 - v_s)} \quad (2.5)$$

The formulation is valid only if there is no loss of fine particules during the drainage process. Meaning the sample should be taken as soon as possible after the event and taken undisturbed. Finally, the bulk density of the fluid ρ_f from Equation 2.3 can be estimated using the value of v_{fines} in Equation 2.5. Evaluation of the fluid part properties are important. The fine grains present in the pore fluid affects the effective viscosity, one of the most important properties for debris flow modeling.

Einstein (1906) deduced a relation to express the effective viscosity of a fluid containing rigid, chemically inert and non aggregating suspended particles. The relation was valid for low fine volume fractions ($v_{fines} < 0.1$) and slow flow regime (Reynolds number $Re \ll 1$). The expression has been adapted by Thomas (1965) for volume fractions of ($v_{fines} < 0.5$). They realized the higher the volume fraction is, the higher the error occurred. The authors collected experimental data, they used non linear techniques and extrapolation to find an empirical relation minimizing the error induced by the non-Newtonian behavior, inertia and non-homogeneous suspension effects. The relation is valid for particle size variation and is given by Equation 2.6:

$$\frac{\mu_f}{\mu_w} = 1 + 2.5v_{fines} + 10.05v_{fines}^2 + 0.00273e^{16.6v_{fines}} \quad (2.6)$$

The effective viscosity of the fluid phase is observed to increase rapidly with the volume fraction of fines. Even if the relation is empirical and does not take into account more complex effects as particle shape (deviation from spherical shape, roughness, etc.), or chemical interaction, it gives a first estimation of the effective viscosity of the fluid phase and an interesting guideline to determine values of the viscosity based on grain size distribution and bulk densities (v_{fines} can be estimated by Equation 2.5).

To conclude this section, (Richard M. Iverson 1997) gives an estimation of typical values of the physical properties encounter in general debris flows. They can be recalled at Table 2.1.

Table 2.1: Typical values of basic physical properties of debris flow mixtures (Richard M. Iverson 1997).

Property and Unit	Symbol	Typical Values
<i>Solid Grain Properties</i>		
Mass density (kg/m ³)	ρ_s	2500--3000
Mean diameter (m)	d	10 ⁻³ --10 ⁻²
Friction angle (deg)	ϕ_g	25--45
Restitution coefficient	e	0.1--0.5
<i>Pore Fluid Properties</i>		
Mass density (kg/m ³)	ρ_f	1000--1200
Viscosity (Pa · s)	μ_f	0.001--0.1
<i>Mixture Properties</i>		
Solid volume fraction	γ_s	0.4--0.8
Fluid volume fraction	γ_f	0.2--0.6
Hydraulic permeability (m ²)	k	10 ⁻¹³ --10 ⁻⁹
Hydraulic conductivity (m/s)	K	10 ⁻⁷ --10 ⁻²
Compressive stiffness (Pa)	E	10 ³ --10 ⁵
Friction angle (deg)	ϕ	25--45

2.3 The three main phases of a debris flow

2.3.1 Initialization and triggering mechanisms

Debris flows' initiation is once again complex and depends on numerous factors. The triggering effects can be induced by snow or ice melt, earthquakes (soil liquefaction), high flood runoff entraining an important amount of sediments or soil slope failure (this list is non exhaustive). The section recalls the principles involved useful for understanding the behaviors and risk evaluation.

Slope failure debris flow requires three mobilization conditions (Richard M. Iverson 1997): first, the mass of material should be unstable and prone to failure, second, there should be a water condition able to saturate the mass of material and third, a transition from gravitational potential energy to internal kinetic energy should be possible and sufficient to transform the sliding motion of the mass into a behavior similar to a flow. The first condition does not imply that the water content condition has to be met before the failure. In some cases, a mass movement can meet a water source during its fall and become saturated enough to transform into a debris flow. The majority of debris flow results from a change in pore pressure because of important rainfall events or snowmelt.

According to (Richard M. Iverson, Reid, and LaHusen 1997), there are three main triggering processes. They can happen at the same time and reinforce each other or behave independently: (1) *widespread Coulomb failure within a soil mass*, (2) *liquefaction of a soil mass by high pore-fluid pressures*, and (3) *production of granular temperature in a soil mass that becomes agitated as it moves downslope*.

Coulomb failure

The well known Coulomb's criterion is usually used to evaluate the stress on a slope where frictional failure are expected. The relation between τ , the shear stress on the slope, the normal stress σ (positive in compression) and the cohesion c is given by Equation 2.7:

$$\tau = (\sigma - p)\tan\phi + c \quad (2.7)$$

Where p is the pore fluid pressure and ϕ the friction angle of the material. The shear stress τ triggering a failure is depending linearly on the effective normal stress ($\sigma' = \sigma - p$) and the cohesion c . The friction angle of such mechanism is divided into two components: the friction angle of the grains ϕ_g and the arrangement of the grains following the rupture surface. This means, the value of ϕ varies during the movement as rearranging is inevitable. When the failure occurs, it is assumed that the material on

the failure plane loses its cohesion ability (Skempton 1964). The theory (reinforced by experimental measurements) predicts that no stresses states exceed the equation above and the material body adapts to fail wherever the condition is surpassed.

Soil liquefaction

The pore fluid pressure also drives the liquefaction potential of the soil. Liquefaction is a state change from a solid soil to a liquid like behavior. It can occur due to cyclic loading (e.g earthquake) or static loading (also called "quick condition"). This process occurs when the pore water pressure increases enough to reduce the effective normal stress to zero ($\sigma - p = 0$). The undrained condition help to reach this state by preventing water to flow out of the soil and thus increasing the pore pressure. Considering a geostatic stress (equilibrium state where the soil is subjected only to its own weight) and a completely saturated soil (water-laden), liquefaction condition is verified when the pore water pressure gradient equals the vertical gradient of stress in the soil (Equation 2.8) (Richard M. Iverson, Reid, and LaHusen 1997). This leads to a complete liquefaction of the considered volume.

$$\frac{\partial p}{\partial z} = \frac{\partial \sigma}{\partial z} = \gamma_{tot} \quad (2.8)$$

Where γ_{tot} is the total specific weight of the soil (solid and fluid phase). The condition means the pore pressure increases with depth at the same rate as the total stress. Here geostatic stress and water-laden condition are necessary. If the soil is partially saturated or if the stress state is not geostatic (e.g stress nonzero at the surface), the effective stress would not be zero even though the p and σ gradients are equals. Cohesive soils can reach this liquefaction state and loose their shear strength if their cohesive bounds are broken (Richard M. Iverson, Reid, and LaHusen 1997). Understanding liquefaction and Coulomb's failure criterion helps to understand the initiation of debris flows for a sliding mass to a fully fluidized behaviour, typical of debris flow.

Generally, it is known that soils with high compaction have tendency to dilate during failure. At the opposite, for loose soils, contraction of the material can occur while failing. The contraction leads to a pore pressure increase and weaken the soil in case of undrained condition or at least a slow fluid movement compare to the failure speed. The process helps to generalize the failure to a global mobilization of the soil body and could also lead to liquefaction if the pore pressure increases enough.

Granular temperature

The granular temperature (also abbreviated in "temperature" for granular material contexts) is a concept used to describe the mobilization energy contained in a granular material. It defines the kinetic energy associated with the disorderly grain velocity (i.e degree of agitation of the solid grains) leading to the fluidization⁴ of the soil mass. The concept was introduced by (Ogawa 1978). The name "granular temperature" illustrates the analogy with the molecular temperature in gases. The granular temperature T is defined in Equation 2.9 and becomes important when the soil mass is already in motion and leaves the quasi-static state.

$$T = \langle (\vec{v} - \vec{v}_x)^2 \rangle \quad (2.9)$$

In Equation 2.9, $\vec{v}$ is the instantaneous grain velocity vector and v_x the average velocity in the direction toward the slope (downstream direction). The symbol $\langle \rangle$ the appropriate averaging operation of all grains. The temperature as defined in Equation 2.9 can represent twice the kinetic energy per unit mass associated with the disorderly motion of the grains. The energy is dissipated by inelastic collision between grains and the energy related to the granular temperature is decreasing if no external energy factor is supplied (e.g no slope) (Ogawa 1978). The authors illustrate the dissipation of granular temperature for a granular material as follows: *shaking up a granular material-for example, a bottle of aspirin or a box of cereal-induces random temperature-like motions between the particles, but once the shaking stops, the particles will almost instantaneously lose their granular temperature and cease all of their motion. To*

⁴Fluidization is a dynamic state in which the particles are suspended in a fluid composition in a way to that the whole the mixture also behaves like a fluid, sometimes dense.

maintain the temperature, energy must be continually added into the random components of velocity to balance that which is lost to the dissipative collisions. The granular temperature is a key concept in the mobilization and fluidization of a soil mass. Increasing the temperature reduces the effective stress σ' and thus facilitates the fluidization. Furthermore, it helps to propagate the movement to the whole soil mass and transform it to debris flow.

A granular temperature rise brings an effective stress reduction. At a resting sheared state, the soil body is maintained in place by the fictional contacts between grains. When mobilization starts, relative speed appears between grains and if the mobilization requires not only a failure surface but a global movement⁵, non uniform speed field appears. Granular temperature means that the grains contacts are shorter and separate between shocks. It means that the granular skeleton structure loosens ($\sigma \searrow$) and effective stress ($\sigma' = \sigma - u$) decreases. The bulk density is also affected.

Bingham and soil mobilization

The three processes presented above characterize the main mechanisms around the mobilization of debris flows. Models, as the Bingham flow model, could be crossed with some processes developed earlier. The Bingham theory (precised in the following section 4) is the heart of the present thesis and the mathematical model is purely based on this theory. It expresses the behavior of non-Newtonian flow, meaning, unlike Newtonian flows (water is usually considered as a Newtonian flow) the material can support a certain shearing value before any motion. This value is called yield strength and corresponds to a threshold between solid behavior (for shear stresses lower than a given value) and liquid behavior (when shear exceeds it). In a sense, this theory is similar to the Coulomb's criterion (Equation 2.7), the material is at rest if the shearing stress applied to it is lower than a value. But unlike Coulomb's formulation, Bingham theory can express a flowing fluid whereas Coulomb describes the triggering between static and dynamic. The Bingham model's yield strength value is supposed to be intrinsic to the material and does not depend on its conditions (as water pressure, porosity, granular temperature, ...). One could think both theories work together to reconstruct a debris flow mobilization. Coulomb's criterion would dictate the yield strength needed for Bingham transformation from solid to fluid. This would lead to a dilemma when soil conditions change to reach a failure: Coulomb advances that when the shear stress exceeds the shear strength, the material breaks and a surface failure appears meaning no shear can surpass the strength, but for a Bingham movement, the shear needs to surpass the value for a certain area in order to transform it into a liquid phase and fail. It means the Bingham's theory would give a deeper failure mass than Coulomb's if the same yield resistance is assumed. Figure 2.6 shows this dilemma (Richard M. Iverson, Reid, and LaHusen 1997). The value of shear strength and yield strength have to be equal for both models as they both divide rest to motion behavior. However, on the left part, Coulomb's theory dictates that as soon as the shear stresses equal the soil strength, a failure surface occurs meaning it does not allow the soil to overpass the strength value and there is a sort of stress redistribution. At the opposite, on right part of the figure, Bingham theory requires the stresses to overpass the yield strength so the solid can "transform" to a fluid movement. It means Bingham's failure would dictate a deeper failure than Coulomb.

The problem can be overcome by considering a changing value for yield strength and field stress in the failure zone. They both should vary with soil porosities, pore pressures and friction angles as those values can be expected to change during failure (Richard M. Iverson, Reid, and LaHusen 1997). Some advanced Bingham theories are developed in this way.

Initialization through erosion

In some cases, particularly in streams of the Alpine region, debris flows can be initiated by a conjunction of bank slides and bed erosion (Davies 1986). It can create a chain reaction where bank slides provide material to the flow which increases its erosive power and thus erodes the bed more efficiently and leads to debris flow mobilization from regular a stream flow. This process is extremely difficult to predict due to the number of parameters involved (bank erosion capability, flow discharge, slope, etc.) (Coussot and Meunier 1996).

⁵If granular temperature generated along a landslide slip surface is transferred sufficiently into adjacent soil, the localized slip can cause more widespread slip, which can mobilize the landslide into a flow. (Richard M. Iverson, Reid, and LaHusen 1997)

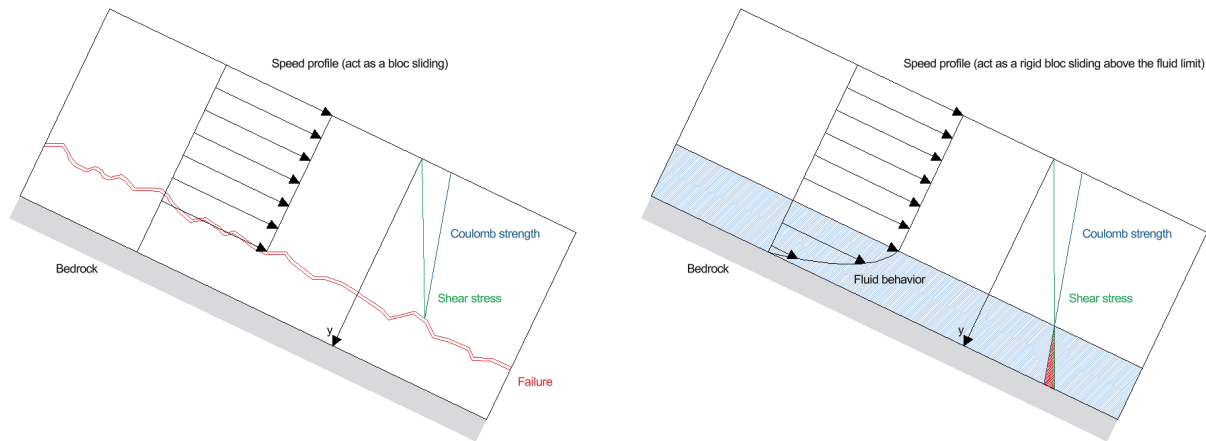


Figure 2.6: Incompatibility of both direct Coulomb and Bingham approach to model a debris flow mobilization, inspired from (Richard M. Iverson, Reid, and LaHusen 1997).

2.3.2 Flow dynamics and behavior

Once the debris flow is mobilized and the fluidization is achieved, the flow behavior can be analyzed. Debris flows can travel long distances, reaching high velocities (up to $10 - 30\text{m/s}$) and can erode the bed and banks of the channel. The flow dynamics is complex and depends on numerous parameters (slope, flow composition, channel geometry, etc.). The flow is usually following a preferential line with a minimum bed elevation, as a stream bed until all the energy is dissipated.

Surges and pulsations are often observed in debris flows, these pulsed waves can travel faster than the front flow itself (Vallejo 1997). One can see in the following video *debris flow surges footage by the San Bernardino County Public Works in California, USA, September 2022*, an example of debris flow surges during a field event. This interesting behaviour can be explained by the capability of debris flows to sustain a given shear stress until a critical state is reached. When the critical state is obtained, the mass is stopped. The stopped material is remobilized by an increase of shear stress, coming for an overpressure from upstream for example, the flow can suddenly accelerate again and create surges. The variation of pore pressure in the flow body can also create pulsations, remobilizing a part of the flow that was close to deposition (Savage and R. M. Iverson 2003)(Kean, McCoy, et al. 2013). Surges are represented in figure 2.7.

Regarding the flow shape, generally, debris flows possess an abrupt front with a steep face composed by the biggest solid particles (e.g boulders, trees, cars, etc) followed by a more watery tailing body with a gentler slope at the end. Between both front face and tail, the body is gradually varying (Richard M. Iverson 1997). The front face of the flow experiences pore pressure (measured at the base of the flow) close to zero and the rest of the flow are about (or overpass) the pore pressure necessary to reach liquefaction (balancing total normal stresses at the base as discussed in the previous section). The biggest clasts accumulate at the front via two ways, either by direct entrainment as soon as the flow front reach them and are torn out or they can be transported from somewhere else in the flow body by kinetic sieving (Richard M. Iverson 1997). In this context, kinetic sieving is the process by which smaller particles falls into the voids created by the movement of the bigger ones. The shear gradient created by the flow movement helps to create voids. In addition, the base of the flow is usually slower than the top (due to friction with the bedrock and the link between shear stress distribution and speed velocity vertical profile). Bigger clasts are thus transported to the front face, as they are likely to stay at the top and travel faster than the smaller ones more concentrated at a slower speed close to the flow base. Classical front segregation is clearly visible in Figure 2.8.

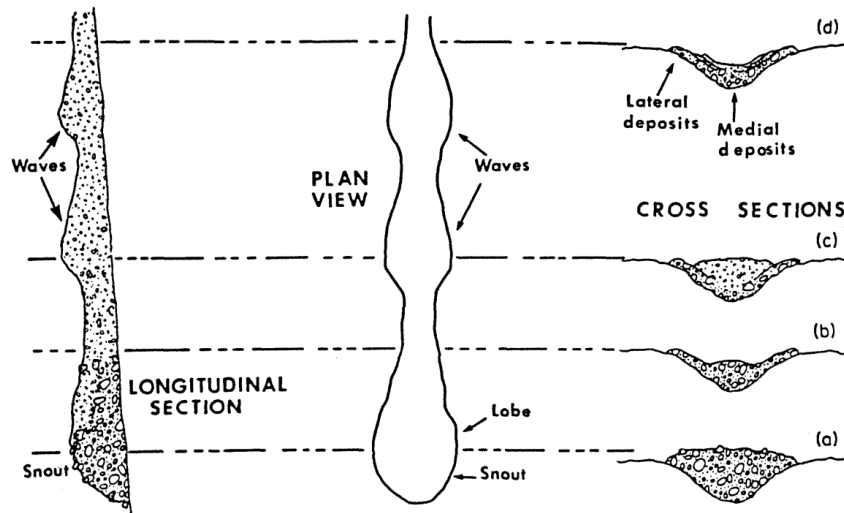


Figure 2.7: Schematic illustration of debris flow surges. The figure is from (Vallejo 1997).

Agitation affects the flow composition by size segregation and pore pressure as mentioned earlier but also by density segregation. The agitation can be characterized by the fluctuating velocity components of the grains around a mean velocity value. The fluctuating component is $v' = v - \langle v \rangle$. The way to quantify the agitation in a flow is to calculate the kinetic energy associated with these fluctuating components and is directly linked to the granular temperature $T = \langle v'^2 \rangle$ (see section 2.3.1). A higher granular temperature means a higher agitation of the grains but also a decrease of bulk density as the grains are more separated. In laboratory experiments, fluctuation is a difficult quantity to measure. The experimental campaign carried out by (Richard M. Iverson 1997), used a 500cm^2 sensor plate to measure stresses at the base. This technique can not capture the small scale fluctuations of grains but only the mean variations. However, the authors also used a smaller sensor plate of 1cm^2 to capture the fluctuations at a smaller scale, highlighting stresses variation not captured by the bigger plate. These fluctuations are linked to the granular temperature and thus to the agitation of the grains. Debris flow implied grain by grain movement but can also imply agglomerate particle moving together as "clusters".

Granular temperature plays a key role in debris flow dynamics, combined with the pore pressure excess, it helps to maintain the flow in a fluidized state. Granular temperature quantifies the agitation of the grains and thus the shocks between them, which reduces the contact time but also the retaining force or friction between them. The fluid movement creates pore pressure exceeding hydrostatic pressure and reduces the effective stress. The reduction of effective stress helps to maintain the fluidized state. The two processes are therefore linked and reinforce the fluid mobility.

Analyzing field observations, it is clear that those flows are majorly laminar (Coussot and Meunier 1996) despite their chaotic behavior at macroscopic scale.

2.3.3 Deposition process

Deposition of a flow is the process by which the flow loses its mobility and comes to rest. The deposition process is a key aspect of debris flow. This behavior is reached when the energy of the flow degrades enough to not be able to maintain the fluidized state anymore. It is accompanied by a radical decrease in the grain temperature toward zero. The front flow part is the first to deposit and creates a natural dam for the rest of the flow (Richard M. Iverson 1997). When deposition occurs, consolidation of the solid network happens and water (i.e the fluid composition: see 2.2) is expelled from the pores by drainage. The consolidation process increases the effective stress and thus the shear strength of the soil. This results in greater resistance of the flow to remobilization compared to its resistance leading to stoppage. Usually the drainage is vertically dominant because length and width scales are greater than the height scale (Jonathan J. Major 1996). Lateral parts of the flow cross-sections in streams are often slower than the rest of the flow because of friction with the banks when concentrated in gullies. This leads to a lateral



Figure 2.8: Debris flow and front segregation. Biggest rocks are likely to be transported with the front abrupt face. The figure is from (Sumner 2017).

consolidation first and lateral levees formation for the tail part. The front part consolidates and usually creates a higher deposit in the center of the channel (Jonathan J. Major 1996). Deposits are thus quite recognizable in the field by their shape and composition dispersion. In some cases, previous deposits of debris flows can be remobilized by a stream flow and lead to a new debris flow event.

2.4 Chapter summary

In this chapter, a literature review around debris flows have been presented, including their characteristics and composition as well as their mechanism from initiation toward deposition. These three mechanisms represent the life cycle of debris flows, and the aim of this thesis is to describe them as a whole in a simplified single numerical model.

Chapter 3

Mathematical models and numerical resolution techniques

The numerical modeling techniques used to represent the behaviour of mudflows are usually risks prediction systems used to simulate a potential event and its consequences rather than being early warning tools that can predict the occurrence of an event. They are useful to understand the dynamics of the flow and to predict the runout characteristics. The module expected to be built in Matlab meets the same requirement. Some early warning systems still exist, for example Imaizumi et al. (2021) who defined a critical rainfall threshold susceptible to trigger a disaster in a specific area.

Numerous numerical models have been developed in the literature to simulate the behavior of mudflows and debris flows. Ranging from simple empirical models to complex physics-based simulations, these models aim to predict their three key characteristics: initiation, dynamics, and deposition patterns (Richard M. Iverson 1997). Models are generally built for risk simulations; evaluating the risk associated with a potential mudflow, determining the traveling distance of a catastrophic runout, assessing the destructive potential of a disaster and determining the geometry of the deposition volume.

The first known model was developed by Voellmy (1955) to simulate snow avalanches. Since then, the model has been adapted and extended to simulate mudflows and debris flows more accurately. Models are now more and more complex, detailed and precise. Advances are possible through an increase in physical knowledge and growing computational capabilities. Latest models are able to simulate 3D and multiphase flows with high resolution. The increase in prediction accuracy is important for effective risk management and mitigation strategies in vulnerable areas (Trujillo-Vela et al. 2022). But those precise models are often computationally expensive and up to now, associated with smaller scale simulations.

By definition, mountainous mass movement are multiple phases. They are composed of a solid, fluid and air fraction (the latter is usually negligible because of saturation). Ideal models should be able to encapsulate these multiple phases. Recent researches seem to focus on this complexity. Nevertheless, it can be relevant to mainly consider the phase that dissipate the most important amount of energy (Trujillo-Vela et al. 2022), fully liquid for homogeneous mudflows and dry grains for debris flow with low liquid component.

Building a solid model asks for some assumptions: the materials are supposed to be continuum and a mass conservation can be applied. The materials do not provide chemical reactions within them and the heat production is negligible. Finally, no abrasion between particles are considered and the flow is usually assumed to be incompressible. Some exceptions exist and models were built without some of these assumptions. (Trujillo-Vela et al. 2022) recalled the associated literature. Several mathematical models exist in the literature and help to build appropriate numerical tools for each behaviour and dimensions targeted in the studied problem. They can be divided into three categories, starting from simple single-phase models, with less calculation cost and robust numerical resolution towards mixtures models and multiphase models. These latter are more complex and cost heavy but increasingly accurate as they capture the real behaviour of each component of the flow.

The choice of a numerical model system is based on three steps. First, the phases to be considered must be assessed. Next, the constitutive relationship are determined. Finally, the numerical resolution method must be selected.

3.1 Phases representation

This section follows the classification proposed by (Trujillo-Vela et al. 2022) which divides the phases representation in: single-phase, mixture and multiphase.

Single-phase models

Models based on single-phase assumption (i.e purely liquid-like) provide a continuum set of equations from mass and momentum conservation. Only one material component is considered, additional material sources are thus modelled as being the exact same constituents as the flow itself. For example, erosion provides a material source identical as the one already present within the flow. Mass and momentum conservation are written in this way:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = \mathbb{E} \quad (3.1)$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \rho \mathbf{u} \otimes \mathbf{u} = \rho \mathbf{g} + \nabla \cdot \boldsymbol{\sigma} + 2\lambda^b \mathbf{u} \mathbb{E} \quad (3.2)$$

ρ is the mass density, $\mathbf{u}$ is the speed vector, $\mathbb{E}$ is the erosion or localized deposition source term, τ^b is the base friction and $\mathbf{g}$ are the volume forces. This model illustrates a single material and works best when applied to a flow largely dominated by one of the component: typically for relatively homogeneous mudflows where the fluid part is significantly larger than blocks on individual larger particles carried out within it. These models are numerically stable and are the lightest models as only one phase and one set of equation are taken into account.

Mixture models

Mixture models, as opposed to single-phases consider a fluid composed both by a fluid part and a solid part (and air if any). To avoid calculating two speed fields, one for each of the components, these models assume that the interaction between phases are neglected and the relative velocities are not separated : $\mathbf{u} = \mathbf{u}_s = \mathbf{u}_f$ (s and f respectively denote the solid and fluid composition), meaning both larger blocks and fluid phase move with identical properties. To appropriately include both behaviours in the total mixture, phases components are scaled by their fraction volume. Equation 3.4 and 3.2 are adapted to respect $\rho = \alpha_f \rho_f + \alpha_s \rho_s$, $\rho \mathbf{u} = \alpha_f \rho_f \mathbf{u}_f + \alpha_s \rho_s \mathbf{u}_f$ and $\boldsymbol{\sigma} = \alpha_s \boldsymbol{\sigma}_s + \alpha_f \boldsymbol{\sigma}_f$. α_i being the fraction volumes and the total sum of each fraction volume should be one (here neglecting air). Mixture models work best for flows which are respectively homogeneous for each of their component, would it be the solid part or fluid part. To capture concentration variations in solid particles as segregation, mixture models add a new mass conservation to the set of equations, Equation 3.3. These equations follow the solid composition mass variation in space and time, capturing changes inside the material (S_s is the solid source term). The simple mass conservation of the total composition $\rho = \alpha_f \rho_f + \alpha_s \rho_s$ will not be able to capture solid concentration variation within the mixture material as it only rules the global mixture with external sources, not internal composition variation.

$$\frac{\partial \alpha_s \rho_s}{\partial t} + \nabla \cdot \rho_s (\alpha_s \mathbf{u}) = S_s \quad (3.3)$$

The erosion and deposition are here better captured as the source terms add directly solid materials (e.g rocks or soil grains) which is usually the process observed in nature. When the flow tears the soil away at its base, solid particles join the flow rather than an already prepared mixture of mud identical to the one already part of the flow. Unlike single-phase model, erosion is better captured by mixtures. The cost consequence to solve them is higher than single-phase calculation but they are a great compromise concerning accurate representation of nature and numerical efficiency. They are able to capture more complex phenomena such as segregation or changes in concentration of solid particles and agglomerates. But they can not represent speed variation, for example large grain concentration at the front of the flow. This phenomena happens because of speed difference between larger particles and interstitial fluid (see Section 2.3.2)

Multiphase models

The most precise way to account for each individual behaviour is to build a set of equation for every part of the flow. It increases the complexity of the model and the numerical cost but also increases the accuracy and captures better the real natural effects. As opposed to mixture models, relative velocities can be observed. However, multiphase models ask for a representation of interaction between phases through forces equations. The final complex model is then based on:

$$\frac{\partial \alpha_k \rho_k}{\partial t} + \nabla \cdot (\alpha_k \rho_k \mathbf{u}_k) = \mathbb{E}_k \quad (3.4)$$

$$\frac{\partial (\alpha_k \rho_k \mathbf{u}_k)}{\partial t} + \nabla \cdot (\alpha_k \rho_k \mathbf{u}_k \otimes \mathbf{u}_k) = \alpha_k \rho_k \mathbf{g} + \nabla \cdot (\alpha_k \boldsymbol{\sigma}_k) + \mathbf{F}_k + 2\lambda_k^b \mathbf{u}_k \mathbb{E}_k \quad (3.5)$$

k represents each phases of the flow, α_k their volumetric fraction and the forces between them are captured by $\mathbf{F}_k$ which respects Newton's third law as drag forces for the solid particles traveling in the fluid part. Two-phases and three-phases models are now becoming increasingly frequent in the literature (Trujillo-Vela et al. 2022). These models are chosen when the interaction between the phases plays an important role and can't be ignored. The cost to be paid is a significant computational resources consumption and a more complex set of equations to be treated.

3.2 Constitutive relationships

Once the phase model is established, the constitutive relationships have to be highlighted. Constitutive relationships are equations linking two physical quantities. In this case it is appropriate to use stress and strain (or strain rate). A clear relation between stress and strain determines the specific behaviour of each materials and their interaction. Stress tensor is then defined and the term $\nabla \cdot \boldsymbol{\sigma}$ or $\nabla \cdot (\alpha_k \boldsymbol{\sigma}_k)$ in Equation 3.2 or 3.5 is adapted accordingly. Depending on the phases considered, different constitutive relationships can be applied. They should be chosen carefully for each phase to correctly capture the physics involved. All together, constitutive relations of each phase and relations linking them, defines the global rheological model. The constitutive relationship is the local law of the material; the rheological model is the coherent set used to describe its macroscopic behavior, encompassing each behavior specific to each material.

For single phase models, only one constitutive relationship is needed, it directly defines the rheology, as presented in Section 4, this has the advantage to capture the whole behaviour in few parameters and to lower the number of equations.

When using mixture models or multiphase models, different relations can be applied to each phase. The stress tensor is then the sum of each phases constitutive relationship component (scaled by their volumetric fraction). In this case, solid and fluid parts are often distinguished. The solid part are the larger grains. The distinction between larger grains and the effective fluid were introduced earlier in Equation 2.2. As a reminder, larger grains are distinguished from the effective fluid phase if their settling time in water is short in comparison to the characteristic duration of the flow.

Solid: a soil element

The solid part is often modelled as little element of soil, obeying laws such as *elastoplasticity* with *Hooke's law* for the elastic part, *Mohr-Coulomb* for the failure criterion and *plastic flow rule* for plastic deformations for example. Numerous other constitutive models are available; perfectly plastic or even more recently μ -rheology. The latter states a non constant friction coefficient and dependent on the speed of deformation and confined pressure; the slower the flow, the more resistant is the friction coefficient and at the opposite, when the flow is fast, it is more fluid.

Fluid: interstitial

As far as an element of fluid is considered, the constitutive relationship is based on volumetric and

deviatoric stress tensor decomposition (pressure and shear stress). The fluid pressure is mainly modelled as a hydrostatic pressure, but more complex pressure models still exists, accounting for small compressibility for example. The shear stress is often modelled as a link between with shear rate through viscosity. Newtonian and non-Newtonian fluids are widely used in the literature for this purpose. A lot of models are available. Two of them are presented in Section 4, they are exposed as a rheology, ruling the flow at a global scale but those relations can also be used here, for multiphase application to account for the shearing part of the interstitial fluid constitutive relationship.

Interaction between phases

Finally, to connect each phases in multiphase models, interaction constitutive relationships are needed. Volumetric component links the total stress of the solids σ_s and the fluid pressure p for its volumetric part. A typical expression is the *Terzaghi's effective stress principle* defined by the total stress lowered by the pore pressure: $\sigma_s = \sigma'_s - p\mathbf{I}$ (fluid mechanics and soil mechanics have different convention for positive pressure directions). σ'_s is the effective stress tensor, p the pore pressure and $\mathbf{I}$ the identity matrix. This principle is widely used in soil mechanic to represent the static interaction between solid particles and interstitial fluid. The deviatoric term is often a drag force, proportional to the relative velocity between phases. In most cases, it is empirically determined (Trujillo-Vela et al. 2022).

Constitutive relationships are a key part of numerical models as they define the behaviour of each phase and their interaction through the equations of motion. For multiple phases models, the organization of such relationships are summarized and illustrated in Figure 3.1. The figure provides a small overview of the possible relations. Numerous models were not presented here.

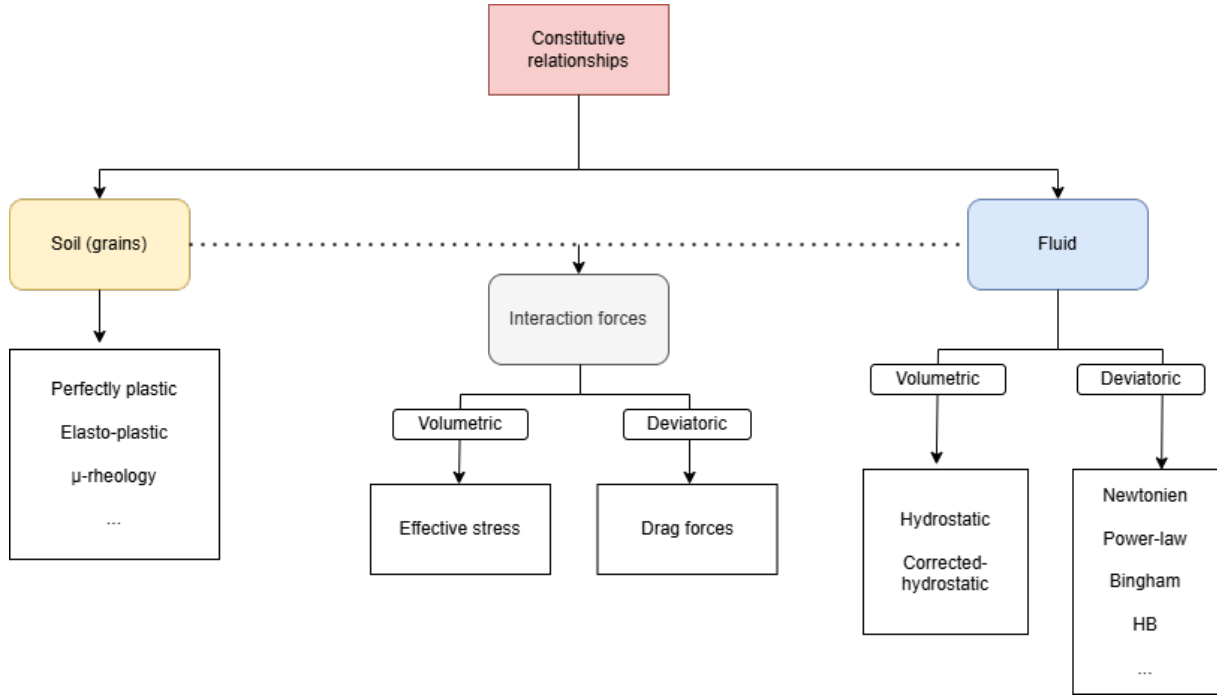


Figure 3.1: Schematic representation of constitutive relationships organization for multiphase models. Adapted from (Trujillo-Vela et al. 2022).

3.3 Numerical scheme

As mentioned earlier, numerical models are based on mass and momentum conservation equations adapted to the phases representation presented above. These equations should be solved for a given number of dimensions (1D, 2D or 3D) and over a specific domain. 3D models are the most precise but definitely too expensive for any practical use at large scale. 2D models are the most commons. They are simplification

of the real 3D problem. They can be depth-averaged, assuming that the vertical dimension is negligible compared to the horizontal ones. In this case, all the vertical variations of horizontal variables are averaged by integration over the depths and all vertical velocities are neglected. They are called *depth-averaged models*, *depth-integrated models* or *shallow water equations* (SWE) (Trujillo-Vela et al. 2022).

Variables such as horizontal velocities and stresses are then only function of horizontal coordinates and time. They are now considered to be uniform over the depth and calculated as:

$$\bar{f}(x, y, t) = \frac{1}{h(x, y, t)} \int_{z_b}^{\xi} f(x, y, z, t) dz \quad (3.6)$$

Here $\bar{f}$ is the depth-averaged variable, h the flow depth, z_b the bottom elevation and ξ the free surface elevation. The overbars denote depth-averaged quantities but this notation is often omitted in the literature for simplicity.

The Figure 3.2 schematically represents the depth integration of velocities. On the left side of the figure, is represented a random velocity profile with decomposed components along every axis; vertical speed along z in green and horizontal speeds in red and blue (u, v and w). When doing depth integration, the speed along z , w is neglected and horizontal components are considered to be constant along the depth with a value respecting the depth integration: $\bar{u}$ and $\bar{v}$. This is represented in right hand side of the figure.

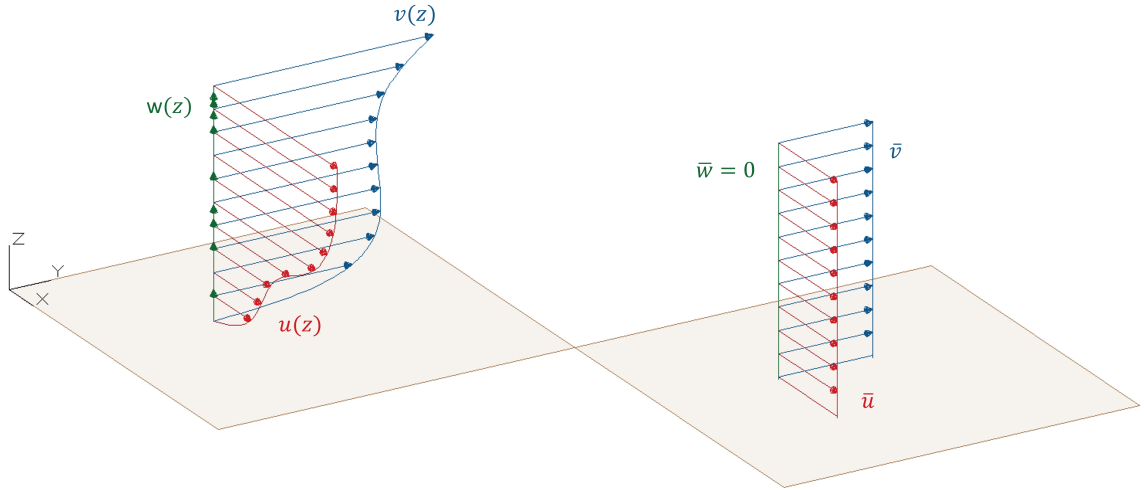


Figure 3.2: Illustration of the depth averaging of velocities.

Equations 3.4 and 3.2 following the 2D depth integration become:

$$\frac{\partial}{\partial t} (\alpha_k h) + \frac{\partial}{\partial x} (\alpha_k h \bar{u}_k) + \frac{\partial}{\partial y} (\alpha_k h \bar{v}_k) = E_k \quad (3.7)$$

$$\frac{\partial}{\partial t} (\alpha_k h \bar{u}_k) + \frac{\partial}{\partial x} (\alpha_k \beta_{k(xx)} h \bar{u}_k^2) + \frac{\partial}{\partial y} (\alpha_k \beta_{k(xy)} h \bar{u}_k \bar{v}_k) = S_{k(x)} \quad (3.8)$$

$$\frac{\partial}{\partial t} (\alpha_k h \bar{v}_k) + \frac{\partial}{\partial x} (\alpha_k \beta_{k(xy)} h \bar{u}_k \bar{v}_k) + \frac{\partial}{\partial y} (\alpha_k \beta_{k(yy)} h \bar{v}_k^2) = S_{k(y)} \quad (3.9)$$

$$S_{k(x)} = \alpha_k b_x h + \left[\frac{\partial}{\partial x} (\alpha_k h \bar{\sigma}_{k(xx)}) + \frac{\partial}{\partial y} (\alpha_k h \bar{\sigma}_{k(xy)}) \right] + \alpha_k \bar{\sigma}_{k(xz)} \Big|_{z=b}^{z=\xi} + h \bar{F}_{k(x)} + 2\lambda_k^b \bar{u}_k E_k \quad (3.10)$$

$$S_{k(y)} = \alpha_k b_y h + \left[\frac{\partial}{\partial x} (\alpha_k h \bar{\sigma}_{k(xy)}) + \frac{\partial}{\partial y} (\alpha_k h \bar{\sigma}_{k(yy)}) \right] + \alpha_k \bar{\sigma}_{k(yz)} \Big|_{z=b}^{z=\zeta} + h \bar{F}_{k(y)} + 2\lambda_k^b \bar{v}_k E_k \quad (3.11)$$

These equations are hyperbolic-parabolic and valid for single-phase, mixtures and multiphase by choosing the appropriate k index⁶. $\beta_{k(ij)}$ are the momentum correction factors from depth integrating the advective terms (Trujillo-Vela et al. 2022). In practice, they account for squared mean speed profile errors: $\beta_{xx} = \frac{\bar{u}^2}{\bar{u}^2}$, $\beta_{xy} = \frac{\bar{u}\bar{v}}{\bar{u}\bar{v}}$, ... The terms b_i are body accelerations (gravity projected along the slope), since the governing equations are written per unit mass. By assumptions on the stress tensor $\bar{\sigma}_{k(ij)}$ (being the depth-averaged stresses normalized by the density), these equations can become purely hyperbolic and the process is governed by the convection. The appropriate assumptions on this tensor, can neglect the divergence of the deviatoric stresses in the horizontal plan, the system therefore becomes purely hyperbolic. Otherwise, turbulent or viscous diffusion introduces a parabolic character through second-order spatial derivatives.

Finally, 1D integrated models could be obtained by doing the same integration process. But they are less used as the versatility of this approach is relatively low. It only fits for very narrow applications with insignificant width variation of a flow which is considered long in its propagation direction.

3.4 Numerical resolutions

First, it is important to note that analytical solutions exist for some severely restricted applications. The hyperbolic-parabolic system (or purely hyperbolic) can be solved with several differential equations tools as Methods of Characteristics, Separation of Variables and Similarity Solutions (Trujillo-Vela et al. 2022). This work does not aim to recall those developments but they can be used in very simple case to validate numerical codes before applying them to more representative applications.

Numerical methods require phases establishment, constitutive relationship and spatial scheme for determining the equations to be solved. In a second step, they ask for a discretization, both in time and in space. In space, the discretization can be mesh-based or mesh-free.

Mesh-based:

The first one approach the domain by dividing it into smaller elements and the equations are solved on each one of them at each time step. Several calculation resolution are based on this discretization: the Finite Difference Method (FDM), the Finite Element Method (FEM) and the Finite Volume Method (FVM) are typical examples.

FDM is the most intuitive one. The calculation is settled on the connection nodes between the cells and derivatives are established by Taylor approximation over those connection points. The difficulties with this method lie in its unstable nature, its indirect application on unstructured mesh and the instability of shocks modeling, the FDM postulates that the solution is sufficiently regular to be differentiable (mudflows are particularly affected by shocks with dry front propagation or surges and unstructured mesh are widely used in large scale calculation).

FEM is suitable for unstructured meshes and the unknowns are approximated by continuous shape functions over the mesh elements. Although widely used in solid and fluid mechanics, the classical FEM formulation is not adapted for sharp variable variation or shock-like discontinuities. The continuity of shape functions and the underlying weak form of classical FEM stipulates a certain smoothness of the solutions. Accurate shock capturing therefore requires stabilizing techniques or modified FEM formulation, as the classical Galerkin is not well adapted in this context.

Finally, FVM is particularly well suited for the numerical simulation of mudflows and debris flows. The resolution is based on the conservation of physical quantities over each control volume, defined by

⁶Note that the governing equations are written after division by the density.

the mesh element filled with material. The governing equations are written in an integral form and the resolution lies on fluxes equilibrium throughout the interfaces of the mesh elements. This locally conservative formulation makes FVM robust for handling strong gradients and discontinuities, while remaining applicable to unstructured meshes. Its advantage rely on the fact that FVM does not attempt to smooth the solution; it balances fluxes.

Mesh-free:

The second technique, mesh-free, partitions by acting directly on the material and considering particles or points evolving in time and space with a lagrangian approach. The particles or points are carried within the flow flux and the resolution follows directly the moving particles. The two most common methods are Smooth Particle Hydrodynamics (SPH) and the Discrete Element Method (DEM).

SPH represents particles as if they carried all the surrounding information within them. Each particle transport mass, speed, pressure or energy information. They are not actual grains but rather calculation points in a continuous environment. To recover a variable on a point, a weighted mean is performed with its neighboring cells. This technique allows great deformation and handle well rupture, impacts and sharp transition. However, many calculation points are needed for a correct representation and the cost of such calculation on each point is particularly heavy.

In contrast to SPH, DEM uses directly a grain assumption. Each particle are assumed to be a little element of material possessing a size, a mass and a given rigidity, allowing for particles to interact with collision and friction. The method follows each grain individually. It asks for an extremely large amount of small particles to model a representative real scale disaster. Moreover, time discretization is very fine and prevents the modeling of long processes at a reasonable computational cost. Being precise for small scale and short representation, this method is not adapted for large scale application. Furthermore, it is intrinsically designed to handle purely granular flows, without any interstitial liquid component. This is far from the application for liquid mudflows.

3.5 Single phase, 2D depth-averaged and FVM: the chosen model

Over the variety of models and available equations presented in this section, a single phase, 2D depth-averaged model and FVM resolution approach was chosen. Up to date, depth-averaged equations are still the best compromise and largely dominant in the most researches around this topic. As argued before, Finite Volume Method are particularly adapted for mudflows and debris flows representation, with a great compromise between the good capture of governing physical processes (i.e handling shocks for dry front and surges representation) and the reasonable cost calculation in comparison to other numerical methods.

In addition to its advantages, Shallow Water Equations (SWE) solved with FVM was already implemented in the program Watlab developed at UCLouvain (Hydraulics Group, UCLouvain 2025). It resolves the equations for water dominant behaviours. Therefore, the challenge in this thesis was not to implement a resolution from scratch but rather to find the appropriate tools to be added in the existing program in order to change the material to be modelled. The program should be able to simulate water or mudflows depending on the choice of the user with a simple command. This approach ensures both numerical robustness already proven and practical applicability.

Finally, several hypothesis were made from this global model to precise it:

Single phase material was adopted. This is the simplest approach and it stipulates purely homogeneous behaviour, which differs from reality, particularly for granular debris flows. From now on, the document will then exclusively focus on *mudflows*, assuming a continuous and homogeneous material with constant properties all along the life of any simulations. In order to be as close as possible to the physical model with the numerical model. In this context, relatively homogenous mudflows are not affected by segregation of

granular front as the grain size distribution is narrow and no larger blocks are assumed. According to the homogeneous assumption, the model precision will be acceptable and not far from reality. Furthermore, single phase was the main choice as it will prepare ground rules and establish global behaviour for further and more precise works, that will, hopefully follow this thesis.

Next, it was assumed that no erosion is possible. The experiments used to validate the numerical model were carried out a wooden ground base (the experiment will be presented in the Section 5), preventing any erosion from occurring. Therefore, the model is, in theory, perfectly capable to represent the experimental campaign targeted. However, this assumption is the most problematic for large scale application. Mudflows have a high erosion capacity and the increase in the moving volume should not be neglected for real scenarios. In further works, this aspect may be treated in priority⁷.

The last assumption from the 2D depth-averaged and FVM is the neglect of the divergence of the deviatoric stress tensor. The depth-averaged stresses $\bar{\sigma}_{k(ij)}$ include both an isotropic pressure contribution and deviatoric stresses. Under the hydrostatic assumption, the isotropic part yields the classical hydrostatic pressure term, while the divergence of deviatoric stresses in the horizontal plan is neglected. These terms represent internal momentum diffusion and are usually omitted to preserve a purely hyperbolic system and not to deal with second order derivative, harder to be implement numerically. In contrast, the basal shear stresses $\bar{\sigma}_{k(yz)} \Big|_{z=b}^{z=\zeta}$ ⁸ are retained as they represent an external momentum exchange with the soil bed and control dissipation and flow arrest.

In other words, the depth averaged stresses can be rewritten:

$$\bar{\sigma}_{k(ij)} = -p\delta_{ij} + \tau_{ij} \quad (3.12)$$

p is the pressure, δ_{ij} the identity matrix and τ_{ij} the deviatoric stresses. Under the hydrostatic assumption, pressure variations in the vertical direction dominate over horizontal variations. The depth-averaged pressure can be expressed as a function of the flow depth. The depth averaged of the hydrostatic pressure is:

$$\begin{aligned} \frac{\partial p}{\partial z} &= -\rho_k g \\ p(z) &= \rho_k g(\zeta - z) \\ \bar{p}_k &= \frac{1}{h} \int_b^\zeta p(z) dz = \frac{1}{2} \rho_k g h \end{aligned} \quad (3.13)$$

So in the momentum equation along the x axis it gives:

$$\frac{\partial}{\partial x} (\alpha_k h \bar{\sigma}_{k(xx)}) + \frac{\partial}{\partial y} (\alpha_k h \bar{\sigma}_{k(xy)}) = \frac{\partial}{\partial x} (-\alpha_k h \bar{p}_k) = \frac{\partial}{\partial x} \left(-\alpha_k h \frac{1}{2} \rho_k g h \right) \quad (3.14)$$

$\alpha_k = 1$, and $\mathbf{F}_k = 0$: the assumption of single phase reduces the number of equations to those presented in Equation 3.15 and no interaction forces between phases can occur if only one is modelled. The traction at the free surface is neglected, the internal stresses are also neglected and no erosion or deposition are considered. A slight nuance should be added to the previous statement; no deposition is taken into account in the sense of E_k . This term adds or removes material from the flow. The deposition represented by this term is a sort of absorbing mass concept if not combined with an equation

⁷Erosion modules already exist in Watlab and, at first glance, the implementation process described below, particularly for LHLLC fluxes treatment and stopping criteria, could be similar to the required information for erosion-deposition. However, a hydrodynamic model with a sediment transport law is necessary to correctly use the Exner relationship, given the different transport capacities of mudflows compared to conventional river flows.

⁸The surface ($z = \zeta$) stresses are neglected for open surface flow and the basal stresses are sometimes noted as t_i^B in the literature.

ruling the bed elevation (e.g Exner). Deposition in the sens of a material stops is possible with the formulation presented in Equation 3.15. This approach of the deposition will be explained and justify later.

The β_{xx} , β_{xy} and β_{yy} , were assumed to be 1. This assumption is already made in Watlab and was not modified in this work but could be criticized for mudflows. These coefficients, often called *Boussinesq coefficients*, express the correction of rations between integrated velocities: $\beta_{xx} = \frac{\bar{u}^2}{\bar{u}^2}$, $\beta_{xy} = \frac{\bar{u}\bar{v}}{\bar{u}\bar{v}}$,... For turbulent water profiles behaving in a plug-like shape (sharp transition close to the base and quasi-constant over the depth), taking Boussinesq at 1 is a good approximation but for laminar flows, as the majority of mudflows (a quick analysis of Reynolds number gives an idea of the laminar tendency of mudflows), the speed profile is no more a barely perfect plug but more a quadratic shape (sort of half Poiseuille profile) fused with a constant part, this velocity profile is typically illustrated in Figure 6.3. For example, considering a purely quadratic profile (with no constant part) will give $\beta_{xx} = 1.8$, this development can be verified in the Appendix A.3. $\beta = 1.8$ actually designates a laminar flow without any shear resistance at rest; a highly viscous Newtonian flow. This is the extremity of the spectrum that could take β . Therefore, the values should be evaluated between 1.8 and 1, and should respect as much as possible the flow conditions. This quick reasoning simply illustrate the variability of the Boussinesq coefficients without considered a purely plug formation and could also be an area for improvement in the future.

The global equations described above (see Equations 3.7, 3.8, 3.9, 3.10 and 3.11) are simplified in the following way:

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) + \frac{\partial}{\partial y}(h\bar{v}) = 0, \\ \frac{\partial(h\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(h\bar{u}^2 + \frac{1}{2}gh^2\right) + \frac{\partial}{\partial y}(h\bar{u}\bar{v}) = b_x h + t_x^B, \\ \frac{\partial(h\bar{v})}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}\bar{v}) + \frac{\partial}{\partial y}\left(h\bar{v}^2 + \frac{1}{2}gh^2\right) = b_y h + t_y^B. \end{cases} \quad (3.15)$$

The final set of equations, based on the posed hypothesis represents to mathematical model to be numerically implemented. It is actually the one initially coded in Watlab and respects both the behaviour water and of homogeneous mudflows. The adaptation from water modeling to mudflows modeling takes place in the last term of the two momentum conservation equations: t_x^B and t_y^B . The constitutive relationship specific to mudflows are hidden in this friction term.

Now that the global mathematical model is introduced, next will be presented the constitutive relationship for homogeneous muddy flows, with only one liquid phase. It will enable the present model to account for the expected behaviour.

Chapter 4

Single-phase rheology of debris flows

Constitutive relationships were introduced before. One important highlight is that the constitutive relationship of a single phase material corresponds to its rheological model, as only one behaviour is considered and no interaction between multiple phases will be supposed. The choice of a single phase model in this work simplify the model but is not the only way to express the rheology. As exposed, more complex model can assemble different constitutive models to obtain the global rheology. By considering a single phase, we do not assume the material to be perfectly homogeneous but we try encapsulate the little local physic interactions at a grains-scale approach and translate it into a global flow behaviour at a macroscopic scale. Therefore single phase means that the mathematical model thinks only one continuous material is modelled. But its mechanical properties were appropriately chosen to encapsulate the real physical characteristics with grains and fluid interaction. This simplification is valid when the scale of observation is large compare to the grain size. It is hence assumed here. Generally grains can measure up to several centimeters (for mudflows, debris flows are subjected to larger objects) and the simulations can reach hundred of meters or kilometers long.

Rheology is the study of the flow deformation under applied stresses. It is a key aspect of mudflows as it rules the majority of their characteristics (mobility, erosion capability, deposition, etc.) and are the basis of most numerical modeling including the one developed in this thesis.

Unlike water, which is considered to be a Newtonian fluid, meaning the viscosity is independent on the forces applied to it, debris flows are usually modeled as non-Newtonian fluids. It supposes that their viscosity is not especially constant and depends on the shear stress applied. They are characterized by their ability to sustain shear stress without flowing; modeled by a yield strength value. The yield strength is the stress value that has to be exceeded for the material to start flowing (or stops the flow if the stress decreases below this value). The strength can come from the clay particles interaction network (Coussot and Meunier 1996) or from the effective frictional forces between larger grains and the interstitial fluid (Vallejo 1997). These cohesive bounds must be broken to allow the flow movement.

Rheological models are usually represented by their relation between shear stress τ and shear rate $\dot{\gamma}$. The shear rate is the spatial gradient of velocity in the perpendicular direction to the applied shear stress. For water (Newtonian fluid), the relation is linear: $\tau = \mu\dot{\gamma}$, where μ is the dynamic viscosity. It means whatever shear stress applied to the fluid, it will start flowing, as variation in speed will occur for any non zero shear stress.

Several rheological models exist to represent debris flows. The most used are the Bingham model (Bingham and Green 1919) and the Herschel-Bulkley model (Herschel and Bulkley 1926). Considering a simple shear flow (velocity u in the x direction and shear gradient in the y direction), those two models are defined as follows:

- **The Bingham model** is characterized by two parameters: the yield strength τ_y and the plastic viscosity $\mu_{pl} = \mu$. The relation between shear stress τ and shear rate ($\dot{\gamma} = \partial u / \partial y$)⁹ is given by Equation 4.1:

$$\tau = \tau_y + \mu\dot{\gamma} \quad \text{if} \quad \tau > \tau_y \quad (4.1)$$

$$\dot{\gamma} = 0 \quad \text{if} \quad \tau \leq \tau_y \quad (4.2)$$

⁹This expression can be easily demonstrated by looking at a simple sheared cube of fluid: γ is the deformation angle, Δx is the displacement of the upper face, Δy the height of the cube and Δt the time needed for the displacement. For small displacements, $\tan \gamma \approx \gamma = \Delta x / \Delta y$. The shear rate is then $\dot{\gamma} = d\gamma / dt = d(\Delta x / \Delta y) / dt = du / dy$ (y is independent from time).

- **The Herschel-Bulkley model** is a generalization of the Bingham model by adding a power law behavior for the shear rate. It allows to represent shear thinning or shear thickening. The model is characterized by three parameters: the yield strength τ_y , the consistency index K and the flow behavior index n . The relation between shear stress τ and shear rate $\dot{\gamma}$ is given by Equation 4.3:

$$\tau = \tau_y + K\dot{\gamma}^n \quad \text{if } \tau > \tau_y \quad (4.3)$$

$$\dot{\gamma} = 0 \quad \text{if } \tau \leq \tau_y \quad (4.4)$$

Both models represent well debris flow behaviour. The Herschel-Bulkley model is more versatile due to its power law component but Bingham model is simpler to implement and decreases the calculation cost (this point is exposed further). The simplicity of Bingham leads to a wider use in debris flow numerical modeling. Models are represented in Figure 4.1, where they are compared with Newtonian-type model.

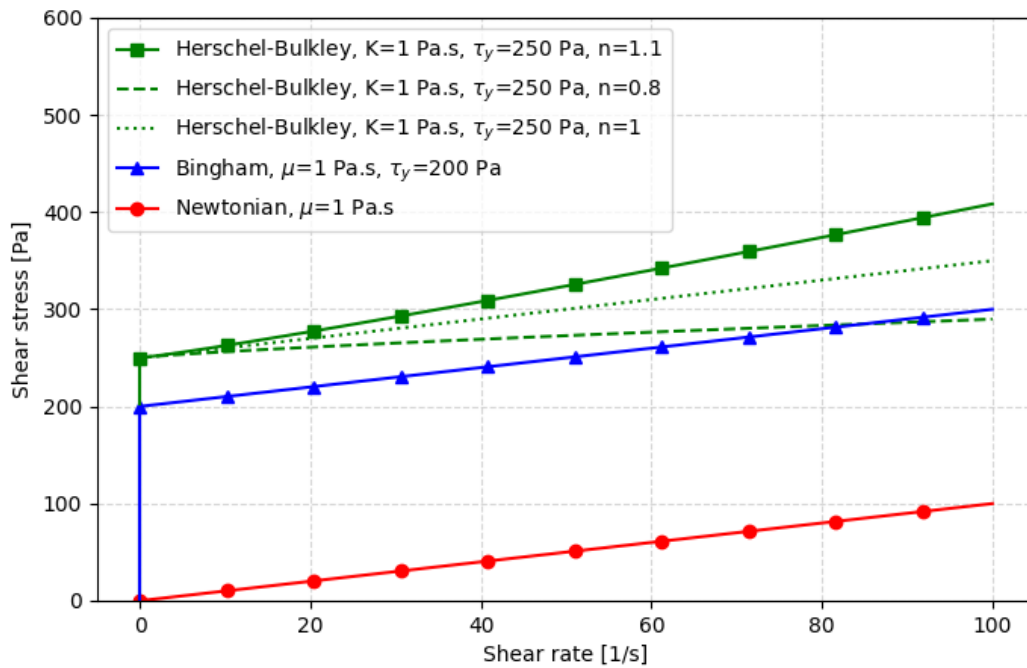


Figure 4.1: Schematic representation of different rheological models.

Bingham is in reality a particular case of Herschel-Bulkley where $n = 1$ and $K = \mu$ and some articles use the name of the latter to designate both. The models are often used for muddy debris flows or mudflows, where the flow is dominated by the viscosity of the pore fluid. Granular debris flows (see 2.1), are often modelled with another method: Bagnold. The model is valid for granular flows where collisions dominates, so called the grain inertia regime. Bagnold (1954) gives the relation in Equation 4.5 (ρ_s is the grains bulk density, d is their mean representative diameter and k is an empirical coefficient depending on the concentration). Later, its work was reviewed and the equation slightly adapted to better fit the data measurements carried out during the first work (Hunt et al. 2002).

$$\tau = k\rho_s d^2 \dot{\gamma}^2 \quad (4.5)$$

The Bagnold model describes the behavior of granular flows dominated by particle collisions and inertia. It is also a single phase model, where the domination comes from the solids and not the effective fluid anymore. In this regime, for granular debris flows, both air and interstitial fluid have a negligible effect,

and the stress increases with the square of the shear rate, as shown in Equation 4.5. However, such purely inertial regimes are less representative of typical natural debris flows, which often contain a significant fraction of fine particles, clay, and water. These components create interparticle bonds and viscous effects that introduce a finite yield strength. One can see that no yield strength is present in Equation 4.5 preventing the model to capture stops or starts from a resting state.

When cohesive or viscous forces become significant, the flow transitions from a grain-inertia regime toward a viscoplastic regime, where stress is no longer purely quadratic in the shear rate. In this case, Bagnold's law gradually loses validity, and constitutive models that incorporate a yield stress and an effective viscosity such as the Bingham and Herschel–Bulkley (HB) laws provide a more appropriate description of the rheological behavior. These models are thus widely adopted for mudflows, where the continuous phase controls the mechanical response, while the Bagnold regime remains relevant for dry or coarse-grained flows where smoother grain collisions dominate.

Rheological model incorporating a yield strength are particularly relevant for debris flows and allows the model to represent the stopping ability but also the initiation of the flow (as explained in section 2.3.1), although it should be carefully crossed with other theories to avoid incompatibilities. All three phases of debris flow (initiation, dynamics and deposition) are then dictated by this single rheology.

Furthermore, debris flows can exhibit a block-like behavior at the surface. The yield strength is responsible for this solid motion in the upper part of the fluid column, it is also called "plug" formation. In reality, there are no changes from solid to liquid phase but only a speed gradient variation with depth. HB laws help to model this behavior. As the shear stress evolves with depth, the base of the flow is subjected to high shear stress due to friction with the bedrock and thus, in reaction, the base apply a large shear stress on the flow, while the surface (for surface free flows) is subjected to null shear stress. From null shear stress to high shear stress at the base, there is a depth where the shear stress equals the yield strength. Above this depth, the shear stress is lower than the yield strength and in consequence no shear rate appears. The speed is constant above this depth, the material moves therefore as a solid block; the plug formation. Below this depth, the shear stress exceeds the yield strength and the material flows with a shear rate depending on the shear and the effective viscosity.

Figure 4.2 illustrates this movement for a Bingham rheology, compared to a Newtonian flows (as water). We consider a linear¹⁰ shear stress distribution ($\tau(z)$) with a yield stress (τ_y). Until the shear stress is higher than the strength of the flow, there are no speed (u) variation. The speed is not null but cease to vary as $\tau(z) < \tau_y$ so $\partial u / \partial z = 0$. If the shear stress decreases enough, the maximum value is lower than the strength, the flow stops.

Above a certain depth, the surface behaves as a solid block (constant velocity over the considered depth) sliding over a layer deformed by shearing (with speed variation). When the bottom shear stress decreases (as can happen while the slope decreases), the entire flow can stop as the shear stress is lower than the yield strength (slope decrease and the stopping process is illustrated on the right part of the Figure 4.2). The left part of the figure shows a Newtonian behavior for a simple sheared flow while the right part represents a Bingham flow for the same shearing conditions. When τ_0 is higher than τ_y , the material is not strong enough to resist to the shear stress that is applied to it. When the slope decreases, τ_0 decreases and when the applied stress reach the value of the resistance, the material is strong enough to stop the movement. *Remark: On this representation, the slope decrease can lead to hydraulic jump formation. Mudflows are also known to create such jumps when hydraulic regime changes (Coussot and Meunier 1994).*

¹⁰In reality, the shear stress distribution is not necessary linear with the depth but depends on the flow conditions as this linearity asks for several hypothesis: steady flow and uniform flow (constant mean speed in the flow direction). In this case, assuming no head losses, a uniform steady flow, the base shear stress per unit of width is equal to the weight component along the slope: $\tau_0 = \tau_b = \gamma_t h \sin \alpha$ with h the flow depth and α the slope angle. The shear stress distribution is then linear with depth: $\tau(y) = \gamma_t (h - z) \sin \alpha$. We assumed a free surface with null shear stress at the top: $\tau(h) = 0$ (negligible wind or other sources of shear stresses).

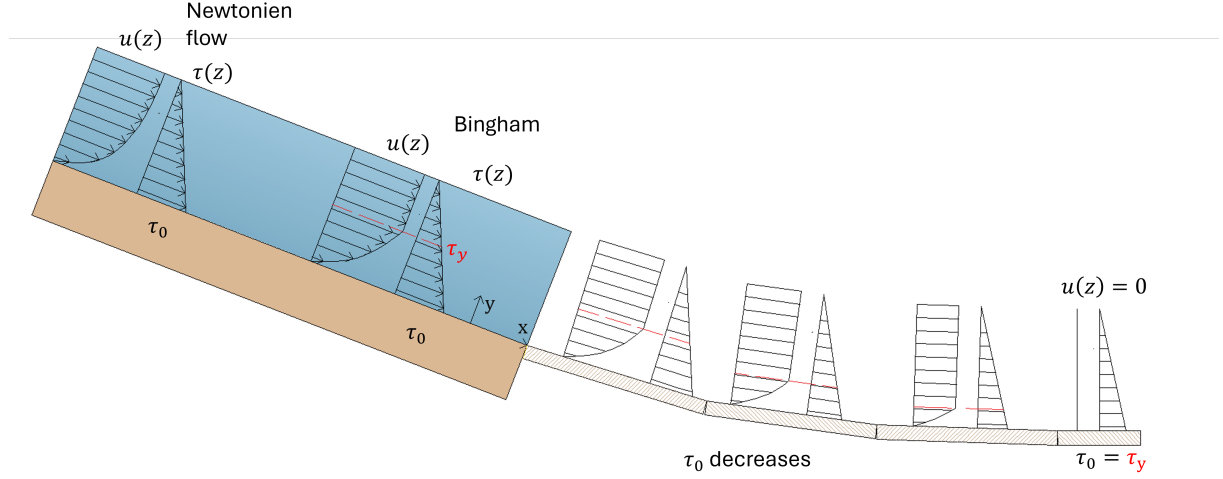


Figure 4.2: Schematic illustration of a block movement at the surface for a Bingham flow.

A direct practical application of fluid rheology possessing a yield stress is the estimation of the yield strength value from field measurements of flow deposits. If one consider a debris flow deposition, by assuming that no change in the material properties has happened, one can deduce a minimal value of the yield strength τ_y by measuring the deposition height h_{stop} and the slope angle α at this location: $\tau_y \geq \tau_0 = \gamma_t h_{stop} \sin \alpha$. Considering the flow stopped at the limit of its shear strength, the basal shear stress τ_0 is equal to the yield strength τ_y , meaning the resistance of the flow reached maximum of the shear applied to its base.

4.1 Linking material parameters to rheology

Rheology governs the overall behavior of the events that are the main concerns of this work. However, the rheological parameters are relevant only if they can be linked to the material properties as they are supposed to encapsulate the complex interactions and mechanical characteristics within the material. This section discusses how rheological parameters such as yield strength and viscosity can be related to the physical parameters contained in the debris flow material.

HB and Bingham models require the definition of a yield strength value τ_y , and effective viscosity parameter μ (or K and n for HB).

It is commonly accepted that the relation between both viscosity and yield strength with the volumetric concentration C_v (also denoted ϕ_T), has an exponential behavior (O'Brien and Julien 1988)(Coussot and Meunier 1996).

$$\mu = \alpha_1 e^{\beta_1 C_v} \quad (4.6)$$

$$\tau_y = \alpha_2 e^{\beta_2 C_v} \quad (4.7)$$

In Equations 4.6 and 4.7, coming from (O'Brien and Julien 1988); α_1 , β_1 , α_2 and β_2 are empirical coefficients depending on the material properties and are fitted to some collected data. For instance, this relationship is used in HEC-RAS debris flow module (U.S. Army Corps of Engineers 2025b) to estimate the yield strength from the solid volume fraction.

These relations are widely accepted in the literature to represent the influence of fine particles on the rheological behavior of debris flows. Although they are empirical, they provide a practical way to estimate

rheological parameters from measurable material properties.

4.1.1 Effective viscosity

First, the effective viscosity is chosen to be modelled with: (U.S. Army Corps of Engineers 2025a). They provide a useful observation for the effective viscosity evaluation. The dynamic viscosity, equivalent to effective viscosity, is related to the viscosity of water μ_w as the following:

$$\frac{\mu_m}{\mu_w} = e^{\beta_1 C_v} \quad (4.8)$$

μ_m is called the dynamic viscosity or the effective viscosity and can be estimated with size distribution data and empirical parameters. The expression lowers the number of empirical parameters and also gives a comparison of the observed viscosity from known water value $\mu_w \cong 10^{-3} Pa \cdot s$.

U.S. Army Corps of Engineers (2025a) gives typical values for β_1 respecting the formulation in Equation 4.8:

Material	β_1
"Typical soil"	8
Kaolinite	8
Sensitive Clays	5

Reported values of β_1 in the literature typically range between 4 and 12 depending on the fine content conditions of the pore fluid. The values adopted in HEC-RAS therefore represent conservative average estimates rather than universal constants. Nevertheless, for the experiment analysis in the following section, the value of $\beta_1 = 8$ was retained.

4.1.2 Yield stress

The yield stress is more sensitive to the solid concentration and Equation 4.7 can lack theoretical representativeness to generalize them for all kinds of flows. For example, τ_y of a low concentration of fine sediments should tend to zero, as the flow should behave as a granular flow without cohesion (as explained for Bagnold's formulation). However, in Equation 4.7, τ_y does not tend to zero when no fines are present. It is actually only valid for a certain range of C_v . More theoretical developments were carried out to precise the link between material properties and rheology.

In this section, three types of formulation are introduced to calculate τ_y based on the solid concentration. There is no single yield-stress formulation valid over the entire range of solid volume fraction; each empirical law is relevant for a specific concentration regime, even if explicit bounds are rarely stated with confidence in the literature. The distinction between low, middle and high values of C_v is left to the user's discretion. However, an order of magnitude could be: $C_v = [0; 0.40]$, $C_v = [0.30; 0.70]$ and $C_v = [0.70; 0.80]$, respectively for each category.

Low values of C_v

In Schippa (2021), they extended Equation 4.7 with the granular distribution. The formulation gives non constant values for both α_2 and β_2 . Authors based their reasoning on the volumetric concentrations of fine particles ϕ_f ¹¹ and coarse particles ϕ_g (so that $\phi_T = \phi_f + \phi_c$). And postulated that $\phi_M = 0.64$, the maximal concentration for random close packing. The value of 0.64 corresponds to a close packing

¹¹ $\phi_f = \frac{V_{sf}}{V_{sf} + V_{sg} + V_w}$, $\phi_g = \frac{V_{sg}}{V_{sf} + V_{sg} + V_w}$, with V_{sf} , V_{sg} and V_w being respectively the fine volume, the coarse grain volume and the water volume in a sample of the chosen material. The total solid volumetric concentration is expressed as $\phi_T = \frac{V_{sf} + V_{sg}}{V_{sf} + V_{sg} + V_w}$.

of non-swell grains with uniform spherical size. This is not the case for usual mudflows. The value of $\phi_M = 0.8$ mentioned by (Richard M. Iverson 1997) will be preferred. α_2 and β_2 are expressed as follows:

$$\alpha_2 = \beta_2 = \psi_1 \frac{\phi_T}{\phi_M - \phi_T} \quad (4.9)$$

Where $\psi_1(\phi_g/\phi_f)$ is the *grading function* depending on the ratio between coarse and fine particles volumetric concentrations. It is noteworthy that when $\phi_T = 0$, α and β tends to zero, leading to a null yield strength as expected.

Finally, this grading function ψ_1 was fitted on several experiments involving a large range of granular distributions and concentrations:

$$\psi_1 \left(\frac{\phi_g}{\phi_f} \right) = k_1 e^{-k_2 \frac{\phi_g}{\phi_f}} + k_3 \quad (4.10)$$

Where k_1 , k_2 and k_3 are empirical coefficients fitted on the experimental data and they are found to be respectively 3.5, -5 and 2.9. The final expression obtained from 4.7, 4.9 and 4.10 gives a surface for τ_y depending on both fine and coarse particles volumetric concentrations. The model is able to represent a wide range of distributions particularly precise for low solid concentration. During the campaign of experiments, the authors positioned the material in a horizontal plan, they measured the height of the formed deposit h_0 , then the slope was slowly increased until the deposit started to move. The angle is measured and the trigger shear is measured: τ_{c1} . The material is able to flow and stops again thanks to its yield strength. The new characteristic height of the deposit h_f is measured at this slope angle and the stopping shear τ_{c2} is deduced. Note that this value is a minimum as it is the value able to stop the flow. In reality, shear strength evolves quickly when the flow stopped. The increase comes from a granular rearrangement that solidifies shear strength. Results showed that these shear values τ_{c1} and τ_{c2} were different, with about a coefficient of 1.4 between both (τ_{c1} being the highest) at normal granular distribution (not excessively concentrated in solid or very dilute). This observation is similar to the theory of a block sliding on a rough surface. The static friction coefficient is higher than the dynamic friction coefficient as considered as separated in the article of (Schipa 2021).

This formulation is a combination of exponential, increasing rapidly the values of τ_y for high solid fraction volumes in general.

Middle values of C_v

For middle values of C_v , the combination of exponential tends to give values that are too high. Here, constant value of α_2 and β_2 are preferred. U.S. Army Corps of Engineers (2025b) gives a relationship of τ_y similar to 4.7. The coefficients α_2 and β_2 are reported as¹²:

Material	C_v	α_2	β_2	Range of τ_y (Pa)
Typical soil	0.65 -- 0.80	0.005	17.2	375 -- 5000
Kaolinite	0.4 -- 0.5	0.005	20.7	200 -- 1600
Sensitive clays	0.35 -- 0.60	0.3	23.0	950 -- 300 000

The formulation of Equation 4.7 is not appropriate for small τ_y tendency. The correction proposed by Schippa (2021) elegantly completes the calculation to refine the yield stress calculation when the values are small. This is usually met when the fraction volume is low. The classical exponential approach still works best with different values of α_2 and β_2 when τ_y is relatively high; for middle values of C_v . Both methods complete each other and one should choose appropriately between the two depending on the condition of the material.

¹²There should be a slight typo in the web page mentioned here. In the table provided by HEC-RAS, C_v is the solid fraction volume, not the "liquid limit" as written in the web page.

High values of C_v

When the fraction volume is extremely high, reaching 0.8, the exponential relationships are both too steep and the yield stress can be evaluated with a power law. Migniot (1968) gave the following equation:

$$\tau_y = aC_v^b \quad (4.11)$$

a depends on the soil characteristics and $b = 3$ is often accepted. This final expression completes the spectrum that the solid fraction volume can take and the techniques that could be used to determine τ_y .

Unlike τ_y , viscosity does not seem to be greatly affected by significant changes in C_v and the Equation 4.6 remains valid for most C_v concentrations.

A synthetic table refers to the choice calculation for the Bingham parameters depending the the solid concentration of the sample: Table 4.1.

Table 4.1: Summary of Bingham rheological parameter formulations depending on solid fraction concentration C_v

Bingham param.	Low C_v	Middle C_v	High C_v
μ (effective viscosity)	Eq. 4.6, $\beta_1 \sim 4\text{--}12$	Eq. 4.6, $\beta_1 \sim 4\text{--}12$	Eq. 4.6, $\beta_1 \sim 4\text{--}12$
τ_y (yield stress)	Eqs. 4.7, 4.9, 4.10	Eq. 4.7, Table 4.1.2	Eq. 4.11

For single phase assumption, rheology was presented as the key element of debris flow behavior, governing their initiation, dynamics and deposition, encapsulating the complexity of the material properties at small scale into a few parameters. Finally, some relations linking rheological parameters to material properties have been discussed to provide a fast estimation of rheology from measurable data. Bingham parameters τ_y and μ are estimated using empirical laws linking rheology and granular structure. The values highlighted here will be used as first guess for practical application of the model.

Chapter 5

Experimental campaign

The experimental campaign used to validate the model was performed in Politecnico di Milano in the Gap2 lab (Lecco campus, Italy) by Professor Laura Longoni, PhD doctoral researcher Lorenzo Panzeri and Master's Candidate Yixin Guo (Guo 2025).

The use of some results from the campaign in the context of the present thesis relies on an objective of verification and validations of the numerical simulations. The data were collected in a clever way to be effectively compared with a model under laboratory controlled environment. Even if the final goal is to model larger natural disaster, successfully modeling laboratory observations will provide a solid foundation of confidence for more real case numerical applications. Furthermore, larger scale applications rarely possess precise data to provide a thoughtful comparison of the model's results.

In this section, first, a brief description of the setup and experimental protocol will be provided. Next, the material used in the experiment is characterized and some typical results obtained will be exposed. Finally, an analysis of the materials used will enable the rheological parameters to be determined for use in the numerical model. All the work and complete result analysis can be read in Yixin Guo's thesis: *Mudflow mechanisms analysis through downscaled test for a model validation*, Yixin Guo, promoter: Prof. Laura Longoni, Politecnico di Milano (Guo 2025), which presents in very detail the campaign they carried out and a deep analysis of it. This section aims to expose the general experimental conditions for an understandable comparison with the model.

5.1 Experimental set-up and protocol

Set-up

The set-up is a wooden channel placed on an inclined plane spilling the mudflow onto a deposition plan. The channel is placed on the landslide simulator of the Gap2 lab (Hojat et al. 2019), enabling easy incline of the channel at a large variety of angles. Materials are stored behind a wooden inclinable gate. The gate is suddenly opened and materials flows inside the channel, follows a deposition in the horizontal plan: the deposition area.

The canal was painted to take advantage of the color difference between the brown mud and the yellow paint. The data was partly collected by a series of photos taken at intervals of Δt seconds (usually 0.1s or 0.2s), and a ratio between the number of pixels colored in yellow and the number of pixels colored in brown makes it possible to track the geometric changes in the mud with precise real time measurements. Number of pixels are converted to dimensions through a scaling process. Before any experiment, pictures with reference graduated ruler were taken to scale the pixels. The second tool used to collect data was an optical fibre placed both at the gate and at the exit of the channel (just before entering the deposition area). This device provided a very good estimate of the average speed of the mud front in the channel. Where the series of photos were taken at a fix Δt interval, the optical fibre recorded the vibration of the gate opening and the passage of the mud front with a better accuracy. The latest recorded data came from a laser scanning of the deposited material. The pictures helped to track the horizontal propagation of the material (in the deposition area) and the laser scanning provided the height dimension when the material had stopped moving.

Schematic drawing of the main measurements can be seen in Figure 5.1 Pictures of the experimental set-up can also be found in Figure 5.2a, 5.2b and 5.2c. .

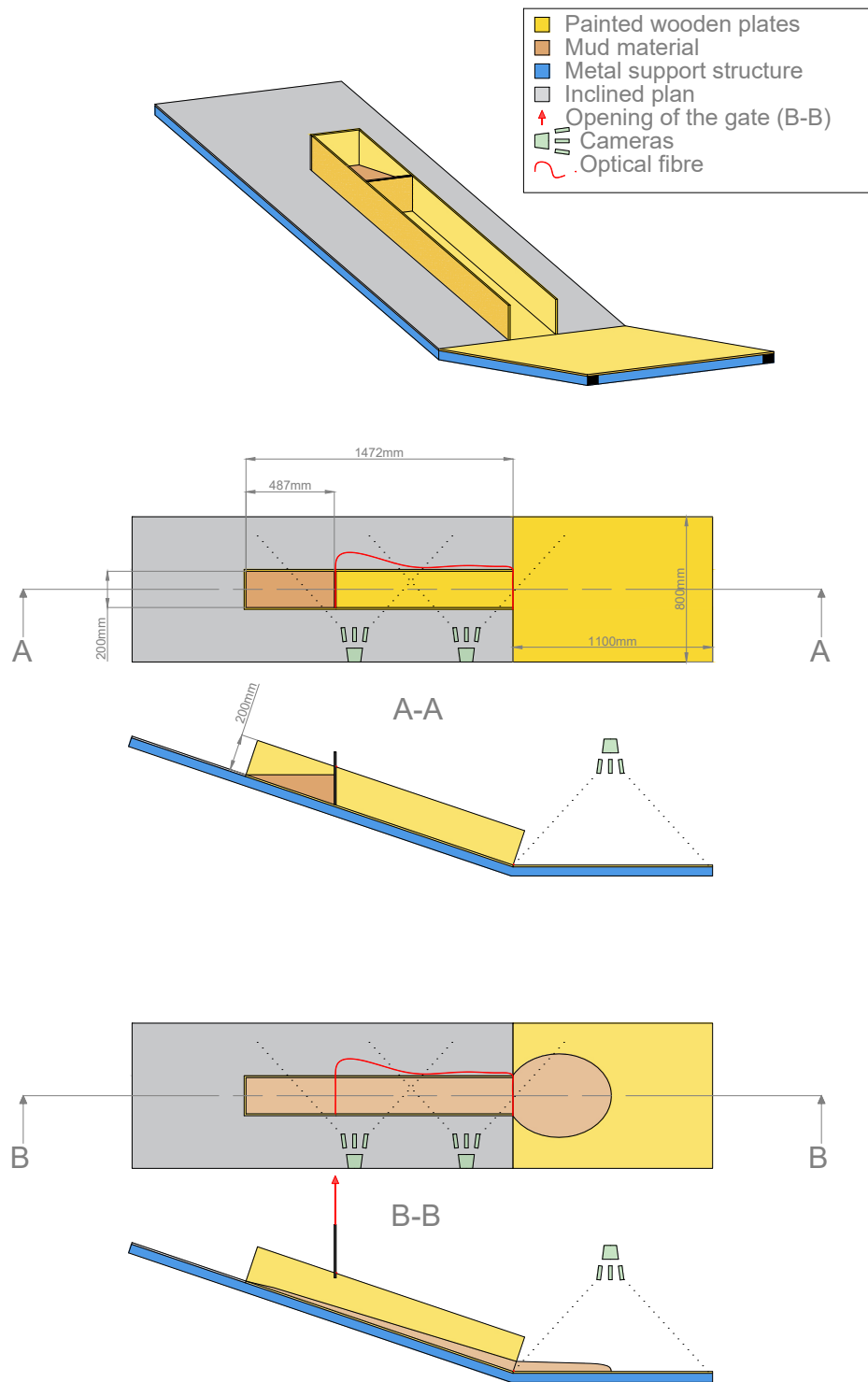


Figure 5.1: Representation of the set-up used in the experimental campaign and representative measures.



(a) Experimental set-up ready to be used and filled with material. Negligible liquid leakage passes through the gate.



(b) Detail of the inclinable gate and optical fibre used to precisely record the opening time. The material is stored upstream of the gate prior to release.



(c) Final deposit after flow cessation on the horizontal plane.

Protocol

Mudflows material are prepared in external boxes, a soil sample is mixed with a given quantity of water to reach the targeted texture. The material is weighted and placed behind the gate with the channel horizontally placed (not inclined at first). The level of the mud filling the stored box created by the gate is measured. Knowing the dimensions of this box, the volume of material can be easily obtained and data concerning water content or bulk density can be retrieved.

The channel is then inclined up to the expected angle. The gate is rotate in order to be fully vertical and not simply perpendicular to the channel. This make sure the numerical code can represent correctly this geometry. Depth averaged equations provide a unique height for each cell of the mesh, if the gate was not vertical and simply kept perpendicular to the channel, some mesh elements would be impossible to model. A quick justification is present in Appendix A.4

When each tools were activated, the material is finally released by opening the gate. This wooden plate was removed in a quick upward motion. The speed of the opening was measured to be approximately $19m/s$ compared to a mean velocity of the front wave around $1.8m/s$ at maximum. The effect of the opening gate was then neglected and assumed instantaneous. The material flows in the channel and deposits in the horizontal plane. During all the experiment, cameras (GoPro-type) recorded series of photos both in the channel and on the deposition area. The positions of the cameras are represented in Figure 5.1. The fiber optic cable recorded the exact time the gate opened and the exact time the mud exited the channel.

Finally, when the material has stopped to move, the laser scanning was processed. After that, the material can be removed and the canal cleaned for a new experiment.

5.2 Analysis of the materials

The laboratory tested the material with a granulometric study¹³. Using a sieve analysis, it was found that the material was a clayey to silty sand.

Table 5.1: Grain size distribution of the material used in the experimental campaign of Politecnico di Milano

Grain size fraction	Size range (mm)	Percentage (%)
Coarse gravel	19 -- 75	0.00
Fine gravel	4.75 -- 19	13.34
Coarse sand	2.00 -- 4.75	16.71
Medium sand	0.425 -- 2.00	30.26
Fine sand	0.075 -- 0.425	17.05
<i>Fine particles</i>		
Silt	0.005 -- 0.075	15.45
Clay	< 0.005	7.19
Grains density $\gamma_s = 2651[kg/m^3]$		

The grain size distribution curve, in Figure 5.3 is the result of combining sieve analysis for coarse particles and hydrometer analysis for fine particles. Resulting in a continuous cumulative distribution spanning from gravel to clay sizes. The median value is around $0.72mm$, which corresponds to the lower values of a medium size sand.

¹³Data ordered by Politecnico di Milano - Prof. Laura Longoni, PhD doctoral researcher Lorenzo Panzeri for Gap2 lab (Lecco campus, Italy) and proceeded by ALTAIR SRL certificated UNI EN ISO 9001 ICIM n. 11353.

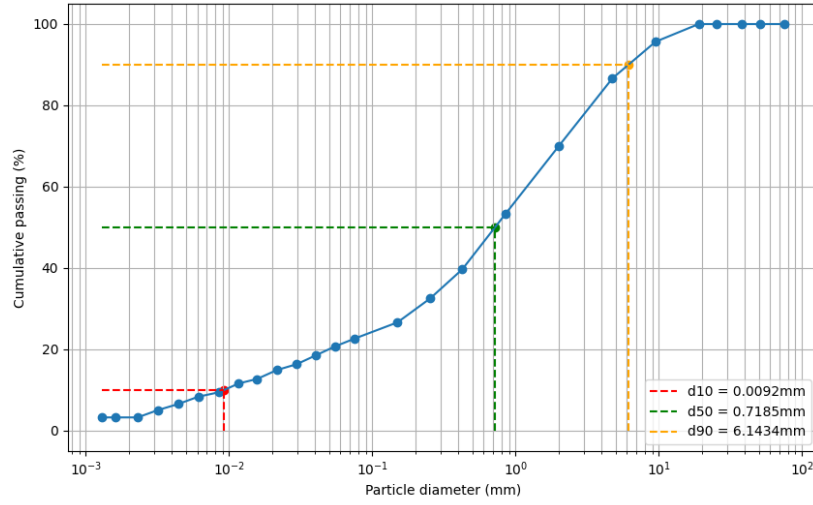


Figure 5.3: Grain size distribution curve of the material.

The wide grain size distribution suggests that the material may exhibit both a fluid-like and a granular flow behaviour. However, the size distribution is relatively fine compare to what debris flow can experience. The assumption of a muddy flow with single phase is therefore quite defensible.

5.3 Typical results

This work had access to some of the results provided by the campaign and more precisely Yixin Guo's thesis: (Guo 2025). They are represented in Table 5.2. For the five results (from $T06$ to $T10$) available in the comparison of the model, the following data are available:

- Initial conditions: recall the name of the current experiment, the slope reached before opening the gate, the volume of material included in the experiment and its density.
- Deposition: these data contain information about deposition. The deposition area is the area of the deposited material in the deposition plan (without accounting for the deposited material in the channel) and the runout distance is the distance from the gate to the farthest deposited mud.
- Speeds: they are measured in the channel and in the deposition plan. The mean speed in the channel V_m is detected by the optical fibre and account for the mud front from the gate opening to the outlet of the channel. The peak velocity in the deposition are V_f is the maximum speed detected when the material spread on the horizontal plan before it stops moving. V_f is calculated using a photogrammetry technique.
- Heights in the channel: the heights detected in the channel are the maximal height captured by the cameras. Once the brown mud passes through the controlled points (lower point: 10cm from the channel outlet, middle point: 40cm from the channel outlet and upper point: 80cm from the channel outlet), the maximum height recognized by the colorimetric difference is recorded.

Furthermore, the duration of the flow (from gate opening to complete stop of the material) were $T06 : 8s$, $T07 : 11s$, $T08 : 8s$, $T09 : 12s$ and $T10 : 14s$.

Results were also provided in a form of a series of pictures from the top view deposition, from Guo (2025)'s work. The pictures were already treated to recognize the advance of the mud front; the colors differences are illustrated by the red curve, such a photo is shown in Figure 5.4 for illustration purposes. It will enable future comparisons between the model results and the experiments to which this work had access.

Table 5.2: Experimental conditions and runout characteristics, from (Guo 2025)

Initial conditions				Deposition		Speeds		Height channel		
Name	Slope	Volume	Density	Dep. area	Runout dist.	V_f	V_m	Lower	Middle	Upper
Units	[°]	[m ³]	[kg/m ³]	[m ²]	[m]	[m/s]	[m/s]	[cm]	[cm]	[cm]
T06	30	0.0082	2273.39	0.055	1.161	0.054	0.316	1.06	1.54	1.73
T07	30	0.0082	2285.62	0.045	1.137	0.062	0.305	1.64	1.71	1.80
T08	33	0.0080	2203.64	0.337	1.666	0.381	1.467	--	1.19	3.35
T09	25	0.0082	2212.28	0.299	1.576	0.216	1.055	0.94	1.53	2.54
T10	20	0.0083	2234.57	0.263	1.484	0.182	0.943	1.29	1.61	2.24



Figure 5.4: Top view example of the evolution in the deposition area. The sequence is from T10 (Guo 2025)

5.4 Derivation of rheological parameters for the model

From data about the granulometric distribution and from the typical results provided by Politecnico di Milano (PoliMi), the single phase rheological parameters of the Bingham model are estimated following the methodology described in Section 4.1.

As an illustrative example, experiment T06 is considered to demonstrate the application of the methodology used to derive the Bingham rheological parameters.

First, the grain size threshold between the effective fluid and the solid partition is deduced by Equation 2.2. This equation provides the limit diameter of fines considered as part of the fluid component. When bigger than the value of δ' , the grains are part of the solid fraction. With the following values, the critical diameter defining the fluid and solid composition δ' is retrieved:

$\mu_w = 10^{-3}[Pa \cdot s]$, $\rho_s = 2651[kg/m^3]$, $h = 1.54[cm]$ (for $T06$ at the middle part of the channel for this example) and $t_D = 8s$.

$$\delta' = \sqrt{\frac{h/2 \cdot 18\mu_w}{(\rho_s - \rho_w)g \cdot t_D}} = 0.0327mm \quad (5.1)$$

Even though the value is small, it represents 17.1% of the granular distribution, 17.1% of the material is part of the fluid and 82.9% of the material is acting as solid particles carried out within the fluid matrix. The limit is represented over the granulometric curve in Figure 5.5.

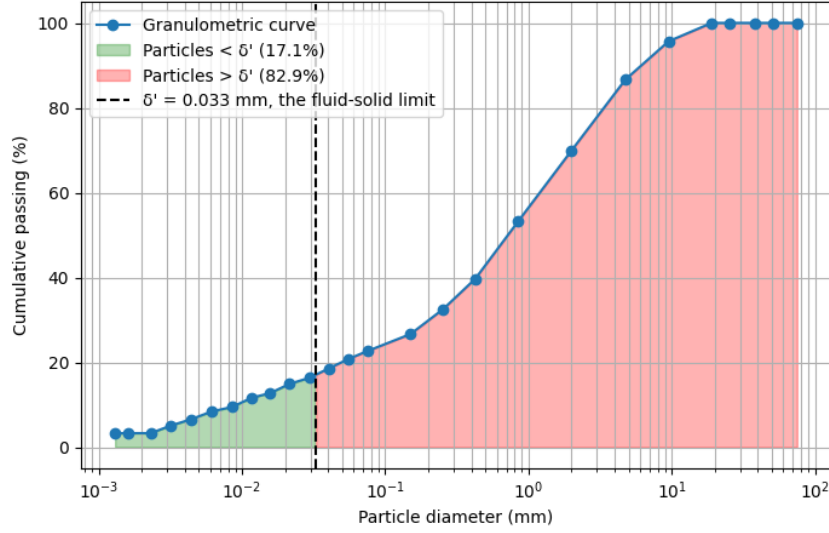


Figure 5.5: Grain size distribution curve with difference between effective fluid and solid composition.

If we make the hypothesis that the sample tested with the granulometric survey is representative, all experiments have a grain density of $\rho_s = 2651[kg/m^3]$; a very common value in sediment studies. We can evaluate C_v , the total solid fraction (fine and coarse) evaluated simply using the total bulk density $\rho_{tot} = 2273.39[kg/m^3]$ (here with $T06$) and the bulk density of water $\rho_w = 1000[kg/m^3]$.

$$C_v = \frac{V_s}{V_s + V_w} = \frac{\rho_{tot} - \rho_w}{\rho_s - \rho_w} = 0.771 \quad (5.2)$$

The obtained value may appear high when compared to classical random packing limits. The maximum random arrangement is commonly accepted around $C_v = 0.64$ (Schippa 2021). But this value, as mentioned earlier corresponds to a close random packing of uniform spherical grains. In this case, this maximum is not valid. Similar values have been reported for dense debris and mudflows (Richard M. Iverson 1997). The authors exposed a maximum of $C_v = 0.8$, which proves that the value belongs to an acceptable range.

To further assess the plausibility of this value, a simplified volumetric analysis was performed using the measured dry density of the material. This analysis, detailed in Appendix A.7, indicates that the measured bulk density cannot be reproduced without significant grain rearrangement when water is added, thereby supporting the high value of C_v obtained.

Now, coming back to the partition between effective fluid and coarse grains, δ' , the limit between them, was calculated. Even though silt and clay are usually both labelled as "fines", in this case, a part of the

silt is bigger than the threshold value δ' . It physically means that these sediments are not fine enough to be seen as part of the effective fluid. The material will see those silts as solid grains. Therefore, the fine and grain fraction volumes are adapted according to the value of δ' and its fraction compared to the rest of the solids. C_v being the fraction volume occupied by all the "non-water" elements, the fraction of fines corresponds to 17.1% of C_v . Same reasoning is applied to the effective grains fraction volume.

$$\begin{cases} \Phi_T = C_v = 0.77 \\ \Phi_f = 17.1\% \cdot C_v = 0.131 \\ \Phi_g = (100 - 17.1)\% \cdot C_v = 0.6391 \end{cases} \quad (5.3)$$

This approach allows a physically consistent partitioning of solids into fines and coarse grains based on sedimentation behavior, rather than relying purely on traditional silt and clay classifications. The fine and coarse grains partitions should be adapted to each flows as they depend on several parameters and are not direct values generalized to any given flow.

Now that the fraction volumes are determined accordingly to the material and the flow conditions (h and t_D), the element exposed in Section 4.1 will help to find the Bingham's rheological parameters of the single phase approach. Equations 4.6 and 4.7 allow the values of the parameters μ and τ_y to be calculated. They rule the Bingham rheology (see Section 4).

The effective viscosity of the single phase approach for PoliMi's experiments is:

$$\mu = \mu_w e^{8C_v} = 0.477 Pa \cdot s \quad (5.4)$$

Here, something interesting to note is that the viscosity of the interstitial fluid can be estimated with the expression introduced by (Thomas 1965) in Equation 2.6. Authors mentioned that it should be valid for a fine concentration of $v_{fines} < 0.5$. Taking $v_{fines} = \Phi_T$ gives $\mu = 0.98 Pa \cdot s$ but would lead to some error according to the paper. The second approach is to use the formulation twice. The first time with the fine concentration inside water to determine the viscosity of the interstitial fluid (fines mixed with water) and a second time to incorporate the larger particles (larger particles mixed with interstitial fluid). The expression gives a viscosity of $\mu_f = 1.53 \cdot 10^{-3} Pa \cdot s$ for the fines. The second use, for which the fluid is not water with μ_w anymore but the interstitial fluid with the fines μ_f , gives a total viscosity of the mixture of $\mu = 0.18 Pa \cdot s$. The three approaches, exponential relation and the two ways of using Equation 2.6, provide the same order of magnitude encouraging confidence in these results.

The yield stress can be developed in three ways. First, by a composition of exponentials, in Equation 4.7, 4.9 and 4.10. It works best when the solid fraction is low. If the fraction volume is relatively high, a simple exponential can be used 4.7 with constant coefficients α_2 and β_2 . Finally, for high solid concentration, a power law is preferred, in Equation 4.11. The experiment T06 has a particularly high fraction volume.

The development of a composition of exponentials can be ruled out. A simple exponential relationship gives realistic values, but they are too high compared to those expected. Ultimately, a power law development is the best approach for this type of concentration. Unlike exponential laws, which explode for high solid volume fraction values, power laws provide a better estimate. Nevertheless, complete development of each of the three methods is presented in Appendix A.6 for more details.

The power-law formulation in Equation 4.11 introduces a coefficient a that depends on the material characteristics. The factor is empirical and should be interpreted as accounting for particles interactions and cohesive effects, including contributions from fine particles, chemical reactions, and attractive forces such as van der Waals interactions.

Not knowing precisely those interactions, it is possible to calibrate the a parameter with the exponential law in a lower solid concentration (i.e a concentration for which the exponential laws are more adapted). In (U.S. Army Corps of Engineers 2025b), taking the middle value of the acceptable range ($C_v = 0.7$) gives:

$$\begin{aligned}\tau_y &= 0.005e^{17.2 \cdot 0.7} = 847 = a \cdot 0.7^3 \\ a &= 2469 Pa\end{aligned}\tag{5.5}$$

With the calibrated value of a , the power law gives a yield stress of $\tau_y = a \cdot 0.77^3 = 1133 Pa$ for experiment T06.

This methodology can be argued but does not aim to produce the perfect values of the rheological parameters, they are ruled by more complex phenomena. For example, the time between which the material is placed behind the gate and the beginning of the flow can settle some bigger particles, modifying the overall effective viscosity and yield stress. It is still interesting to compare values between the 5 experiments of PoliMi as they have different flow properties and bulk densities. Using the same process for each experiment as it was written with T06 gives the following results:

Table 5.3: Rheological and physical parameters for PoliMi mudflow experiments

Experiment	ρ_{tot} [kg/m ³]	δ' [mm]	C_v	μ_{eff} [Pa.s]	τ_y [Pa]
T06	2273.39	0.0327	0.771	0.4783	1133.0
T07	2285.62	0.0294	0.779	0.5075	1165.9
T08	2203.64	0.0288	0.729	0.3411	956.8
T09	2212.28	0.0266	0.734	0.3557	977.6
T10	2234.57	0.0253	0.748	0.3963	1032.5

The values will be used as first guess in the model and iterative process will help precise the right values for a Bingham implementation.

Chapter 6

The numerical model: Watlab and new formulations

All the entry parameters (i.e rheology) and the mathematical model were presented. It is finally time to expose the numerical model.

This section is structured as follows. First, it will briefly explain how the mathematics presented in Section 3.5 are implemented in Watlab and where should it be modified for mudflows. Next, the section will develop the modifications: the physical behaviour and the mathematical model was exposed, this section takes that information to build the model and justifies the modifications brought to Watlab. Finally, the importance of a stopping criterion will be highlighted and the way it was treated in the code.

6.1 Shallow Water Equation: Finite volume resolution

The continuous system of Shallow Water Equations was exposed in Equations 3.15. They are often written in a vectorial conservative form:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} = \mathbf{S}(\mathbf{U}) \quad (6.1)$$

The vector $\mathbf{U}$ is called the *conservation variables*, sometimes referred as *state variables*. It encompasses the variables of water height h (m) and surface fluxes $q_x = uh$ and $q_y = vh$ (m^2/s) which represent the flow rates per unit width across the vertical interfaces of the control volumes. While $u = \bar{u}$ and $v = \bar{v}$ are the depth averaged speeds for which the top bar has been omitted for simplicity's sake in the equations. This vector constitutes the system state and the solutions to the equations.

$$\mathbf{U}^T = (h, q_x, q_y) \quad (6.2)$$

$\mathbf{F}$ and $\mathbf{G}$ are the fluxes along the x and y directions respectively.

$$\mathbf{F}(\mathbf{U}) = \left(q_x, \frac{q_x^2 + \frac{1}{2}gh^2}{h}, \frac{q_x q_y}{h} \right)^T \quad (6.3)$$

$$\mathbf{G}(\mathbf{U}) = \left(q_y, \frac{q_x q_y}{h}, \frac{q_y^2 + \frac{1}{2}gh^2}{h} \right)^T \quad (6.4)$$

$$\mathbf{S}(\mathbf{U}) = (0, b_1 h + t_1^B, b_2 h + t_2^B)^T \quad (6.5)$$

b_i denotes the volumetric body force per unit mass and corresponds to the projection of gravity along the bed slope, written as $b_i = gS_{0,i}$, where $S_{0,i} = -\partial z_b / \partial x_i$ is the bed slope.

The basal traction is expressed in terms of a friction slope $S_{f,i}$ such that the basal shear contribution per unit mass reads $t_i^B = -ghS_{f,i}$.

The source term, without mass addition or retrieve leads to:

$$\mathbf{S}(\mathbf{U}) = (0, gh(S_{0x} - S_{fx}), gh(S_{0y} - S_{fy}))^T \quad (6.6)$$

The friction slope is particularly important here. It accounts for head losses with friction laws as *Manning* or *Chezy* expressions but can also integrate the rheological laws into the SWE system!

6.2 Schematic representation of the calculation process in Watlab

The continuous mathematical model has now been presented. The numerical solution scheme for a Finite Volume code expects these continuous equations to be discretized over the domain. The discrete form is obtained by discretizing the integral form of the equations over a control volume. As this thesis does not aim to modify the structure of resolution, only the final discretized equation is shown in Equation 6.7 (without accounting for mass additions or removals). The global domain is divided into triangular cells, on which the discretized equation will be solved for each time step. The triangular cells are defined as three node points and three edges. The edges are interfaces between neighboring cells

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{A_i} \sum_{j=1}^{n_b} \mathbf{T}_j^{-1} \mathbf{F}_j^*(\bar{\mathbf{U}}_j) \mathbf{L}_j + \mathbf{S}_i \Delta t \quad (6.7)$$

$\mathbf{U}$ are still the state variables, n denotes the present time of the simulation and $(n + 1)$ the next time incremented by Δt . The index i is the spacial index referring to the studied cell. n_b is the number of cells in the problem. $\mathbf{T}$ is a rotation matrix, helping to project the fluxes $\mathbf{F}_j^*$ traversing the j -th interface. $\mathbf{L}_j$ is the length of this interface. Finally, $\mathbf{S}_i$ are the source terms (Hydraulics Group UCL 2025).

The schematic resolution in Watlab is available at the Figure 6.1. At time t , the code calculates the fluxes that will pass through each interfaces. Those fluxes are based on the difference of height between the left and the right cell of the concerned interface and on the speeds of the two cells. At the same time, the source terms with the friction are calculated. Once it is done, the update of the state variables (height and speeds of each cell) can began. The steps of the computation are important, the reason will be detailed in Section 6.6.1. First, the new height and speeds are updated by taking the contribution of the fluxes that want to pass through the interfaces of the evaluated cell. The direction of the new found speeds are calculated and stored. Next, the contribution of the source terms (including the friction) is added to the new speeds, and height if the source term act on this variable: it is not the case for friction but can be for rain source term for example. Finally, the direction of the new speeds is evaluated.

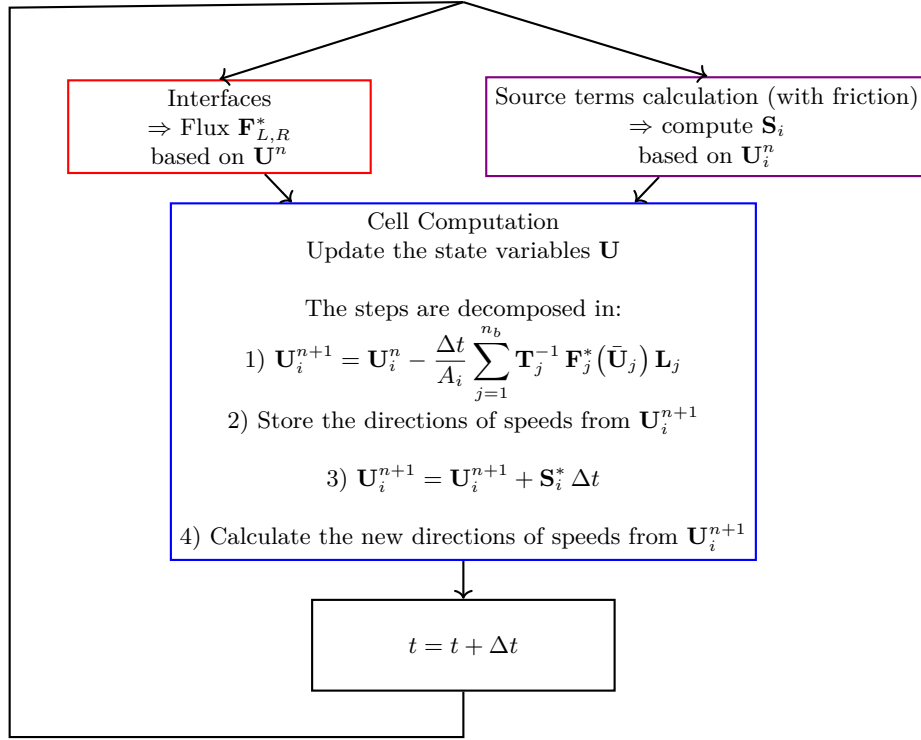
6.3 The friction terms $S_{f,i}$

6.3.1 Why modifying the friction terms?

The adaptation of the classical SWE to mudflows mainly consists in modifying the friction source term. Physically, it seems the friction at the bottom of the flow only represents the interaction between the fluid and the solid ground (i.e the shear stress exerted by the fluid on the solid bed). However, for depth averaged models, the dependency of any behavior on the vertical direction is lost by the integration over the depth. Thus, variation of any parameter or properties —such as velocity, viscosity, or shear rate— at a precise depth cannot be represented anymore and is reflected across the entire height. Varying the friction at the bottom of the flow is then a way to modify the flow behavior in the entire depth. It does not only affect the flow base in contacts with the ground but the entire flow column. Therefore, adjusting the friction term becomes a practical way to incorporate the influence of the rheology of the material within the SWE framework. By modifying this source term, one can effectively mimic how the internal resistance of the fluid (e.g., yield stress, non-Newtonian viscosity) influences the overall flow behavior, even though the model does not explicitly resolve the vertical structure of the flow.

6.4 Water friction term

The friction slopes in Watlab were initially calculated with Manning's formula. This expression attempts to encompass all elements that cause head losses in the system. It models interaction of water with its environment, as the irregularities of the river bed inducing friction. The development of $S_{f,i}$ for water

Figure 6.1: Simulation algorithm for Hydroflow, the *C++* core of Watlab

and their assumption are important as the same process of reasoning will be applied to mudflows and will try to incorporate the rheology of the material through the process. It can be seen in three main steps:

- Linking S_f with the base shear stress τ_b
- Hypothesis on τ_b to be linked with the depth averaged speed V
- Putting together $S_f(V)$

The relation $S_f(\tau_b)$:

A balance of forces is fundamental to this stage. We study the balance between the driving forces caused by the slope and the restrictive forces caused by friction.

The equilibrium asks for an hypothesis of a steady and uniform flow. For a $1m$ large flow with regular water level parallel to the river bed and flowing toward the right hand side direction with a speed V constant along its path, the equilibrium is written in this way:

$$\tau_b P \Delta s = \gamma A \Delta s S_0 \quad (6.8)$$

P and A are the wetted perimeter and the wetted section. Usually, the ratio between these two quantities is rewritten to be $R = A/P$, the hydraulic radius. Under the assumption of small depth compared to horizontal dimensions ($h \ll L$), $R = h$: the hydraulic radius is the water depth.

The uniform flow, for which the bed slope S_0 and the friction slope are parallel gives the relationship between the friction slope and the basal shearing¹⁴:

$$S_f = \frac{\tau_b}{h \rho g} \quad (6.9)$$

¹⁴ $\gamma = \rho g$

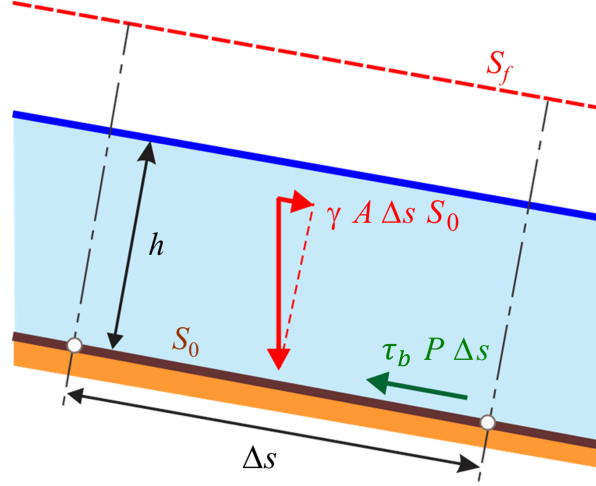


Figure 6.2: Forces equilibrium between the driving gravity forces and friction. The figure is from Soares Frazão (n.d.)

This is the first relationship that we were looking for.

The relation $\tau_b(V)$:

Another hypothesis is required here. The basal shearing is proportional to the square of the average speed (depth averaged: $V = \bar{u}$): $\tau_b \propto V^2$. It recalls a turbulent rough fully developed assumption.

$$\tau_b = kV^2 \quad (6.10)$$

The relation $S_f(V)$:

Combining both relationships:

$$S_f = \frac{kV^2}{h\rho g} \quad (6.11)$$

The constant value of k/g integrated into Chezy and Manning formulation. The Chezy formulation is ruled by the Chezy coefficient C .

$$S_f = \frac{V^2}{C^2 h} \quad (6.12)$$

The Manning formulation shows a factor n such as $C = R^{1/6}/n$. The formulation of Manning is then

$$S_f = \frac{n^2 V^2}{h^{4/3}} \quad (6.13)$$

The formulation of Manning is the one implemented in Watlab. For a two dimension speed field $u = \bar{u}$ and $v = \bar{v}$, in the x and y horizontal directions, the formulation is slightly modified:

$$\begin{aligned} S_{fx} &= \frac{n^2 u \sqrt{u^2 + v^2}}{h^{4/3}} \\ S_{fy} &= \frac{n^2 v \sqrt{u^2 + v^2}}{h^{4/3}} \end{aligned} \quad (6.14)$$

The last expressions allow the friction to act in the opposite direction of the flux. The hypothesis behind it is that magnitude of the friction slope is governed by the scalar Manning law (in Equation 6.13), while its direction is given by the normalized velocity vector. So, with $\mathbf{u} = (u, v)$ and $V = \|\mathbf{u}\| = \sqrt{u^2 + v^2}$, the non directional friction slope is $S_f = \frac{n^2 V^2}{h^{4/3}}$ and integrating the direction gives $\mathbf{S}_f = S_f \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{n^2 V}{h^{4/3}} \mathbf{u}$.

The value of the friction is governed by Manning or Chezy and the direction is the one opposed to the local speed. Here $S_{f,i}$ has the same direction as the speed but when doing $S_{0,i} - S_{f,i}$, from Equations 6.6, the action of the friction is opposed to the speed.

The previous steps and hypothesis are summarized in Table 6.1. This table serves as a comparison and a guideline to what will be presented for the modifications of Bingham and Herschel-Bulkley. It highlights the elements that will be modified.

Table 6.1: Structure of the friction modeling for different rheologies uncompleted.

Rheology	$S_f(\tau_b)$	$\tau_b(V)$	$S_f(V)$	Stop. criterion
Water Newtonian fluid	$S_f = \frac{\tau_b}{\rho gh}$ Uniform steady flow Forces balance	$\tau_b \propto V^2$ Turbulent flow	$S_f = \frac{n^2 V^2}{h^{4/3}}$ Manning formulation	None
Bingham	?	?	?	?
Herschel--Bulkley	?	?	?	?

6.4.1 Could Manning or Chezy model mudflows?

A first intuitive way to model mudflows with SWE without any modification to the model is to increase the bottom friction coefficient, by using a Manning or Chezy formulation with a high value of the coefficient (high for Manning and a low value for Chezy). This method is simple to implement as it does not require any modification of existing SWE models but lacks physical meaning for many reasons. First, the Manning coefficient n is an empirical coefficient that depends on the roughness of the bed. The Chezy coefficient C has the same purpose so the same comments apply. In order to represent highly viscous mudflows, very high values of n should be used, which does not correspond to any realistic roughness of the bed. Indeed, in some cases, the value of n needed for the flow to behave as viscous as mudflows would require a higher grain size than the actual height of the flow. This has no physical sense. Some formulations link the Manning coefficient with the grain size distribution as Yilmaz (2019) in Equation 6.15 where d_{50} is the median grain size of the river bed in $[m]$:

$$n = \frac{d_{50}^{1/6}}{21.1} \quad (6.15)$$

In order to have a Manning coefficient of $n = 0.1s/m^{1/3}$ or 0.2, which are usual values for mudflows (Chen and Chuang 2014), the grain size should be between $d_{50} = 88m$ and $d_{50} = 5647m$. Those values are obviously not realistic for any natural bed.

Some studies, have circumvented this problem by breaking down Manning's coefficient into two distinct components. One component accounts for the actual physical roughness of the bed, while the other represents the internal viscosity of the mudflow, Zhu, B. Liu, and Y. Liu (2020) defines the external roughness coefficient n_1 and the internal roughness coefficient n_2 . Both terms are calculated separately with empirical relation fitted over experimental data and summed to obtain the total Manning coefficient $n = n_1 + n_2$. Even if methods like this bring more physical meaning to the model, there are still major issues with using Manning or Chezy formulations to represent mudflows especially at low speed.

Increasing the roughness of the bed while using a high value of the total Manning's value n to represent a mudflow's viscosity can work for relatively high speeds or confined flows but fails to represent the behavior of the fluid at lower speeds and particularly the stopping motion of the flow due to shear resistance. Indeed, when the velocity tends to 0 (see Equation 6.13).

Thus, when $\bar{u} \rightarrow 0$, $S_f \rightarrow 0$ too. Decreasing the bottom friction at low speeds tends to increase the flow velocity again instead of stopping it.

6.5 Building the model: Adapting the friction terms to mudflows

As presented above, the Bingham and Herschel-Bulkley theories are one of the mosts used to represent mudflows and more generally flows possessing a yield shear stress. The following developments are based on Pastor et al. (2004), who established a relationship with a Bingham rheology for linking the base friction τ_b with the state variables $\bar{u}_i$ and h . This relationship will replace the one provided in the third column of Table 6.1: $\tau_b(V)$ for water. In a second time, the process of thought of Pastor et al. (2004) will be applied to Herschel-Bulkley, the more complex but general theory. Bingham will have the advantage to give an explicit formulation where Herschel-Bulkley will be implicit and will ask for iterative methods.

6.5.1 Bingham friction source term

The assumption of uniform steady flow remains a valid approach for mudflows. The balance of forces is kept and equation 6.9 remains unchanged.

However, the turbulent flow hypothesis is not valid anymore. Mudflows possess a higher viscosity than water which usually entails a laminar-like behaviour. The second relation $\tau_b(V)$ should be adapted accordingly. Shearing appears in the rheological model and this characteristic is used to incorporate rheology inside the SWE.

The first assumption is that the shear stress varies linearly with depth, it is maximum at the bottom τ_b and zero at the free surface. The second assumption is simply the rheological model itself. Both relationship involves the shear stress and the base shearing can be isolated and linked to the depth averaged speed $\bar{u}$ in absolute value. The sign will be retrieved when the friction term is finally assembled. To develop this, we must take a step back, before the depth averaged calculation of the SWE. We consider a positive speed profile flowing in the x_1 direction and the axis perpendicular to the base is x_3 , with h the depth: $u_1(x_3)$.

Figure 6.3 illustrates the shear stress distribution and velocity profile for the given Bingham fluid (h_s is the height for which the shear stress equals the yield strength).

The shear stress profile (now called τ to lighten the equations) is supposed linear and the rheology is supposed to follow Bingham. The system of equation to be solved, where $\tau_b(\bar{u})$ is the unknown, is given in Equations 6.16.

$$\begin{cases} \tau(z) = \tau_{13}(z) = \tau_b \left(1 - \frac{x_3}{h}\right) \\ \tau(z) = \tau_y + \frac{\partial u_1(x_3)}{\partial x_3} \quad \text{if } \tau > \tau_y \\ \frac{\partial u_1(x_3)}{\partial x_3} = 0 \quad \text{if } \tau \leq \tau_y \end{cases} \quad (6.16)$$

First, the equation must get rid of the partial derivative of velocity. The velocity profile is obtained by integrating the relation between shear stress and deformation rate. It has two components. The first one is quadratic and corresponds to the sheared layer (where $\tau \geq \tau_y$):

$$u_1(x_3) = \int_0^{x_3} \dot{\gamma} dx'_3 = \int_0^{x_3} \frac{\tau - \tau_y}{\mu} dx'_3 = \int_0^{x_3} \frac{\tau_y \left(1 - \frac{x'_3}{h}\right) - \tau_y}{\mu} dx'_3 \quad \forall x_3 \in [0, h_s] \quad (6.17)$$

$$u_1(x_3) = \frac{\tau_b}{\mu} \left(x_3 - \frac{x_3^2}{2h} - \frac{\tau_y}{\tau_b} x_3 \right) \quad \forall x_3 \in [0, h_s] \quad (6.18)$$

The second component is the plug flow (where $\tau < \tau_y$):

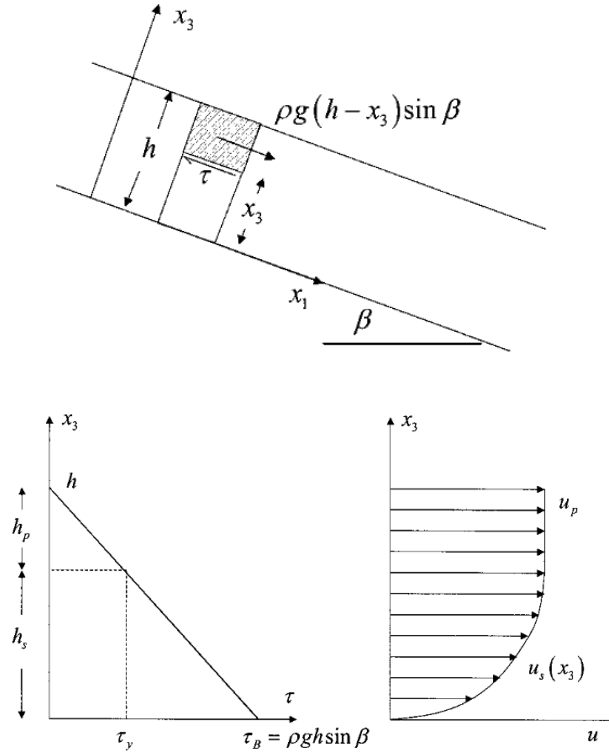


Figure 6.3: Shear stress distribution and velocity profile for a Bingham fluid in steady uniform flow over an inclined plane, from Pastor et al. (2004)

$$u_1(x_3) = \frac{\tau_b}{\mu} \left(h_s - \frac{h_s^2}{2h} - \frac{\tau_y}{\tau_b} h_s \right) \quad \forall x_3 \in [h_s, h] \quad (6.19)$$

Similar triangles give the relation between h_s and τ_y : $h/\tau_b = (h - h_s)/\tau_y$ so $h/h_s = 1 - \tau_y/\tau_b$. The speed profile is finally written as:

$$\boxed{\begin{aligned} u_1(x_3) &= \frac{\tau_b h}{\mu} \left(\frac{x_3}{h} - \frac{x_3^2}{2h^2} - \frac{\tau_y}{\tau_b} \frac{x_3}{h} \right) \quad \forall x_3 \in [0, h_s] \\ u_1(x_3) &= \frac{\tau_b h}{\mu} \left(1 - \frac{\tau_y}{\tau_b} \right)^2 \frac{1}{2} \quad \forall x_3 \in [h_s, h] \end{aligned}} \quad (6.20)$$

Based on the complete velocity profile, the depth-averaged velocity $\bar{u}_1$ is calculated as:

$$\bar{u}_1 = \frac{\int_0^{h_s} u_1(x_3) dx_3 + \int_{h_s}^h u_1(x_3) dx_3}{h} = \frac{\tau_b h}{6\mu} \left(1 - \frac{\tau_y}{\tau_b} \right)^2 \left(2 + \frac{\tau_y}{\tau_b} \right) \quad (6.21)$$

The relation between the bottom shear stress τ_b and the depth-averaged velocity $\bar{u}_1$ is now established through equation 6.21. However, the relation involved a third order polynomial for isolating τ_b . For expressing the SWE friction source term S_f , it is preferable to have an explicit relation of τ_b as a function of $\bar{u}_1$ and h . This step is a challenge. First, developing $\bar{u}_1$ and defining the third order polynomial P_3 with variable $\xi = \frac{\tau_y}{\tau_b}$ called the relative depth of the plug and the parameter a gives:

$$a = \frac{6\mu\bar{u}_1}{\tau_y h} \quad (6.22)$$

$$\begin{aligned} \overline{u_1} &= \frac{\tau_b h}{6\mu} \left(2 - 3 \frac{\tau_y}{\tau_b} + \left(\frac{\tau_y}{\tau_b} \right)^3 \right) \\ \left(\frac{\tau_y}{\tau_b} \right)^3 - 3 \left(\frac{\tau_y}{\tau_b} \right) - \frac{6\mu \overline{u_1}}{h} \frac{1}{\tau_y} \left(\frac{\tau_y}{\tau_b} \right) + 2 &= 0 \\ \boxed{P_3(\xi) := \xi^3 - (3 + a)\xi + 2 = 0} \end{aligned} \tag{6.23}$$

The parameter a is storing all the information of the flow: the viscosity, the speed, the height and the yield stress and transform the relationship into dimensionless polynomial. It becomes the variable of calculation for ξ and precise how close the flow is to rest because it is linearly dependent on the speed.

At this stage, the problem is reduced to finding a physically admissible solution of the third-order polynomial P_3 .

It is interesting to note that the simple shear assumption bounds the value of ξ between 0 and 1, the minimum shear stress is 0 at the surface and τ_y at the base. It is then simpler to evaluate the validity of the solutions found for P_3 with any method as it bounds the valid physical solutions. Any solution of ξ should belong to the interval $[0, 1]$. Four major solving methods are possible to find ξ from P_3 .

- Semi-empirical relations
- Iterative methods
- Polynomial reduction to a second order polynomial
- Analytical solutions of third order polynomials

First, some semi-empirical relations exist in the literature as in Jin and Fread (1997), or in Fread (1988) giving ξ : $\xi = 2/(3 + a)$. Jeyapalan, Duncan, and Seed (1983) also proposed a simple model to find ξ : $\xi = 3/(3 + a)$. Even though this two last models are very simple to implement, they can lead to significant errors. Some authors use iterative methods as Newton-Raphson to find the solutions of P_3 but the calculation cost can be important as the solutions need to be found for each cell at each time step of the SWE resolution. Other works focus on finding approximate analytical solutions by reducing the polynomial order to a second order polynomial as Pastor et al. (2004). This clever approach will be exposed in the present section. Finally, analytical solutions of third order polynomials also exist but are long to develop because the first steps of the resolutions involve solutions in the range of complex numbers which leads to test different values. It was not found in the literature during this work. Authors mentioned a possible analytical solution but no explicit developments were found. However, the analytical solution is the most powerful as it is explicit and without any errors. A proposal of the development for such analytical solution will be exposed here. Accordingly, the two last methods (approximation and analytical solution) are then developed in detail below, highlighting certain difficulties and proposing solutions to them.

Polynomial reduction to a second order polynomial

Pastor et al. (2004) overcomes the problem of the third order polynomial by reducing P_3 to a second order P_2 and minimizing the error encountered E in a very clever way but lacking of physical meaning for some values as will be presented here.

First, let us exposed the resolution of Pastor et al. (2004)'s article.

The expression of the error $E = P_3 - P_2$ is based on an interpolation over three points: ξ_0 , ξ_1 and ξ_2 . P_2 and P_3 are equal at these points: $E(\xi_0) = E(\xi_1) = E(\xi_2) = 0$ and the error at any points is proportional to the product of the distances to these points, giving $E = A(\xi - \xi_0)(\xi - \xi_1)(\xi - \xi_2)$ where A

is a constant. Furthermore, $d^3 E/d\xi^3 = d^3 P_3/d\xi^3 - d^3 P_2/d\xi^3 = d^3 P_3/d\xi^3$, so the value of A is given by: $A = 1/(d^3((\xi - \xi_0)(\xi - \xi_1)(\xi - \xi_2))/d\xi^3) \cdot d^3 P_3/d\xi^3$ giving the Equation 6.24.

$$\begin{aligned} E &= P_3(\xi) - P_2(\xi) \\ E &= \frac{1}{3!} \frac{d^3 P_3}{d\xi^3} (\xi - \xi_0)(\xi - \xi_1)(\xi - \xi_2) = (\xi - \xi_0)(\xi - \xi_1)(\xi - \xi_2) \end{aligned} \quad (6.24)$$

The goal is now to minimize the maximum error E on the interval $\xi = [0, 1]$. To choose the values of ξ_0 , ξ_1 and ξ_2 , Pastor et al. (2004) used the Chebyshev polynomial on the interval $[0, 1]$ ¹⁵. The Chebyshev polynomials are defined on the interval $[-1, 1]$ and represent the optimal interpolation points for minimizing the error of interpolation polynomials. The third order Chebyshev polynomial is:

$$T_3(x) = 4x^3 - 3x \quad \forall x \in [-1, 1] \quad (6.25)$$

With the change of variable $x = 2\xi - 1$, the polynomial is adapted to the interval $[0, 1]$:

$$T_3(\xi) = 32\xi^3 - 48\xi^2 + 18\xi - 1 \quad \forall \xi \in [0, 1] \quad (6.26)$$

The roots of $T_3(\xi)$ are then used as interpolation points for minimizing the error E so the error is given by $T_3(\xi)$ normalized for the cubic term, it does not change the values of the roots and removes the cubic term for the third order polynomial:¹⁶

$$E(\xi) = \xi^3 - \frac{3}{2}\xi^2 + \frac{9}{16}\xi - \frac{1}{32} \quad \forall \xi \in [0, 1] \quad (6.27)$$

The maximal error on the interval $[0, 1]$ is then $\max(E) = 1/32 = 0.03125$ which is acceptable for practical applications.

Finally, the expression of the second order polynomial P_2 is obtained by subtracting the error E from P_3 . The expression minimizes the maximum error on the interval $[0, 1]$ for the value of ξ that is needed to calculate τ_b .

$$\begin{aligned} P_2(\xi) &= P_3(\xi) - E(\xi) \\ &= \xi^3 - (3 + a)\xi + 2 - \left(\xi^3 - \frac{3}{2}\xi^2 + \frac{9}{16}\xi - \frac{1}{32} \right) \\ &= \frac{3}{2}\xi^2 - \left(a + \frac{57}{16} \right)\xi + \frac{65}{32} = 0 \end{aligned} \quad (6.28)$$

Now that the expression is reduced to a second order polynomial, the solutions can be found easily but only one of them should be valid in order to have a meaningful physical interpretation. The solutions of P_2 are given by:

$$\xi_{+,-} = \frac{a + \frac{57}{16} \pm \sqrt{\left(a + \frac{57}{16}\right)^2 - \frac{195}{16}}}{3} \quad (6.29)$$

The smallest value of $a = \frac{6\mu\overline{u_1}}{\tau_y h}$ is 0 (when $\overline{u_1} = 0$, as a reminder, we placed the axis x_1 toward the speed to always have positive values). When calculating $\xi_+(a = 0) = 1.42$, the value is out of the physical range

¹⁵There should be a slight mistake in the article of Pastor et al. (2004), the third order Chebyshev polynomial is not $T = 4x^3 - x^2$ but $T_3(x) = 4x^3 - 3x$.

¹⁶The last term of Equation (40) in Pastor et al. (2004) should probably be $1/32$ and not $1/16$.

$[0, 1]$ so ξ_+ is not a valid solution. For ξ_- , when $a = 0$, $\xi_-(a = 0) = 0.951$ is the maximum value provided by the model. When $a \rightarrow +\infty$, $\xi_- \rightarrow 0$ (happens for small value of height and high velocity). So ξ_- is the only valid solution for ξ in the physical range $[0, 1]$ and is always valid for any physically correct values of a (a should be positive to be valid).

Finally, the expression of the bottom shear stress τ_b as a function of the depth-averaged velocity $\overline{u_1}$ and the flow depth h is:

$$\tau_b = \frac{\tau_y}{\xi_-} = \frac{3\tau_y}{a + \frac{57}{16} - \sqrt{\left(a + \frac{57}{16}\right)^2 - \frac{195}{16}}} \quad (6.30)$$

The expression should fill in the box for $\tau_b(V)$ for the Bingham model. However, it has some problems for low speeds conditions.

Pastor et al. (2004) showed that their expression gives very accurate results when compared to the exact solution of the third order polynomial P_3 (probably calculated from iterative methods). It is illustrated in Figure 6.4.

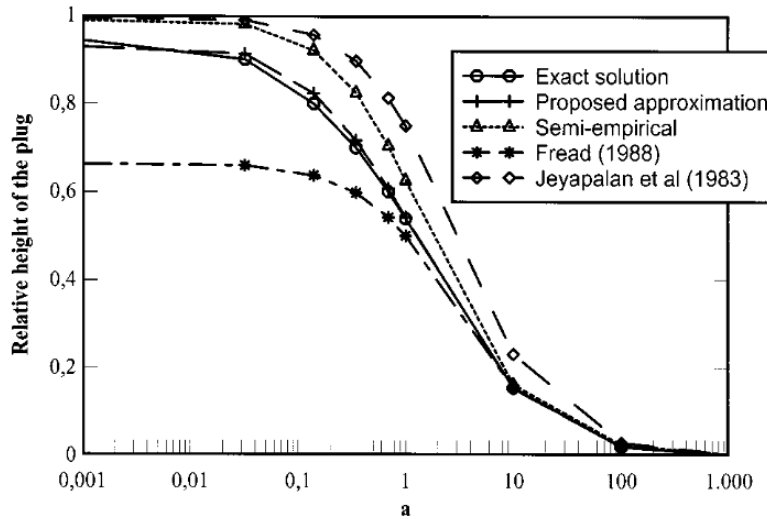


Figure 6.4: Comparison of the exact solution of P_3 and the approximate solution of P_2 for τ_b as a function of a , from Pastor et al. (2004). The exact solution is probably calculated from iterative methods. The proposed approximation is given in Equation 6.30. The semi-empirical relation from Jin and Fread (1997) is also represented for comparison, the expression for Bingham is given by $\xi = \left(1 + \frac{1}{1.396}a\right)^{-(1/1.15)}$. Fread (1988) and Jeyapalan, Duncan, and Seed (1983) proposed respectively the approximated solutions $\xi = 2/(3 + a)$ and $\xi = 3/(3 + a)$.

Even if the expression reduces a lot the calculation cost by giving a direct expression of τ_b and providing results very close to the exact solution, the authors seemed to neglect one important point. The expression never allows to have $\tau_b = \tau_y$ as ξ_- always stays between 0 and 0.951. This means that the model never theoretically allows to have a complete stop of the flow, which leads to a major physical problem as the main objective of the Bingham rheology is to allow a stopping of the system.

To correct this problem, an idea would be to set a threshold value and forcing τ_b to be close enough to τ_y when the velocity reaches 0. However, this also lacks of physical meaning. The best idea would be to develop again the Tchebyshev polynomial while forcing one of the roots to be 1, so the expression would

be exact at this point and still minimize the error on the interval $[0, 1]$. This idea has not been developed yet and could be an interesting perspective to improve and is presented here.

Taking back the expression of the error E (Equation 6.24), if we force one of the roots to be 1, we have:

$$E_{forced} = (\xi - 1)(\xi - \xi_1)(\xi - \xi_2) \quad (6.31)$$

This expression is nul at $\xi = 1$ and at $\xi = \xi_1$ and $\xi = \xi_2$. Representing an error forced to be 0 at $\xi = 1$. It allows to have the approximated polynomial P_2 equal to P_3 at this point, so when $\tau_b = \tau_y$.

To find the values of ξ_1 and ξ_2 minimizing the maximum error on the interval $[0, 1]$, we can use the Chebyshev polynomial of order 2:

$$\begin{aligned} T_2(x) &= 2x^2 - 1 \quad \forall x \in [-1, 1] \\ T_2(\xi) &= 8\xi^2 - 8\xi + 1 \quad \forall \xi \in [0, 1] \end{aligned} \quad (6.32)$$

The expression of the forced error is then the multiplication between $(\xi - 1)$ and the normalized Chebyshev polynomial:

$$E_{forced}(\xi) = (\xi - 1) \frac{(8\xi^2 - 8\xi + 1)}{8} \quad \forall \xi \in [0, 1] \quad (6.33)$$

The maximum error on the interval $[0, 1]$ is then $\max(E) = 0.23$, which is higher than the previous. However, the way of calculating the error can be criticized. It does not find the error on ξ but finds the maximal difference of the sum between polynomials. The error developed by Pastor et al. (2004) for example is said to be $1/32$ at maximum but if we calculate the value at $a = 0$, the model finds $\xi = 0.951$ instead of $\xi = 1$. The error on ξ calculation is different from the error $E(\xi)$.

Finally, the reduced and forced second order polynomial P_2 is:

$$\begin{aligned} P_{2, forced}(\xi) &= P_3(\xi) - E_{forced}(\xi) \\ &= \xi^3 - (3 + a)\xi + 2 - (\xi - 1) \frac{(8\xi^2 - 8\xi + 1)}{8} \\ &= 2\xi^2 - (a + \frac{33}{8})\xi + \frac{17}{8} = 0 \end{aligned} \quad (6.34)$$

And the solutions are given by:

$$\xi_{+, -} = \frac{a + \frac{33}{8} \pm \sqrt{(a + \frac{33}{8})^2 - 17}}{4} \quad (6.35)$$

With this new expression, when the flow velocity tends to 0, a tends to 0 and $\xi_-(a = 0) = 1$ so $\tau_b = \tau_y$. Thus, the model have a correct behavior in slow fluid movement and allows to have a complete stop of the flow. The expression found for τ_b is explicitly expressed in Equation 6.36.

$$\tau_b = \frac{4\tau_y}{a + \frac{33}{8} - \sqrt{(a + \frac{33}{8})^2 - 17}} \quad (6.36)$$

Comparing forced and unforced expressions of ξ with real solutions of P_3 shows that both expressions are very close to the exact solution for medium and high values of a (so high velocities for $a = 10^{-1}$ and higher), even if the new solution seems to be more precise also in the middle part. The differences appear for low values of a (low velocities) as illustrated in Figure 6.5, the forced expression is definitely more accurate than the unforced one and reaches the correct limit when $a \rightarrow 0$.

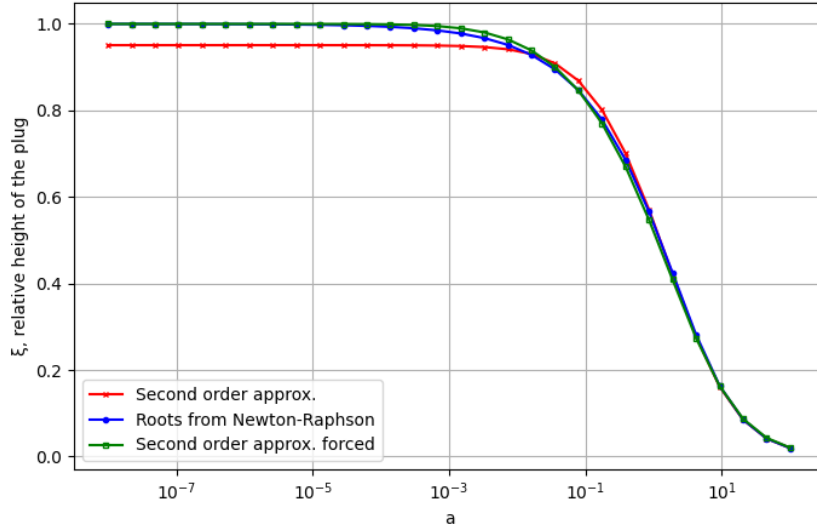


Figure 6.5: Comparison of the exact solution of P_3 (with N-R), the approximate solution of P_2 from Pastor et al. (2004) and the new forced approximate solution of $P_{2,forced}$ developed in this document to calculate ξ as a function of a .

Analytical solutions of the third order polynomial

The best way to find the exact solution of P_3 is to use the analytical solutions of third order polynomials. The thought process is presented in the appendix A.2 to keep the text more concise. The final solution involves three possible real values.

$$\begin{aligned} \Phi &:= \arccos\left(-\frac{3\sqrt{3}}{(3+a)^{3/2}}\right) \\ \xi_k &= 2\sqrt{\frac{3+a}{3}} \cos\left(\frac{\Phi + 2\pi k}{3}\right) \quad k \in \{0, 1, 2\} \end{aligned} \tag{6.37}$$

The numerical model asks for only value of ξ between 0 and 1 to calculate τ_b . If two values are in this range, the model would not be able to choose the correct one. Physically, it means that two different states for the shear of the flow are possible for the same depth-averaged velocity and flow depth. Testing the three solutions for different values of a shows that only one solution is valid. Figure 6.6 illustrates the three solutions for different values of a .

Comparing the exact solution of P_3 with the approximated solution of $P_{2,forced}$ shows that the approximation is extremely close to the exact solution for any value of a , as illustrated in Figure 6.7.

In summary, the Bingham rheology introduces a non-linear, velocity-dependent basal shear stress that naturally allows for flow stopping when the yield stress is reached. This behaviour cannot be reproduced by classical Manning or Chezy formulations. Several methods can be employed to solve the relationship imposed by the Bingham rheology for the SWE friction term. The third order relation can be solved with methods ranging from semi-empirical relations to exact analytical developments while passing by iterative methods or polynomial order reduction. The polynomial order reduction method proposed by Pastor et al. (2004) is an elegant compromise between calculation cost and accuracy but seems to lack physical meaning for low velocities and should be corrected to allow a theoretical stop when base friction reaches the yield strength. The new forced polynomial reduction proposed in this document seems to provide a better fit to the solution and reduces the physical problem of the previous model. Finally, the exact analytical solution of the third order polynomial has been presented, it provides the exact solution but

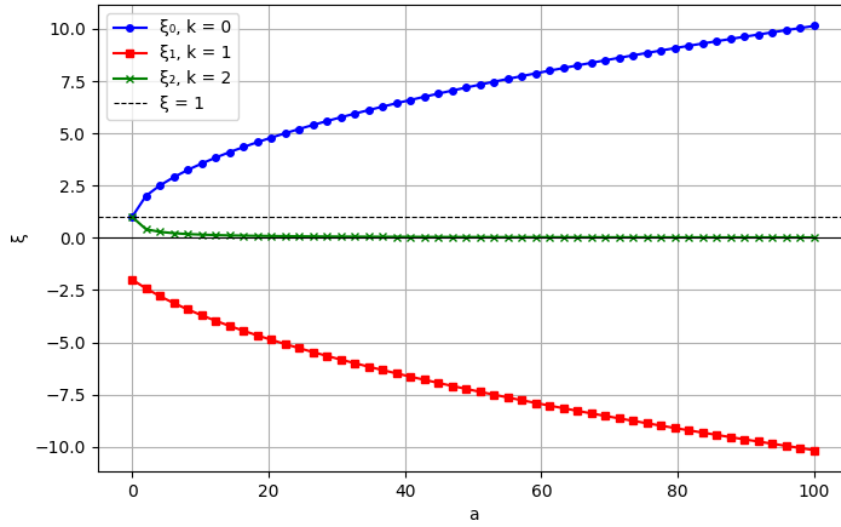


Figure 6.6: Plot of the three solutions of P_3 as a function of a from Equation A.24. The only valid solution is ξ_2 as it is the only one between 0 and 1 for any value of a (for $a > 0$).

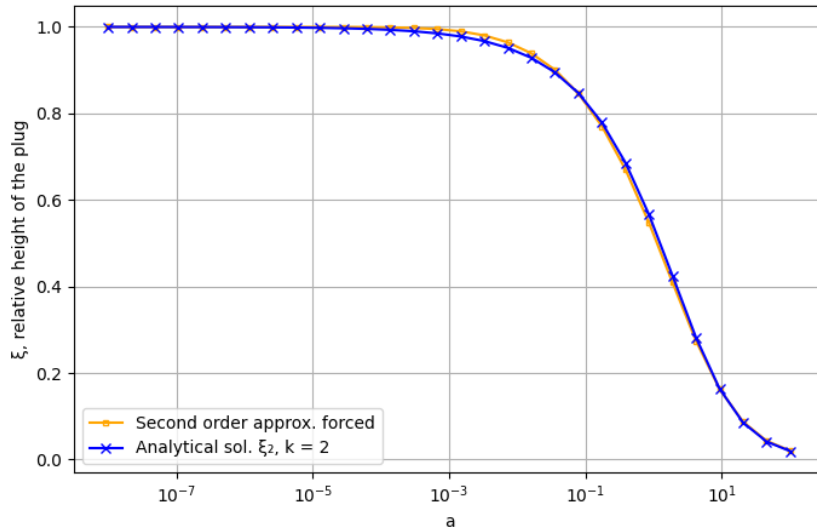


Figure 6.7: Comparison of the exact solution and the force approximation.

involves complex solutions which may explain why it is not commonly used in the literature. However, the physical solution is explicit and does not require any iterative method. Both solution of $P_{2,forced}$ and exact analytical solution ξ_2 can then be used to calculate the base friction coefficient τ_b . The developments exposed in this document would encourage one to use one of these two last expressions for modeling a Bingham fluid with shallow water equation resolution.

The numerical implementation follows strictly the analytical expressions derived above. The basal shear stress for each cell at each time step is computed based on the rheological parameters and state variables $\bar{u}$, $\bar{v}$ and h . The rheological parameters are asked to the user of Watlab when he initializes the simulation and are constant over the total calculation. The state variables evolve during the simulation and the friction source term too.

The previous section, which was rather mathematical, presented the findings of this thesis, inspired by Pastor et al. (2004), on the relationship between basal shear stress and the state variables of the SWE system. The expression presented in Equations 6.36 and A.24 can both fill in the boxes of $\tau_b(V)$ in the Table 6.1. $S_f(V)$ is easily retrieved with $S_f = \tau_b/(\rho gh)$. It completes the SWE for mudflows and a specific rheological behaviour of Bingham.

6.5.2 Comparison of the implementation in Watlab with Pastor et al. (2004)

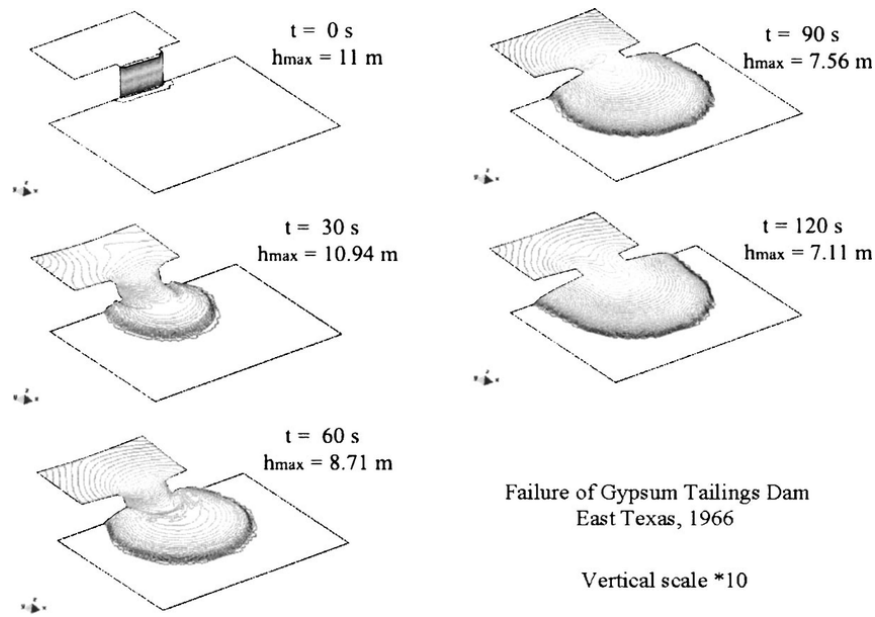
Pastor et al. (2004) provides an application of the formulation they introduced in the paper with large scale simulation: a Gypsum Tailings failure. They modelled the 1966 failure that happened in Texas (USA). An aerial view of such an event can be found in the Appendix A.8. The results from Pastor et al. (2004) were qualitative and no real precise data were explicitly provided. A visual comparison from the result of the article and the results calculated by Watlab with the new Bingham formulation gives an initial idea of the model's capabilities and confirms the accuracy of the implementation. Both simulations used a characteristic mesh of 6 to 10m for a 410m long geometry. The material is supposed to be *non-plastic silts* with Bingham properties: $\mu = 50Pa \cdot s$, $\tau_y = 1000Pa$ and $\rho_{tot} = 1400kg/m^3$. The simulation using Watlab was tried with both $P2$ approximated polynomial expression coming from the article but also the exact solution exposed earlier. A slight difference was noted. The exact solution gave lower velocities of the front propagation and higher material height retained behind the dike. This was honestly unexpected. There is an explanation, it is a bit tricky and not essential in this document, but I refer any curious reader to the Appendix A.10 where the reason is exposed.

The replicability of the simulation is clear, but the maximum height results still highlight a major difference in the early stages of the simulation. Where Pastor et al. (2004) shows a very small maximum height difference in the first 30 seconds, the implementation in Watlab decreases more rapidly the quantity contained behind the failed dike. The limited data mentioned in the article does not allow us to determine the source of this difference with any precision. Geometry is mentioned, but not clearly, and no hypothesis regarding the implementation process in their calculation code is really provided. The difference can come from many different factors but the main one could be the numerical method used to solve the same equations. Pastor et al. (2004) used a FEM, with Taylor–Galerkin algorithm, where Watlab uses a FVM. The present dike rupture is extremely steep, with an 11-meter difference in level between the dry area and the area where the material is contained. The simulation is comparable to a shock wave with very high gradients of velocity and stress. The Taylor–Galerkin FEM used by Pastor et al. (2004) assumes that the unknowns are continuous on the mesh with continuous interpolation functions (even though the resolution scheme is of order 2). The gradients are maybe underestimated and the solutions are generally smoothed to address FEM instabilities during very abrupt modeling. Which probably reduces the maximum speed and steepness of the front.

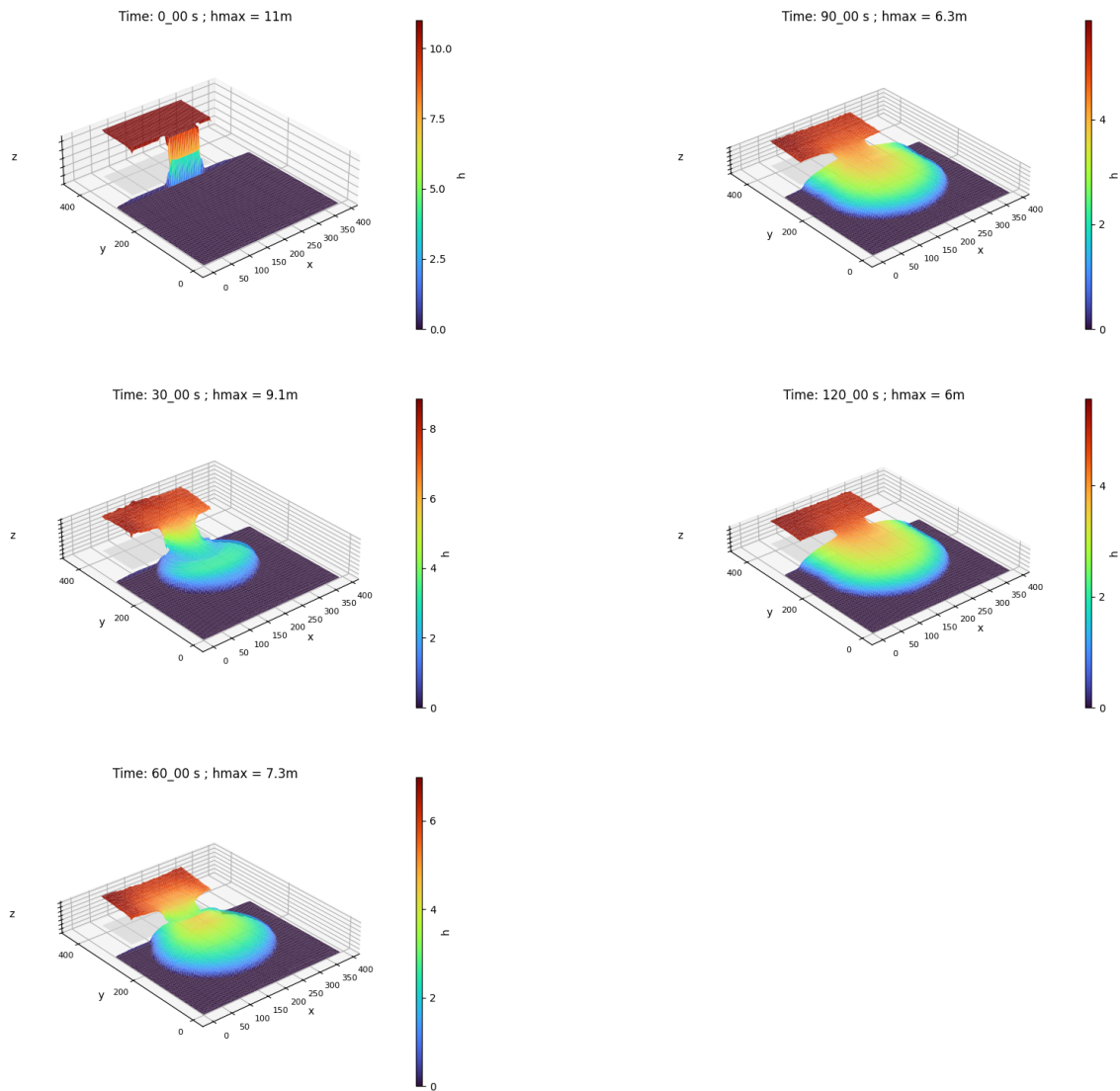
To verify those hypothesis, one should have access to Pastor et al. (2004)'s implementation. However, the overall trend in the deposition geometry and the shape of the wavefront are identical. A rear deposition wave can also be seen between times 30s, 60s, and 90s in both Pastor et al. (2004) and Watlab simulations. It happened because the part of the flow coming out of the breach is very fast as the depth gradient is important. The further away it is, from the breach, the slower the flow. Material starts to accumulate and the height of the front wave increases. The material located upstream continues to arrive at a higher speed and then accumulates in front of this barrier marked by the front. A local increase in height can be observed. This elevation then spreads upstream.

The repeatability underlines the model's ability to produce results that are entirely consistent and physically acceptable.¹⁷ This comparison therefore mainly validates the qualitative behaviour and the order of magnitude of the proposed implementation in Watlab.

¹⁷The reader should pay attention to the color scale change between each figures in Figure 6.8b, it was kept in this way to better apprehend the height variation inside each of the figures and to better see the rear wave propagation. The comparison between each simulation time is highlighted by the maximum material height rather than the color differences.



(a) Results of Pastor et al. (2004): *Sections of free surface at different times by vertical plane passing through centerline of breach.*



(b) Results from Watlab calculation for the Gypsum Tailings failure in Pastor et al. (2004). 56 of 91

6.5.3 Herschel-Bulkley rheology

Herschel-Bulkley model is a generalization of the Bingham model, it adds a power law to the viscous term and keeps a shear strength (see section 4). The exact same reasoning and the exact same assumptions as for the Bingham model can be integrated to the model and to find an expression linking the bottom shear stress τ_b with the state variables $\bar{u}_i$ and h . The relationship is also developed from the velocity profile calculated with the Herschel-Bulkley rheology and a linear shear stress profile assumption. But this time the developments are more complex and longer to write. In this section, only the key equations of the developments are presented, the complete thought process can be found in the Appendix A.1.

The velocity profile for a Herschel-Bulkley fluid in steady uniform flow over an inclined plane is:

$$\boxed{\begin{aligned} u(x_3) &= \frac{n}{n+1} \frac{hK}{\tau_b} \left[\left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{n+1}{n}} - \left(\frac{\tau_b \left(1 - \frac{x_3}{h}\right) - \tau_y}{K} \right)^{\frac{n+1}{n}} \right] \quad \forall x_3 \in [0, h_s] \\ u(x_3) &= \frac{n}{n+1} \frac{hK}{\tau_b} \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{n+1}{n}} \quad \forall x_3 \in [h_s, h] \end{aligned}} \quad (6.38)$$

Integrating the velocity profile over the depth of the flow gives the depth-averaged velocity:

$$\boxed{\bar{u} = \frac{h}{n+1} \left(\frac{\tau_y}{K} \right)^{\frac{1}{n}} \frac{(1-\xi)^{\frac{n+1}{n}}}{\xi^{\frac{1}{n}}} \left(1 - \frac{n}{2n+1} + \frac{n\xi}{2n+1} \right)} \quad (6.39)$$

To verify the expression, when $n = 1$ and $K = \mu$, the Bingham model should be recovered. Indeed, replacing n and K in Equation 6.39 gives:

$$\bar{u}(\xi, n=1, K=\mu) = \frac{h\tau_y}{6\mu} \left(\frac{2}{\xi} - 3 + \xi^2 \right) \quad (6.40)$$

Comparing with Equation 6.23 shows that the expressions are equivalent.

Unfortunately, isolating τ_b in Equation 6.39 is impossible. The only possibility is to use iterations to find the solution. In Watlab, this difficulty was treated with a simple Newton-Raphson (N-R) resolution. Adding time duration for a comparable simulation to Bingham. Equation 6.39, is rewritten in a way to make the parameter a reappear, in a slightly different form but still valid for Bingham. This allows us to compare the evolution of ξ for the two rheological models.

$$6 \frac{\bar{u}}{h} \cdot \left(\frac{K}{\tau_y} \right)^{1/n} = a = 6 \frac{1}{n+1} \frac{(1-\xi)^{\frac{n+1}{n}}}{\xi^{\frac{1}{n}}} \left(1 - \frac{n}{2n+1} + \frac{n\xi}{2n+1} \right) \quad (6.41)$$

The the expression in Equation 6.41 shows the versatility of Herschel-Bulkley compared to Bingham. The way equations have been made dimensionless by the parameter a permits to represent any flow condition in a simple graph and analyze the proximity of ξ with 1 depending on all the parameters contained in a . The proximity of ξ with 1 shows how close the flow state is to being at rest. It can be seen a link between the velocity and the friction at the base but also as the evolution of the plug height depending on the velocity. $\xi = 1$ means the solid block moving over the deformed fluid has grown so much that it filled the height of the flow and everything stopped (it was illustrated in Figure 4.2).

The curve is of course dictated by the velocity and the height but also by the rheological parameters. In Figure A.3, the advantage of a Herschel-Bulkley model is clear; it can represent more versatile situations. When the power index contained in Herschel-Bulkley is lower than 1, the flow will stop more brutally:

the green curve is steeper than other curve close to the stop. This means that the flow will gradually slow down and then suddenly stop when it gets close to stopping. At the opposite, when the power index is high, as the orange curve, the friction will be definitely higher for large velocities (low ξ means high S_f) but the flow will stop more gently. The numerical plateau observed in the curves is an artifact due to the logarithmic scale of the x-axis. In reality, the curves are fairly horizontal and rise steeply as they approach $a = 0$. The same curves without logarithmic axes are presented in Appendix A.9 (it is easier to see the steep increase of ξ for low values of a), but have been kept identical to those presented in the Bingham section to maintain consistency.

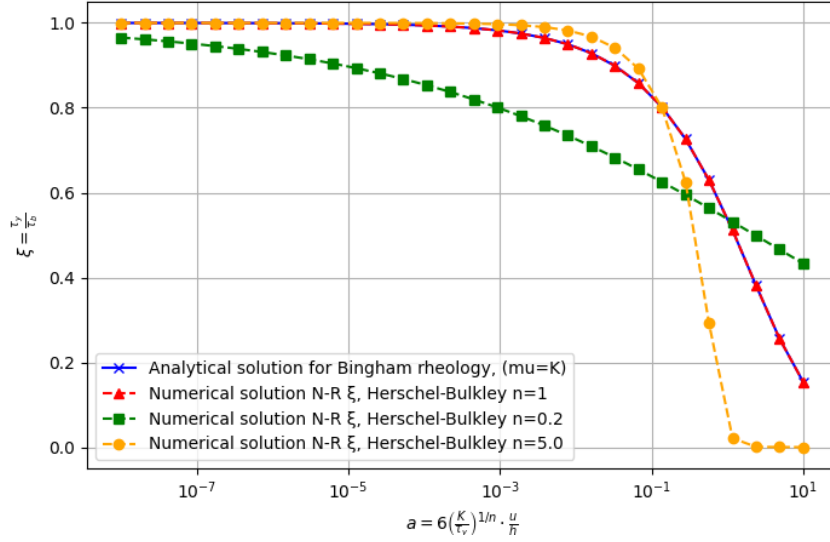


Figure 6.9: Relative plug height ξ and parameter a , they govern the friction.

We can now come back to the main objective: finding the friction source term in SWE for a Bingham rheology and a Herschel-Bulkley rheology. The $\tau_b(V)$ box of Table 6.1 has indirectly filled. Unfortunately τ_b is impossible to isolate from Equation 6.41. The equation acts as a link between τ_b and V and the resolution uses an iterative method; a N-R was implemented for this part in Watlab.

Only two cells are left to be filled in Table 6.1, they are the stopping criteria. Bingham and Herschel-Bulkley are non-Newtonian flows with an ability to stop even when still subject to driving forces. As long as those forces don't exceed the flow strength: τ_y the simulation can reach a stopping state, which is impossible with simple formulation as Manning or Chezy.

6.6 Natural stopping criterion

In order to apprehend the stopping process, the evolution of a , ξ , τ_b and S_f should be put together. First, a is governed by the state variables $\bar{u}$, $\bar{v}$ and h . The state variables results in the calculation of the fluxes through each interfaces of the cells. The flux is calculated from the speed of the considered cell, the speeds of the neighboring cells and the net height difference of the fluid (taking into account the base level variation). So for example if a mudflow flow down a slope that is getting milder, the speed will decrease.

If a decreases, ξ (the relative plug height) increases as can be observed in any of the graphs presented earlier. It was also shown in the section presenting the Bingham theory: Figure 4.2, when the flow slows down, the relative plug height grows and the solid motion takes more place in the vertical section of the flow.

An increase of ξ is a decreases of τ_b (as $\xi = \tau_y/\tau_b$). Is was also illustrated in the same figure. Finally, S_f decreases with the τ_b decrease because $S_f = \frac{\tau_b}{\rho gh}$.

At first glance, this behaviour may appear similar to Bingham or Chezy formulations that also predict a decrease of S_f with a decrease of V . The major difference relies on the minimal value that can take τ_b and so, S_f . The base friction is bounded by τ_y , the resistance of the flow. When the speed comes close to 0, the friction is still acting and continues to slow it down even more. Figure 6.10 shows the comparison between the two theories for the friction source term S_f of SWE.

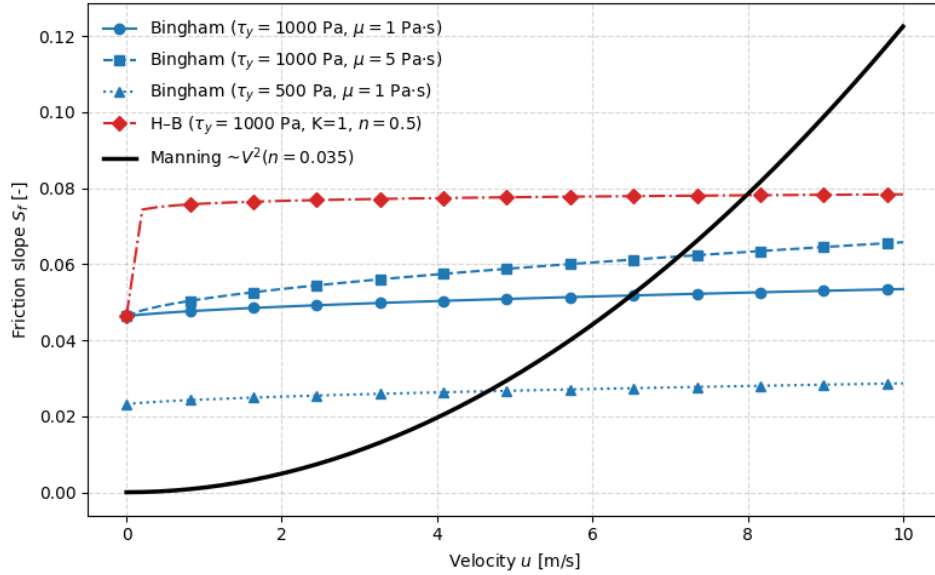


Figure 6.10: Comparison of Manning, Bingham and Herschel-Bulkley for the friction slope calculation S_f .

The Manning formulation shows an abrupt decrease of the friction slope when the speed decreases. An analogy to the friction slope can be a brake pedal in a car. Where the slope is one of the motors of flow, the friction acts as a brake. With no other actions, a decrease of speed will rapidly decrease the brake effect for Newtonian flows. In consequence, when a Manning based flow enter a milder slope, the speed will decrease reasonably. As if you were releasing the brake pedal when trying to stop. At the opposite a non-Newtonian based flow (as Bingham here), in same conditions will stop suddenly. The ratio between the decrease of the speed and the decrease of the "brake pressure" is completely different for both theories. This time it's as if you were slowing down with almost constant pressure on the brake pedal. Behaviour of Newtonian and non-Newtonian flow are thus completely different under velocity changes.

In Figure 6.10, we can also observe the real effect of the rheological parameters on the friction slope. The yield stress τ_y vertically translates the curve (from triangle dots to round dots in Figure 6.10). Following the same analogy, the shear stress dictates how much we push on the brake. It does not influence the change of pressure we exert while the speed decreases. The viscosity rotates the curve (from round dots to square dots): it imposes how much we relax the brake when the flow slows down. High viscosities significantly slow down faster flows but slow them down less and less as their speed decreases. Low viscosities at the opposite, don't change the brake pressure, a null viscosity is a completely horizontal curve. It means whatever the speed, the rheology will slow down the flow with the same friction slope! The power index of Herschel-Bulkley rheology adds a versatility by making the curve steeper at low velocities.

In quick summary, the viscosity makes the friction slope evolve with the speed, the yield stress is the initial condition at null speed (an ordinate at the origin) and the power law makes the curve steeper as it approaches $u = 0$.

How does this really help to understand the effect of the friction on the speed? In reality, during the steps

of the update presented in Figure 6.1, the purple box is found by Figure 6.10. We will call the "friction speed", the contribution of the friction in the new speed at the next time step. This "friction speed" is recovered by multiplying S_f with Δt : during how much time does the friction can slow down the speed. The new speed at the next time step is simply the actual speed, increased by the fluxes component and then decreased by the friction component. This gives the new speed and the process can start again.

The rheology on Bingham and Herschel-Bulkley have a natural tendency to stop the flow when the speed decreases enough to be absorbed by the friction. This process is in reality, the energy dissipation from kinetic energy to friction induced head losses. As explained earlier, the friction is not only the base friction of the flow but thanks to the developments exposed in this section, the rheology and so, the internal friction within the material column also induces an energy dissipation: the approach was to depth integrate the rheology properties.

6.6.1 The no-backflow

A difficulty with the non-nul friction of non-Newtonian flow happens when the flow stops. FVM transforms, in a certain way, the friction source terms into speeds, which are added to the state variables calculated after the fluxes had modified them. Those "friction speeds" are opposed to the actual speeds seen by the model at the earlier time step. That is how they decrease the speed. While the flow stops, the "friction speeds" can be non-zeros which could lead to major un-physical results. If the problem is not treated, the flow could climb uphill for example.

It is often called the "no-backflow" verification. It was already implemented in Watlab as water can some times enter this condition; it suffices that the friction action is greater than the actual velocity. It happens when doing the update if the new speed ($=$ ancient speed $+$ fluxes components $-$ "friction speed"), passes below zero. This is a rather specific and uncommon case for water, but it is common in all mudflow simulations because it is the process that stops the flow. It is very rare for friction to perfectly compensate for ancient speed and fluxes components. Generally, the "friction speed" is greater, and this condition is triggered each time the stop of the movement has to happen: the mudflows comes to rest in this particular cell.

In Watlab, the no-backflow condition simply evaluates the direction of the speed before and after taking the friction into account. If the direction of the velocity has changed, it means the friction tries to reverse the flow, so the code stops the cell.

However, there is a complication: when this condition arises for water, the water speeds is set to zero. So far, nothing unusual. But the negative speed that the no-backflow is treating came from a particular addition. The addition is between the ancient velocity of the cell (earlier time step), the fluxes through the interfaces and finally the "friction velocity", the latter is opposed to the ancient velocity of the previous time step. When adding the flux component, the flux not only gave a speed increment but also a quantity of water, stored in h . If the no-backflow feels that the velocity try to change sign because of the friction, it stops the speed: the current speed is now null. But the water height that was transmit by the flux calculation is not recovered and is not given back to the neighbor cell! For water, this is not a problem and is even intentional. It is the precess from which the kinetic energy is transferred into potential energy and allows the code to model some typical waves¹⁸. But for mudflows, we have seen that they can stops while descending along a slope, thanks to their yield stress. When this condition is triggered in the code, the increment of height is given to a downstream cell and we observe a sort of artificial height diffusion that prevent the simulation from a real stop! The greatest difficulty in this thesis was dealing with the

¹⁸To be more precise, this happens while a water simulation have enough kinetic energy to flow uphill until a certain point. This point is the time where all the kinetic energy was transferred into potential energy. Here, the speed is usually very low just before going backward so the friction can be high enough to reverse the flow direction. As it is not physical for the friction to reverse it, we use the no-backflow and stops the flow a for a unique time step. During this time step, all the kinetic energy is transferred into potential energy and the last drop of kinetic energy is given to the transfer of a small increment of h carried by the flux. This little increment of h helps then the flow to go backward and flow down the slope at the next time iteration.

impossibility for the code to stop the flow in a "natural" way, as expected by mathematics.

For example, without any additional method for stopping the mudflow, the results of the Gypsum Tailing failure presented above would give a completely uniform and horizontal height over the entire geometry of the simulation when it is kept running for a long period. An example is shown in Figure 6.11, for which no additional stopping criterion was used outside of the mathematics presented above.

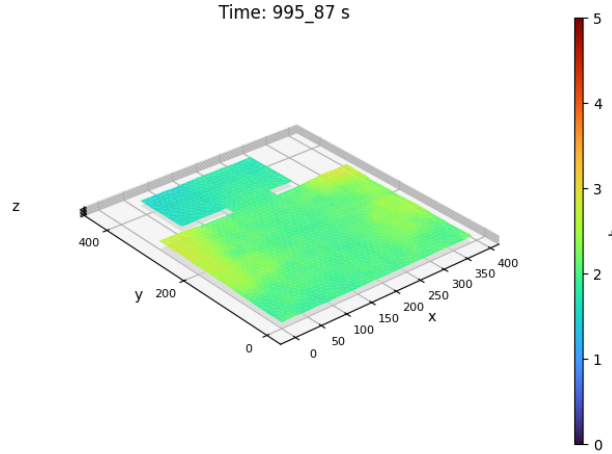


Figure 6.11: Long simulation of a Gypsum Tailing failure without any "artificial" stopping criteria (h is in meters).

To address this issue, some additional numerical conditions were required. This part took a lot of trials to find the ideal conditions. The last proposal, which seems to be the most physical, has two steps. First, a condition on the base friction τ_b helps to avoid numerical noise and too sensible remobilization. This condition is a security and is an additional tool for the stopping motion. If a cell speed is found to be zero thanks to the "no-backflow", it is labeled as "at rest". Then forces acting on this stopped cell are evaluated. If they are strong enough to break the resistance τ_y , then the cell can flow again. It is the "artificial stopping criterion".

6.7 Global view on the friction slope theory for Manning, Bingham and Herschel-Bulkley

Now that all cells were developed, Table 6.2 is complete. The approach for $S_f(V)$ calculation has been established for all the models and the natural stopping character with the mathematical tendency of non-Newtonian rheology was exposed. The challenge relied on the calculation of $\tau_b(V)$. To obtain this expression, we integrated the velocity profile linked to the rheology. It helped to isolate the base friction that account for the behaviour of the material under shear stress (=rheology). The table summarizes the results of the mathematics for this section and helps to obtain a global view and a comparison to what was already present in Watlab for Manning friction source terms.

Table 6.2: Structure of the friction modeling for different rheologies, completed.

Rheology	$S_f(\tau_b)$	$\tau_b(V)$	$S_f(V)$	Stop. criterion
Water Newtonian fluid	$S_f = \frac{\tau_b}{\rho gh}$ Uniform steady flow Forces balance	$\tau_b \propto V^2$; $\tau_b = kV^2$ Turbulent flow	$S_f = \frac{n^2 V^2}{h^{4/3}}$ Manning formulation	None
Bingham	Same expression and same hypothesis $S_f = \frac{\tau_b}{\rho gh}$ Uniform steady flow Forces balance	Semi-empirical relations exist (not accurate) $\xi = \tau_y / \tau_b$; $a = \frac{6\mu u}{\tau_y h}$ Hypothesis: - linear shear stress $\tau(z)$ along the depth - $\tau(z)$ follows the rheology $\tau_y(V)$ is contained in $\xi(a)$. To be solved: $P_3(\xi) := \xi^3 - (3+a)\xi + 2 = 0$ Solution of ξ : to choose from: - Approximated polynomial P_2 : Eq. 6.30 - Corrected polynomial $P_{2,forced}$: Eq. 6.36 - Analytical solution: Eq. A.24	Combining $\tau_b = \frac{\tau_y}{\xi}$ and $S_f = \frac{\tau_b}{\rho gh}$ gives the friction slope.	Natural stopping: $U_{new} = U_{old} + Flux_{speeds} - S_f \Delta t$ $U_{new} \leq 0$ if $S_f \Delta t$ large enough $U_{new} \cdot (U_{old} + Flux_{speeds}) < 0$? -> no-backflow Artificial stopping: Prevents the "diffusion" of h Remobilization condition
H-B	Same expression and same hypothesis $S_f = \frac{\tau_b}{\rho gh}$ Uniform steady flow Forces balance	$\xi = \tau_y / \tau_b$; $a = \frac{6\mu K}{h} \frac{1}{\tau_y}$ Hypothesis: - linear shear stress $\tau(z)$ along the depth - $\tau(z)$ follows the rheology $\tau_y(V)$ is contained in $\xi(a)$ $\bar{u} = \frac{h}{n+1} \left(\frac{\tau_y}{K} \right)^{\frac{1}{n}} \frac{(1-\xi)^{\frac{n+1}{n}}}{\xi^{\frac{1}{n}}} \left(1 - \frac{n}{2n+1} + \frac{n\xi}{2n+1} \right)$ Solved by iterations: N-R.	Combining $\tau_b = \frac{\tau_y}{\xi}$ and $S_f = \frac{\tau_b}{\rho gh}$ gives the friction slope.	Natural stopping: $U_{new} = U_{old} + Flux_{speeds} - S_f \Delta t$ $U_{new} \leq 0$ if $S_f \Delta t$ large enough $U_{new} \cdot (U_{old} + Flux_{speeds}) < 0$? -> no-backflow Artificial stopping: Prevents the "diffusion" of h Remobilization condition
a is the parameter storing the flow information (containing the speed) and makes the relation $\tau_b(V)$ dimensionless. ξ expresses the proximity of the shear stress τ_b and the shear strength τ_y. It is also known as "the relative plug height": the ratio between the vertical part of the flow acting as a solid mass and the fluid deformed by shearing.				

6.8 Artificial stopping-remobilizing criterion

Now a word on what exactly happens in the artificial stopping/remobilizing criterion. This criterion is not derived from continuum mechanics, but rather on intuitive physical behaviour to restore the consistency between the rheological mathematics and the numerical resolution scheme unable to produce a coherent stop for mudflows.

Let us analyze four adjacent cells at the front of the deposition in the PoliMi's set-up, Figure 6.12. They are fictitiously much larger than those used for simulations in order to better visualize them. Two of the cells are still in motion (B and A), the cell C is stopping and the cell D has already stopped and dry.

Each of the cell of the domain has a little boolean tag called: " can_I_move ". By default, all the cells have " $can_I_move = True$ ", they all can move. Once a cell have a null speed the tag is changed to " $can_I_move = False$ ", the cell is supposed to keep at rest, and we should verify if it can move again or not. Being frozen by the boolean means several things: the cell have null speed, the cell can't give away any flux but it is allowed to receive some. A blocked cell receiving flux is illustrated in the Figure 6.12 (side view of cell A and C).

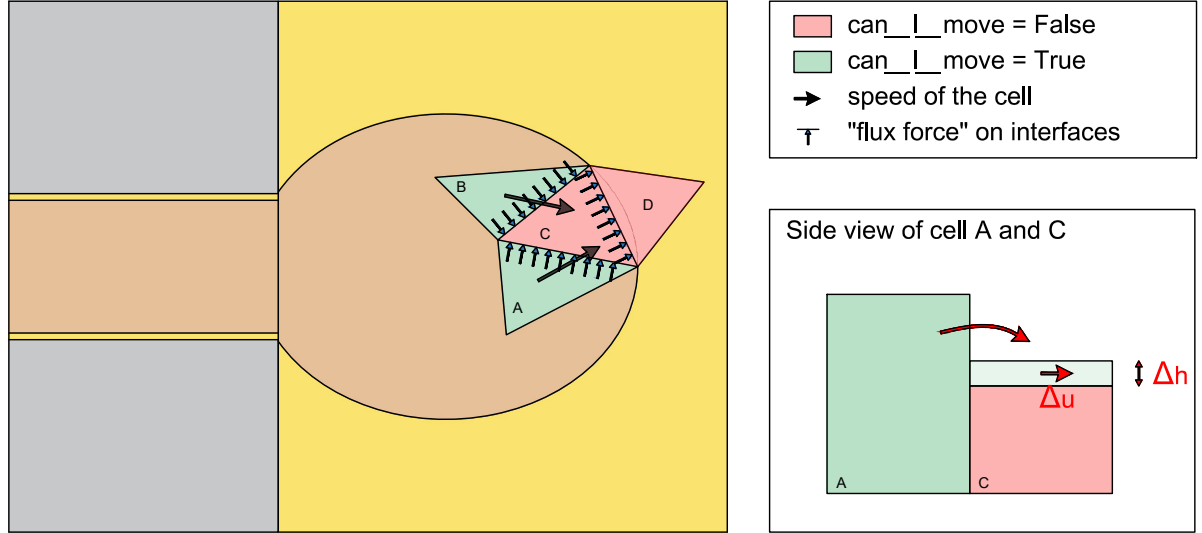


Figure 6.12: Representation of frozen and un-frozen cell for stopping criterion explanation.

Nothing prevents a frozen cell from receiving fluxes, and the process can be viewed as if the moving cell was depositing a small amount of mud on top of the frozen one (circular red arrow in the Figure 6.12). This small amount of mud has a certain small velocity Δu and a certain small height Δh . These two quantities modify the state variables of the frozen cell and can remobilized it if they are important enough. Here, it is easy to understand the little problem this approach brings. Indeed, any little Δu would unfreeze the cell as by definition the cell is frozen only if the speed is perfectly null.

It asks for the security condition. Letting the model work in that way makes it highly unstable and does not allow for a proper stop. The security condition relates to ξ that has been calculated for each cell at each time step. For a cell to come naturally at rest, the basal friction decreases enough to become equal to the resistance τ_y : this is met at $\xi = 1$ and $a = 0$. The security is therefore to send any cell with $xi > 0.9$ to $\xi = 1$ (the value of 0.9 is arbitrary and can be modified). In that way, the little increment of mud deposited on the frozen cell gives $a = \frac{6\mu\Delta u}{\tau_y(h+\Delta h)}$. If this quantity is enough to break the threshold

$\xi = 0.9$, the cell is remobilized at the next iteration. If not, it is kept frozen.

This criterion may seem sufficient at first glance, it dictates if the cell should be kept at rest or not depending on the conditions applied to the frozen cell. In reality, it is not. Indeed, if cells A and B don't give fluxes strong enough to break the freeze, the cell C can act as a kind of vacuum and the cell sucks in the material around it as nothing stops it to grow as much as it wants. It results in a growing height for this particular cell. Such a problem was observed during the first tries of the stopping criterion.

To solve this, the choice of using "force fluxes indicators" was made. These fictive fluxes forces are dimensionally scaled to be compared with τ_y , the material resistance. The fictive fluxes are simply the perpendicular component to each one of the interface of the frozen cell. Here, the force applied to the interface $A - C$ is both dynamic and static. The dynamic component comes from the speed of A pointing toward the interface $A - C$. It is as if the flow of A encounters a rigid wall represented by the frozen cell C , so it pushes against that interface, destabilizing the frozen cell if it is not strong enough to sustain that force. The static component of the forces on interfaces comes from the net difference in height (accounting for bed level variation). In this case, even two frozen cell can move again if the difference in height between them is important enough, or if the bed slope is breaking the resistance τ_y . For example, in the schematic representation, there is a force applied on the interface $C - D$ as C has a certain height and D is dry ($h_D = 0$). The resultant of the driving forces on the frozen cell is simply the sum of the contributions from each of its interface and is compared to the resistance τ_y .

The two processes: small fluxes received and forces on interfaces, work all together to create a smooth, numerically stable stopping motion of the material.

In summary, rheology induces a natural tendency to stop but the numerical scheme in Watlab prevented this to happen. The artificial criterion does not cause the stop, it enables it. And at the same time, it manages the restart. The artificial criterion also allows the user to place different values of τ_y , a stopping value and a mobilizing value, as they can be different. This was observed in experiments of Schippa (2021).

Chapter 7

Validation of the model

This part has two main objectives. First, it hopes to validate the approach chosen by this work, while validating the global processes of initialization, flow behaviour and deposition. The second focus is on the weaknesses of the model and the differences encountered with experimental data. It is hoped that highlighting these weaknesses will lead to better adaptations in the future and perhaps enable guidelines to be established on the main areas for improvement.

First of all, the geometry and test procedure were presented in Section 5. The numerical modeling is as close as possible to the experimental campaign and uses the exact same geometry. The mesh is composed of triangular elements with a characteristic size of 1.5cm distributed evenly but without being a perfectly structured mesh.

The simulation takes in entry the incline angle of the channel, the volume occupied by the material before flowing, the rheological parameters and the bulk density. All of these parameters were collected (Guo 2025) or deduced from the granulometric results, see Table 5.2 and 5.3.

Firsts simulations use a Bingham rheology, as it is the easiest one to track: only two parameters should be changed if needed (μ and τ_y). Simulation of the first available experiment *T06* uses the rheological parameters deduced in 5.3. The final shape deposition viewed from the top shows inconsistency with what was expected, Figure 7.1. Furthermore, instabilities are visible. The red curve is the mud deposition tracked by the color differences in the experimental campaign (Guo 2025) and the variation is height $h[m]$ is represented by the color scale from red to blue.

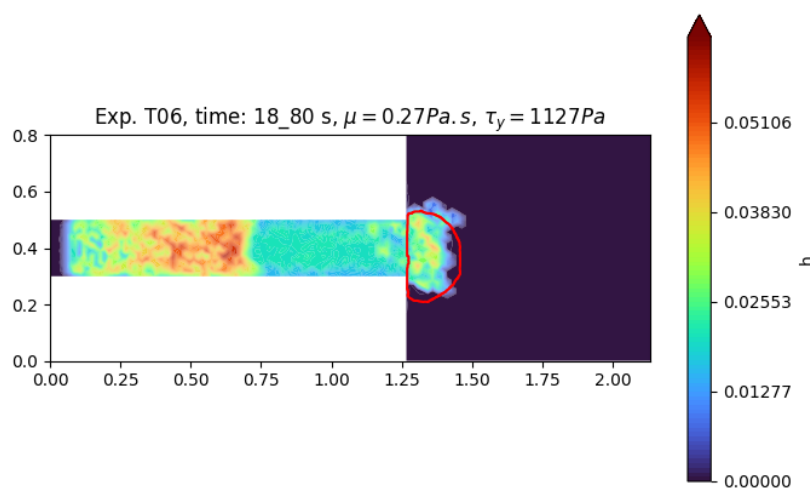


Figure 7.1: Bingham computation of T06 using calculated parameters in Table 5.3.

7.1 Hypothesis for first inaccuracy of the model based on calculated rheology

Regarding the input parameters for Bingham rheology, they were calculated based on a sample studied with granulometry. This assumes that all the material used has the same characteristics, which is debatable. More importantly, both $\mu = 0.4783 Pa \cdot s$ and $\tau_y = 1133.0 Pa$ are individually coherent for a mudflow of $\rho_{tot} = 2273.39 kg/m^3$ bulk density, but don't really work together for this simulation. The single phase Bingham theory presented here predict the friction based on a parameter a that accounts for both μ and τ_y all together. Therefore, their ratio μ/τ_y must be consistent with the experiment rather than estimated individually. Indeed, here the relatively small value of μ allows the material to flow very fast in the inclined channel and this speed prevents it to stop in this part. When the simulation reaches the horizontal plan, the motor of slope is reduced to zero and the "fluxes fictive forces" loose an important component. The resistance τ_y is too high and the artificial destabilizing forces too low: the material stops directly when it reaches the deposition plan. It results in a sort of "glue material" that has quite low viscosity but high resistance. The deposit resembles water that instantly froze in the deposition area after arriving at high speed. Also, the very low ratio of a imposes some cells to be stopped where they should not be. We can see stopped cells in the incline plan and a lot of instabilities of the model due to this low a parameter. The instabilities can be easily smoothen by increasing the threshold of $\xi = 0.9$ as it dictates is freezing condition and is the reason why cells stop where they are not supposed to. However the excessive velocity in the channel can only be adjusted by changing the rheological parameters.

7.2 Rheological parameters found by visual iteration

After visual trial and errors on the value of μ and τ_y and comparing the results from the red curve to the model's result, the best fitted parameters were $\mu = 1.5 Pa \cdot s$ and $\tau_y = 390 Pa$. They are both respectively much larger and much smaller than the first guess. Their ratio is now 16 times higher. Figure 7.2 represents the simulation after the iterative search. The color gradient represents the height, according to the scale shown on the first plot of the figure. The red arrows are the speed vectors and the red circular curve is the advance of the mud front detected by the experiments. The red curve of front detection is only visible in the two lasts plots of the figure, as the numerical model predicts a faster speed inside the channel. The material stopped moving after 8 seconds in the experiments, and the model predicted it would stop after 1.4 seconds. The simulation is really close to expectation regarding the deposition but the speed inside the channel is still much faster than what was measured. We can see that the red circular curve representing the real deposition advancement from the experiments is very close to the simulation at the last time $t = 8s$. However, before that time, the simulation enters the deposition area faster than the experiments and the simulation predicts a deposition time much lower than the reality.

The difference of viscosity between what was calculated and what was iteratively found can come from several sources. The first one is coming from the calculation of μ based on the granulometric test. The formula that was used may not be correct for these experiments, it was found from experimental fitting of other tests in the literature. A second reason could rely on the time between which the material is placed in the canal of PoliMi and the beginning of the experiment. Even though this time is relatively short (maybe 10 to 30 seconds), it can provide enough time for the larger particles to settle down. This would increase the effective cohesion near the channel base and modify a lot the viscosity at the lower part of the mud. Single phase approach probably fails to model the first few seconds of the experiment, while the cohesion of larger particles locally changes the rheological parameters. This may also explain the the model's inability to reproduce the initial much slower velocities of the experiments.

The high speed highlights the weakness in the model's prediction in the first moments of the simulation. When comparing the results of the present model with (Pastor et al. 2004), a difference of speed was also observed. Indeed in the simulations of PoliMi's set-up, when the gate opens, the initial velocities are all zeros for every cells. The friction tries to find the direction that it should take and fails, as it is supposed to oppose velocity ($= 0$). Therefore at the first time step of the simulation, there is no friction and the velocities gain probably their large values from that first time step. During this thesis, some solutions were tried to recover this problem, such as using the opposite direction of the fluxes for the first time step but all efforts lead to instabilities. The solution for future works may be hidden in the calculation steps

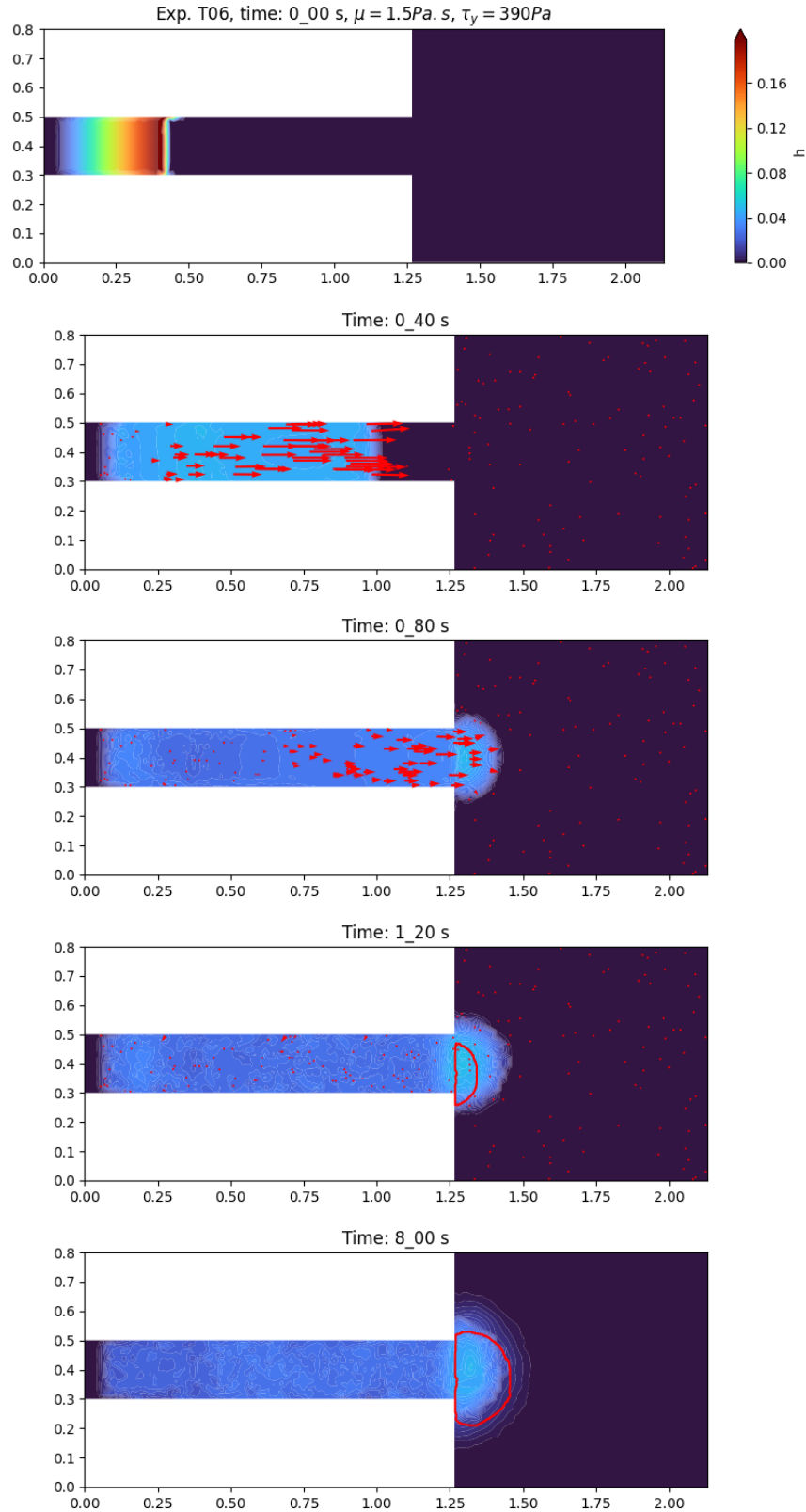


Figure 7.2: Bingham computation of $T06$ using parameters found to visually fit the experiment.

of fluxes and friction inside Watlab or a too large first time step that would let the friction increase too much.

The lack of friction in the early stages of the simulation also prevents the numerical model from achieving a proper initialization. If the yield stress is strong enough to withstand the driving forces induced by the slope or height differences, the simulation should remain at rest. Here, without any friction at the first time step, there is no possibility of retaining the material and a first little movement is always observed.

7.3 Global comparison between simulation results and experiments

Surprisingly Bingham rheology gave better results than Herschel-Bulkley did. Indeed, Herschel-Bulkley is more versatile but it is especially well suited for sudden stops: going from high speed to a stop quickly. This is observable in Figure 6.10: for any value of the power index n , H-B has a tendency to give higher S_f values (for velocities under $10m/s$ than Bingham). It also induces an abrupt transition of S_f close to $u = 0m/s$. Moreover, the rotation of the $S_f(u)$ curve due to the viscosity is extremely attenuated by the power index, whatever its value may be. What happens when using H-B is that the friction acts very strongly down to small velocities. In practice, this rheology combined with the way it was implemented in Watlab produces an abrupt stop.

In the simulation of PoliMi's experiments, the problem of the simulations are related to the speed in the channel which is calculated to be too high. It results in a stopping process more abrupt than what has been measured. In consequence, the use of Herschel-Bulkley instead of Bingham, made this difference even more important and it was found that Bingham gave significantly better results.

The experimental results made available for this thesis are divided into two categories: $T06-T07$ were used for reputability purposes and their results were very similar. $T08-T09-T10$ used the same material at different incline angles. Both categories challenge the model in different ways. $T06$ focuses more on the slow movement in the incline plan and spread over a limited area. Where $T08$ has a higher speed inside the channel but spread in a more complex shape. Illustrative representation of the $T08$ is in Figure 7.3.

In this second simulation, for $T08$, we can deduce that the spreading behaviour of the model is dominated by the speed. Indeed, where the reality spreads in a large part of the deposition area (perpendicularly to the flow axis), the model is narrower. This means the deposition spread advanced thanks to the high speed gained in the channel, as its major component is aligned with the channel axis. The transversal deposition should be dominated by height differences between cells rather than pure velocity entrainment. The model has obviously difficulties to handle this. A second weakness of the model is hence the handle of low speed deposition. It can also be observed that the model is nearly perfectly symmetrical along the channel axis due to the relatively homogeneous mesh that was used. The symmetry is no necessarily present in the experiment as little imperfection in the material or in the channel position can influence a lot on the flow shape.

Results of the mudflows simulation with Watlab and their comparison with the data measured in laboratory controlled environment are shown in Table 7.1.

The model build in Watlab is capable of predicting a first assessment of the runout distance. For simple shape deposition, which does not involve slow mud movement before the deposition, the model is also capable of good area prediction and precise shape definition. However, for a better prediction, corrected formulation for rheological parameters should be searched for. Here the process of parameter determination was based on visual trial and error, to fit as best as possible the physical reality. This approach would be impossible for catastrophic event runout prediction as no comparison would be available. Moreover, the model still over estimates the velocities and deposition at lower speed is also incorrect up to now. Possible solutions have been proposed for these difficulties.

In the present work, validation is understood as the ability of the model to reproduce some firsts runout metrics (runout distance, deposition area and final shape in certain scenarios), rather than the full

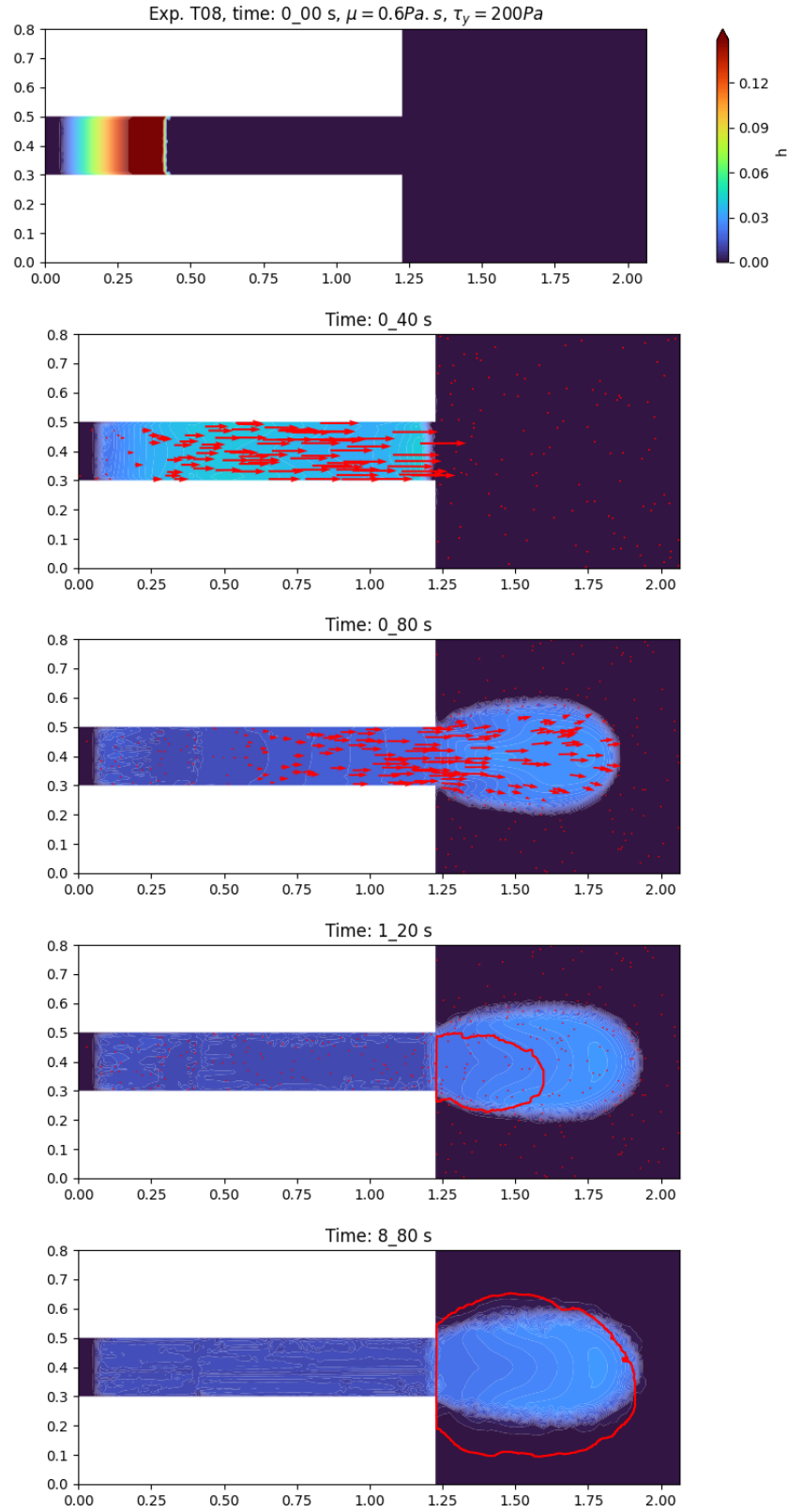


Figure 7.3: Bingham computation of $T08$ with the best fitted parameters possible. The material stopped moving after 8.8 seconds in the experiments, and the model predicted it would stop after 1.2 seconds.

Table 7.1: Experiments and simulation comparison. Experimental measurements are from PoliMi (Guo 2025).

Experiments results				Fitted rheology		Simulation results		
Name	Dep. area	Runout dist.	V_m	Bingh. μ	Bingh. τ_y	Dep. area	Runout dist.	V_m
Units	$[m^2]$	m	$[m/s]$	$[Pa \cdot s]$	$[Pa]$	$[m^2]$	m	$[m/s]$
T06	0.055	1.161	0.316	1.5	390	0.06	1.169	1.64
T07	0.045	1.137	0.305	1.5	400	0.049	1.15	1.56
T08	0.337	1.666	1.467	0.6	200	0.236	1.71	2.0
T09	0.299	1.576	1.055	0.5	150	0.21	1.57	1.89
T10	0.263	1.484	0.943	0.4	140	0.16	1.452	1.56

temporal dynamics of the flow. The Watlab mudflow module is still at an early stage of development for a fully quantitative comparison with the complexity of mudflow reality. Nevertheless, the results obtained suggest that the model provides a consistent and physically meaningful basis for future works and improvements. The limitations and weaknesses identified in this section are hoped to guide future works if this module has the opportunity to grow in the future of Watlab.

Chapter 8

Real case application

In the previous section, the qualitative potential of the model was demonstrated, rather than a perfect dynamic prediction. This chapter continues in the same vein and aims to extend the tests to a larger-scale application with a qualitative perspective.

The chosen application is the glacier collapse that happened in Blatten (Switzerland, situated in Figure 8.1) in the early summer 2025. The village was almost entirely buried beneath a mixture of mud, rock, and ice who slid down the northwest face of the Bietschhorn peak.

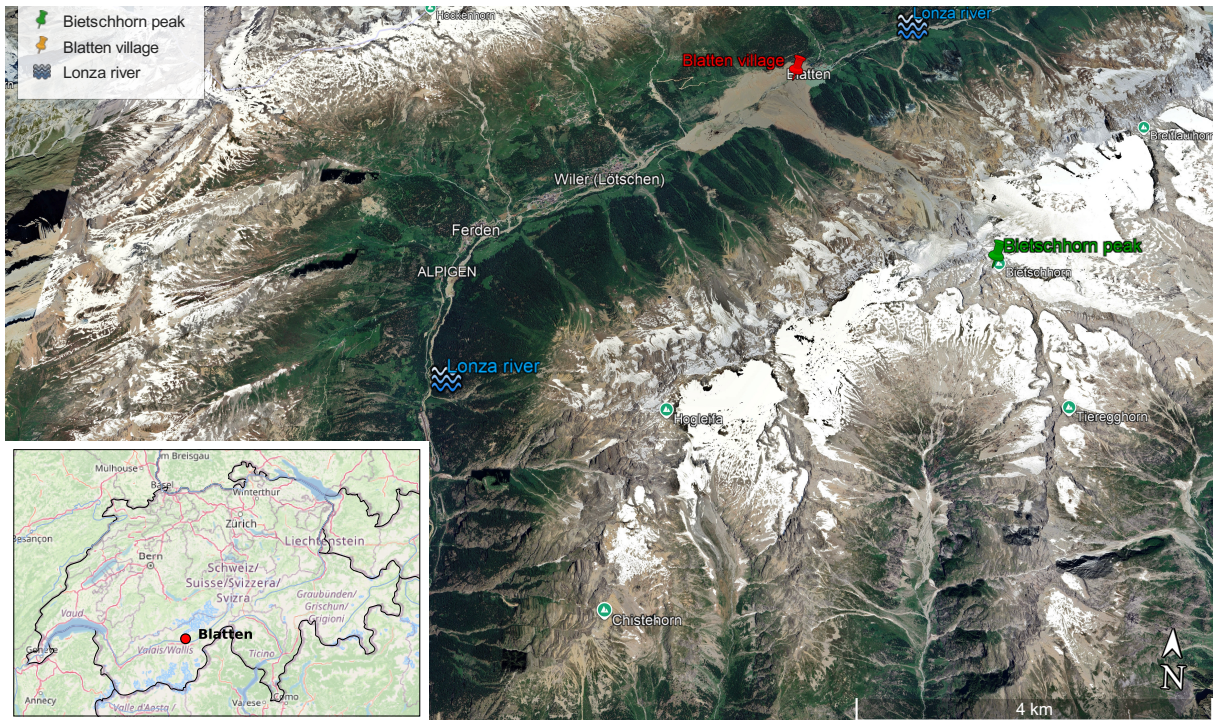


Figure 8.1: Blatten and Bietschhorn peak situation, Swiss Alps (GoogleEarthPro and OpenStreetMap).

Blatten is located not far from Sierre, in the German-speaking part of Switzerland in the Alps massif. The 28th May 2025, a volume of $2.9 \cdot 10^6 m^3$ detached from the Birch Glacier carrying away $6.4 \cdot 10^6 m^3$ of earth and rock in its path. The mixture reached a speed of 200 km/h as it descended 1.200 meters and climbed 200 meters in the opposite slope of the valley (Büntgen et al. 2025). The catastrophic event started days before the final slide with abnormal movement in the Birch Glacier that warned the authorities. Fortunately, the village was evacuated nine days before the rupture. The flow was destructive, Figure 8.5 clearly shows the damage. Almost 90% of the village was devastated, 300 buildings in Blatten and 70 in the nearby settlement. One victim has been reported and the damage is estimated at 320 million Swiss Francs (Büntgen et al. 2025). In addition to the damage caused by the mixture flow itself, the deposition of the material blocked the Lonza River (which originally ran through the village) and created an artificial dam that flooded the part of the village that had been spared by the debris.

These events reinforce the need for early simulation tools that can predict such devastating flows for population safety and risk management. Although the flow can't really be expected to be a mudflows, an extremely high granular temperature probably happened, giving the material a low effective viscosity



Figure 8.2: Front view of the debris deposition after the catastrophic event in Blatten (Geoengineer.org 2025). The debris deposition and artificial lake formed by the Lonza River is visible.

and still a high yield stress. The mudflow module build during this work has a tendency to provide high velocities and brutal stop. In addition to being a recent event that left a lasting impression, this particularly fast event (up to 200km/h) would be suitable for a first test with the model.

The surface area of rupture was visually estimated by aerial photographs. It was assumed to be a triangular shape of 98800m^2 area. From volume estimations, the debris height was deduced. Unfortunately, the erosion is not taken into consideration in this simulation. The initial volume is therefore the ice volume and the rock-soil volume together which causes imperfections in the first few moments of the simulation with a very high material column. The mesh is composed of triangular shaped elements with a regular characteristic dimension of 25m for a 5km by 4km rectangular domain. The Bingham rheological parameters working the best were $\mu = 0.001\text{Pa} \cdot \text{s}$ and $\tau_y = 17 \cdot 10^3\text{Pa}$. As expected, these values are very small and very large, respectively, for mudflows. The bulk density was intentionally lowered compare to what could be expected. The granular temperature induces large local detachment of the grains between them creating a lot of voids and seriously decreasing the bulk density. An arbitrary value of 1900kg/m^3 was chosen.

This real-scale application fundamentally differs from the experiments of PoliMi, in terms of its scale, the material composition and the flow regime. It is not intended as a validation but rather as a qualitative test of the numerical tool for a different scenario.

The simulation was qualitatively able to reproduce the runout that was observed and the extremely high speed was overestimated but kept in the same order of magnitude. A maximum peak of 290km/h was calculated 4 seconds from the beginning. It is overestimated as the initial height is too much important due the unavailability of erosion: all the volume is concentrated in the rupture area, both ice and actual eroded soil-rocks giving a height of 94m . The first plot of Figure 8.3 represents the initial geometry used for material storage at the beginning of the simulation. It is unrealistic but necessary to have the same amount of volume. The mesh is also illustrated on the first plot. Next, follows the steps of the simulation. The simulation does not travel as far as the deposition represented in light brown in the background of the images. This contour reflects the projection of fine particles during the speed of the flow. The major deposition stopped before this contour and the simulation was fitted to approximately respect that final shape. Finally, the deposition shape is compared from simulation 3D views and real photographs and we

can better visualize the real deposition shape on those photographs.

The real case application illustrates that, despite the current limitations, mudflow module developed in Watlab reproduce a consistent runout behaviour at large scale. The results must be interpreted qualitatively as several key mechanism where missing or strong assumptions were used. Nevertheless, this simulation reveals of potential of the model as a basis for new developments and improvements, and supports the coherence of its behaviour outside of simplified experiment environment.

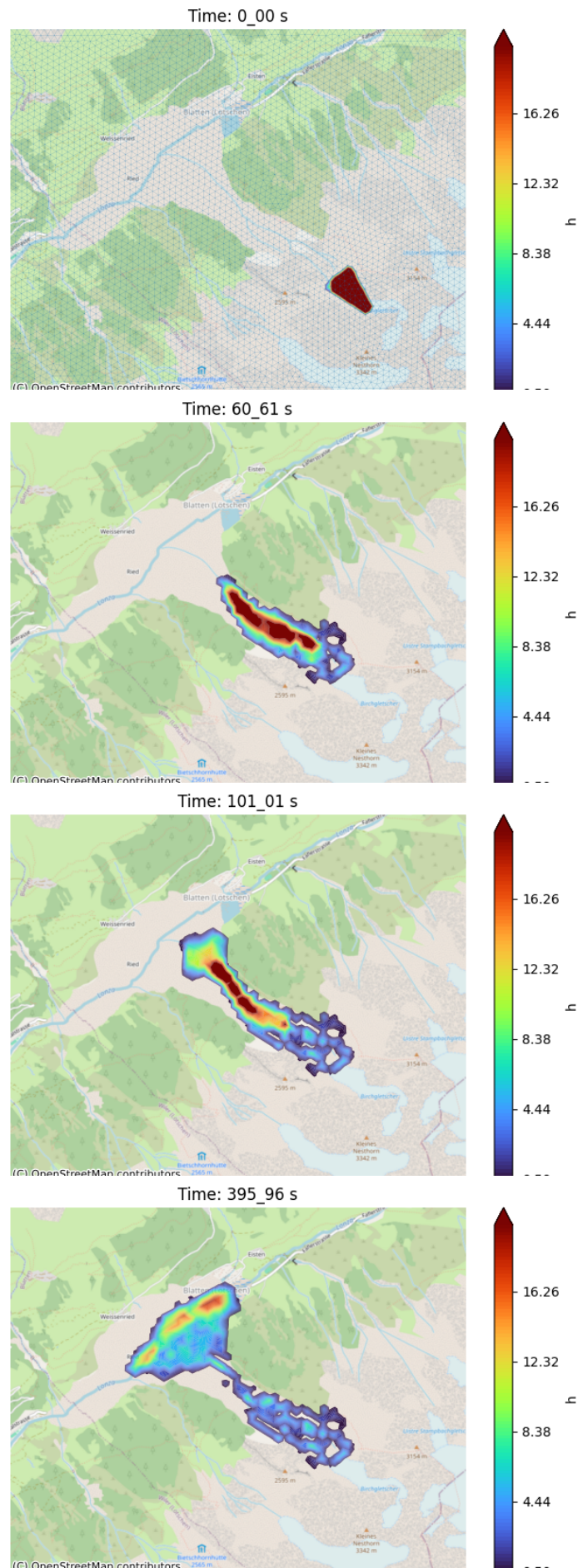


Figure 8.3: Simulation results for Blatten mixture catastrophic flow.

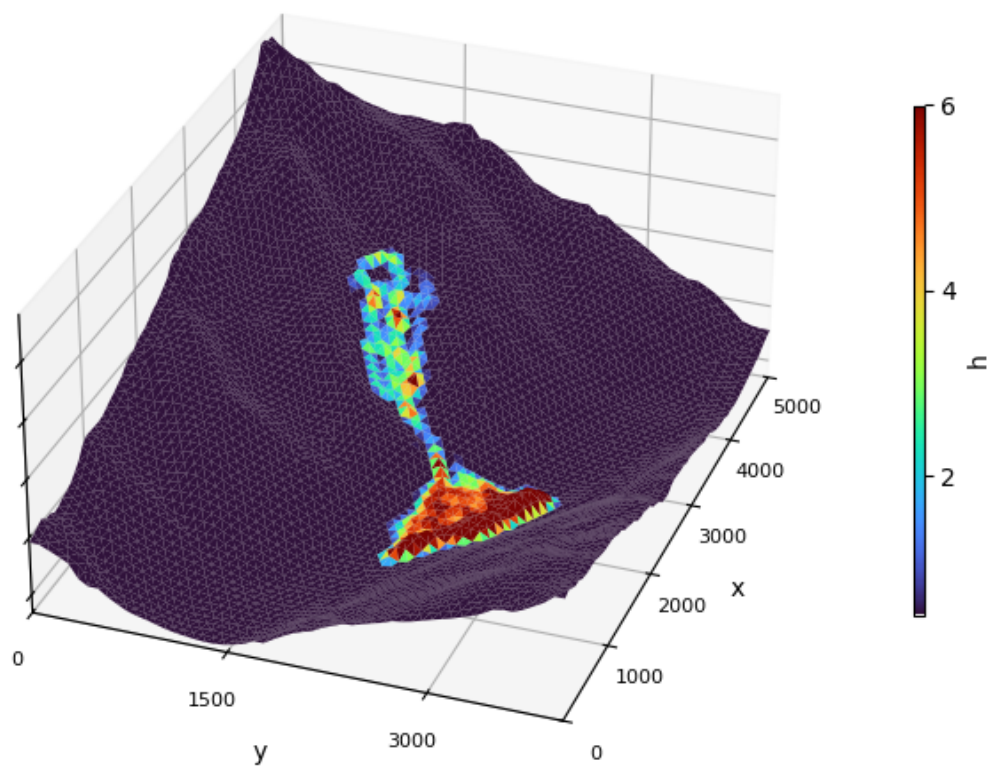


Figure 8.4: Comparison view of the model's deposition and a photograph from northwest (Bradley 2025).

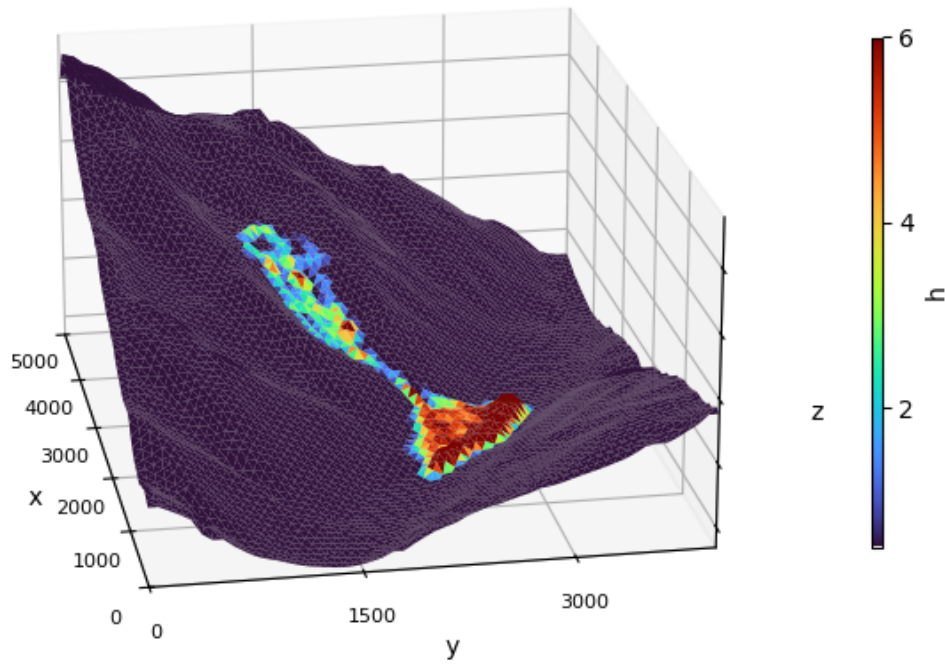


Figure 8.5: Comparison view of the model's deposition and a photograph from northeast (Geoengineer.org 2025).

Chapter 9

Conclusion and perspectives

The purpose of this thesis was to open a new door of functionality in the Watlab program, originally designed for water simulations. The new functionality consisted in creating a type of friction that would enable Watlab to simulate mudflows event.

With a view to this application, initial literature research made it possible to identify the actual behaviours expected and to characterize as accurately as possible the composition of the material to be modeled. The first research aimed to identify the mechanisms behind these complex phenomena and subsequently link the established definitions with the parameters necessary for the model to function.

In a second time, mathematical models generally used were exposed and the SWE model implemented in Watlab was put into perspective within this large family of tools. It has been demonstrated that the resolution model implemented in Watlab is one of the most appropriate approaches for large-scale mudflow simulations and that the single-phase assumption is the first necessary basis for the model to build the foundation of the new module to be implemented.

The calculation of mudflows required specific formulations, and the new elements to be included in the program concern the friction terms. Hence, the developement of such friction was demonstrated. First by using the formulation found by Pastor et al. (2004), then proposing new types of formulations that, seemed to correct an aspect of the latter. Both Bingham rheology and Herschel-Bulkley rheology were implemented in a single-phase structure within the existing code. Allowing Watlab users to simulate several types of flows by providing the program with input parameters for these rheologies.

Those formulation and implementation were put at test, first by comparing the results with qualitative experimental measurements provided by the University of Politecnico di Milano (Guo 2025), then, by testing the resolution at larger scale. The Blatten disaster that happened in May 2025 was the choice for the first real-world application of the model. Despite strong assumptions and certain mechanisms still missing from the module, the results were encouraging.

The applications were the opportunity to test the new module's foundation and verify the approach chosen for such implementation. Similarly, they provided an opportunity to reveal certain weaknesses which, hopefully, will lead to improvements and new methods for expanding on the work that has been carried out in this thesis.

The tracks, open for the future, certainly first and foremost in the correction of the initial instant of the simulations. For the moment, no friction is considered at this stage, and the major errors probably arise from this shortcoming. Next, to enable real-scale applications, the implementation of mudflows could affect Watlab's erosion calculations. Finally, many other rheologies exist and would be consistent with this modeling research. They certainly represent a major opportunity for improvement.

This thesis sincerely hopes that continuity will be found for this work and that the mudflows module will see new implementations in the future of Watlab.

Chapter A

Appendix

A.1 Herschel-Bulkley (HB) speed profile and base friction calculation

As a reminder, the velocity profile is determined by HB's rheological model and by the linearity of shear stresses on the liquid column.

$$\tau = \tau_y + K\dot{\gamma}^n \quad (\text{A.1})$$

The linearity of shear stresses is written as:

$$\tau(x_3) = \tau_b \left(1 - \frac{x_3}{h}\right) \quad (\text{A.2})$$

Integrating $\dot{\gamma}$ from the expression A.1 and A.2 gives the velocity profile:

$$u(x_3) = \int_0^{x_3} \dot{\gamma} dx'_3 = \int_0^{x_3} \left(\frac{\tau - \tau_y}{K}\right)^{1/n} dx'_3 = \int_0^{x_3} \left(\frac{\tau_b \left(1 - \frac{x'_3}{h}\right) - \tau_y}{K}\right)^{\frac{1}{n}} dx'_3 \quad \forall x_3 \in [0, h_s] \quad (\text{A.3})$$

Let's define $\alpha = (\tau_b - \tau_y)/K$ and $\beta = \tau_b/(Kh)$, so that the velocity profile can be rewritten as:

$$u(x_3) = \int_0^{x_3} (\alpha - \beta x'_3)^{\frac{1}{n}} dx'_3 \quad \forall x_3 \in [0, h_s] \quad (\text{A.4})$$

Solving the integral in A.4 gives:

$$\begin{aligned} u(x_3) &= \frac{n}{n+1} \frac{hK}{\tau_b} \left[\left(\frac{\tau_b - \tau_y}{K}\right)^{\frac{n+1}{n}} - \left(\frac{\tau_b \left(1 - \frac{x_3}{h}\right) - \tau_y}{K}\right)^{\frac{n+1}{n}} \right] \quad \forall x_3 \in [0, h_s] \\ u(x_3) &= \frac{n}{n+1} \frac{hK}{\tau_b} \left[\left(\frac{\tau_b - \tau_y}{K}\right)^{\frac{n+1}{n}} - \left(\frac{\tau_b \left(1 - \frac{h_s}{h}\right) - \tau_y}{K}\right)^{\frac{n+1}{n}} \right] \quad \forall x_3 \in [h_s, h] \end{aligned} \quad (\text{A.5})$$

Using the relation acquired from similar triangles $h_s/h = 1 - \tau_y/\tau_b$, the profile can be simplified to:

$$\boxed{\begin{aligned} u(x_3) &= \frac{n}{n+1} \frac{hK}{\tau_b} \left[\left(\frac{\tau_b - \tau_y}{K}\right)^{\frac{n+1}{n}} - \left(\frac{\tau_b \left(1 - \frac{x_3}{h}\right) - \tau_y}{K}\right)^{\frac{n+1}{n}} \right] \quad \forall x_3 \in [0, h_s] \\ u(x_3) &= \frac{n}{n+1} \frac{hK}{\tau_b} \left(\frac{\tau_b - \tau_y}{K}\right)^{\frac{n+1}{n}} \quad \forall x_3 \in [h_s, h] \end{aligned}} \quad (\text{A.6})$$

The depth-averaged velocity $\bar{u}$ is obtained by integrating the velocity profile over the total water height h .

$$\bar{u} = \frac{\int_0^{h_s} u(x_3) dx_3 + \int_{h_s}^h u(x_3) dx_3}{h} = \frac{I'_1}{h} + \frac{I'_2}{h} = I_1 + I_2 \quad (\text{A.7})$$

Calculating I_1 and I_2 separately, with $A = nhK/((n+1)\tau_b)$, $B = ((\tau_b - \tau_y)/K)^{(n+1)/n}$:

$$\begin{aligned} I_1 &= \frac{A}{h} \int_0^{h_s} \left(B - \left(\frac{\tau_b \left(1 - \frac{x_3}{h}\right) - \tau_y}{K} \right)^{\frac{n+1}{n}} \right) dx_3 = \frac{A}{h} \int_0^{h_s} B dx_3 - \frac{A}{h} \int_0^{h_s} \left(\frac{\tau_b \left(1 - \frac{x_3}{h}\right) - \tau_y}{K} \right)^{\frac{n+1}{n}} dx_3 \\ &= I_{1,1} - I_{1,2} \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned} I_{1,1} &= \frac{A}{h} \int_0^{h_s} B dx_3 \\ I_{1,2} &= \frac{A}{h} \int_0^{h_s} \left(\frac{\tau_b - \tau_y - \frac{\tau_b x_3}{h}}{K} \right)^{\frac{n+1}{n}} dx_3 \end{aligned} \quad (\text{A.9})$$

Calculating $I_{1,1}$:

$$\begin{aligned} I_{1,1} &= \frac{A}{h} B h_s \\ &= \frac{A}{h} h_s \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{n+1}{n}} \end{aligned} \quad (\text{A.10})$$

Calculating $I_{1,2}$ (with α and β defined above):

$$\begin{aligned} I_{1,2} &= \frac{A}{h} \int_0^{h_s} (\alpha - \beta x_3)^{\frac{n+1}{n}} dx_3 \\ &= \frac{A}{h} \frac{1}{\beta} \frac{n}{2n+1} \left((\alpha - \beta h_s)^{\frac{2n+1}{n}} - \alpha^{\frac{2n+1}{n}} \right) \\ &= \frac{A}{h} \frac{-hK}{\tau_b} \frac{n}{2n+1} \left[\left(\frac{\tau_b \left(1 - \frac{h_s}{h}\right) - \tau_y}{K} \right)^{\frac{2n+1}{n}} - \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{2n+1}{n}} \right] \\ &\quad as \quad \frac{h_s}{h} = 1 - \frac{\tau_y}{\tau_b} \\ &= \frac{A}{h} \frac{hK}{\tau_b} \frac{n}{2n+1} \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{2n+1}{n}} \end{aligned} \quad (\text{A.11})$$

Assembling I_1 :

$$\begin{aligned} I_1 &= I_{1,1} - I_{1,2} \\ &= \frac{A}{h} \left[h_s \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{n+1}{n}} - \frac{hK}{\tau_b} \frac{n}{2n+1} \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{2n+1}{n}} \right] \\ &= \frac{nK}{(n+1)\tau_b} \left[h_s \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{n+1}{n}} - \frac{hK}{\tau_b} \frac{n}{2n+1} \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{2n+1}{n}} \right] \end{aligned} \quad (\text{A.12})$$

Calculating I_2 :

$$\begin{aligned}
I_2 &= \frac{1}{h} \int_{h_s}^h u(x_3) dx_3 \\
&= \int_{h_s}^h \frac{n}{n+1} \frac{K}{\tau_b} \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{n+1}{n}} dx_3 \\
&= \frac{n}{n+1} \frac{K}{\tau_b} \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{n+1}{n}} (h - h_s) \\
&\text{with } h - h_s = h \frac{\tau_y}{\tau_b} \\
&= \frac{n}{n+1} \frac{K}{\tau_b} \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{n+1}{n}} h \frac{\tau_y}{\tau_b}
\end{aligned} \tag{A.13}$$

Finally, the depth-averaged velocity is:

$$\bar{u} = \frac{nK}{(n+1)\tau_b} \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{n+1}{n}} \left[h_s - \frac{hK}{\tau_b} \frac{n}{2n+1} \left(\frac{\tau_b - \tau_y}{K} \right)^{\frac{n}{n}} + h \frac{\tau_y}{\tau_b} \right] \tag{A.14}$$

Isolating τ_b in equation A.14 requires numerical methods, as it cannot be done analytically. τ_b is substituted by $\xi = \tau_y/\tau_b$, the relative plug depth, as it was introduced for the Bingham model development. The objective is to simplify the numerical resolution as the values of ξ are between 0 and 1. The equation to solve is then:

$$\boxed{\bar{u} = \frac{h}{n+1} \left(\frac{\tau_y}{K} \right)^{\frac{1}{n}} \frac{(1-\xi)^{\frac{n+1}{n}}}{\xi^{\frac{1}{n}}} \left(1 - \frac{n}{2n+1} + \frac{n\xi}{2n+1} \right)} \tag{A.15}$$

A.2 Analytical solutions for the third order polynomial of Bingham friction law

The third order polynomial to solve is recalled here:

$$P_3(\xi) := \xi^3 - (3+a)\xi + 2 = 0 \tag{A.16}$$

Solutions can be found using Cardano's method once the polynomial is expressed in the canonical form:

$$x^3 + px + q = 0 \tag{A.17}$$

The first solution is obtained by solving:

$$\xi_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{R}} + \sqrt[3]{-\frac{q}{2} - \sqrt{R}} \tag{A.18}$$

with $R = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$. In the case of equation A.16, $p = -(3+a)$ and $q = 2$, so that R becomes:

$$R = -\frac{(3+a)^3}{27} + 1 = -\left(\frac{a}{3}\right)^3 - \frac{a^2}{3} - a \tag{A.19}$$

The discriminant R is always negative for $a > 0$. Therefore, three real solutions exist and can be found using trigonometric functions. Equation A.17 can be rewritten in a form to use the trigonometric identity $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$. Changing variable as $x = \xi = 2a\cos(\theta)$ (θ is the new variable and the

expression is verified for any values of ξ as they are bounded by $[0, 1]$, and posing $\alpha = \sqrt{-p/3}$ transforms Equation A.17 into:

$$\begin{aligned} (2\alpha \cos(\theta))^3 + p(2\alpha \cos(\theta)) + q &= 0 \\ 4\cos^3(\theta) - 3\cos(\theta) &= -\frac{q}{2\alpha^3} \end{aligned} \quad (\text{A.20})$$

and θ can be found by solving:

$$\cos(3\theta) = -\frac{q}{2\alpha^3} = \frac{3q}{2p} \sqrt{-\frac{3}{p}} \quad (\text{A.21})$$

In the case of Equation A.16, developing p and q gives:

$$\cos(3\theta) = \frac{3\sqrt{3}}{(3+a)^{3/2}} \quad (\text{A.22})$$

Isolating θ while posing a variable Φ gives:

$$\begin{aligned} \Phi &:= \arccos\left(-\frac{3\sqrt{3}}{(3+a)^{3/2}}\right) \\ \theta &= \frac{1}{3}(\Phi + 2k\pi) \quad k \in \{0, 1, 2\} \end{aligned} \quad (\text{A.23})$$

Recalling that $\xi = 2\alpha \cos(\theta)$ and $\alpha = \sqrt{(3+a)/3}$, the three solutions of Equation A.16 are:

$$\boxed{\xi_k = 2\sqrt{\frac{3+a}{3}} \cos\left(\frac{\Phi + 2\pi k}{3}\right) \quad k \in \{0, 1, 2\}} \quad (\text{A.24})$$

Among these three solutions, only one should be relevant for the problem as ξ represents the relative plug depth and has to be between 0 and 1. The solutions are evaluated for different values of a in the section 6.5.1.

A.3 Boussinesq coefficient calculation for simple semi-quadratic speed profile

Considering a semi-quadratic speed profile in the x direction, recalled as $u(z)$. The speed is null at the flow base; $u(z=0) = 0$ and reach its maximum at the surface h ; $u(z=h) = u_{max}$. The perfectly parabolic speed profile is:

$$u(z) = u_{max} \cdot \left(\frac{z}{h}\right)^2 \quad (\text{A.25})$$

The depth averaged speed profile is:

$$\bar{u} = \int_0^h u(z) dz = \frac{u_{max}}{3} \quad (\text{A.26})$$

The depth averaged of the square speed profile is:

$$\bar{u}^2 = \int_0^h u(z)^2 dz = \frac{u_{max}^2}{5} \quad (\text{A.27})$$

Finally the ratio, and so the associated Boussinesq coefficient, gives:

$$\beta_{xx} = \frac{\bar{u}^2}{\bar{u}^2} = \frac{9}{5} = 1.8 \quad (\text{A.28})$$

A.4 Vertical gate or perpendicular gate? SWE modeling

The depth integrated model gives a unique value of height for each cells of the domain. The height is usually represented as vertical, for clear and comprehensive visualization. In this case, having a perpendicular gate would be impossible to model. The problematic is illustrated in Figure A.2. The element of the mesh underneath the gate (in purple) would have a dry height (red part) and then a wet height (blue part) at the beginning of the simulation. This is impossible to model with SWE. At the opposite, having a vertical gate would not bring any "dry height" and SWE are perfectly able to model such geometry.

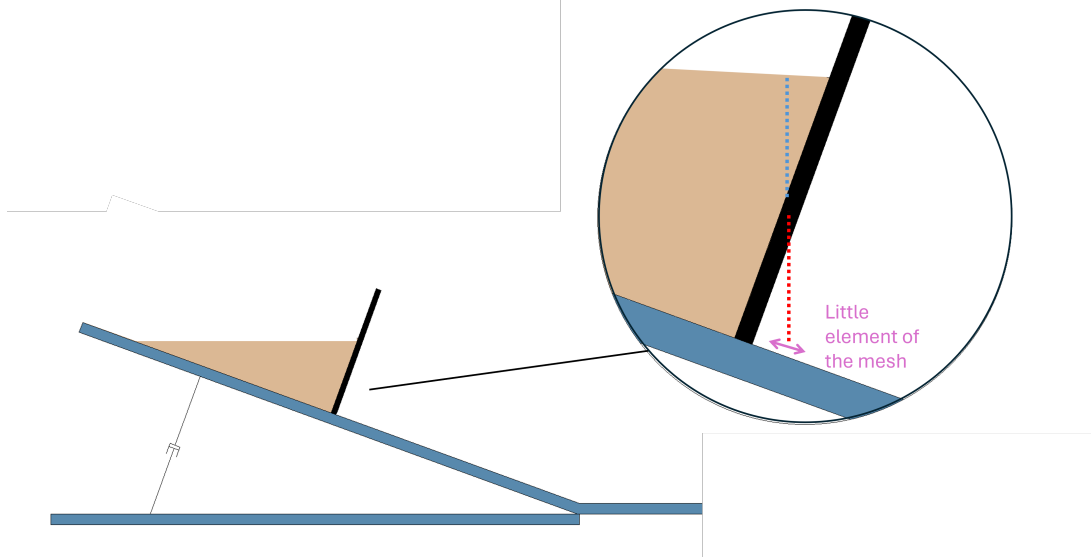


Figure A.1: Representation of a vertical gate and the impossibility to model it with SWE.

To model such geometry, a variation of certain quantity should be possible in the vertical direction. This simply impose to not depth integrate the formulation and asks for a 3D model.

A.5 Solid volumetric concentration, equation development

$$C_v = \frac{V_s}{V_s + V_w} \quad (\text{A.29})$$

$$V_s = \frac{m_s}{\rho_s}, \quad V_w = \frac{m_w}{\rho_w} \quad (\text{A.30})$$

Developing the volumes with densities:

$$C_v = \frac{m_s/\rho_s}{m_s/\rho_s + m_w/\rho_w} \quad (\text{A.31})$$

Considering 1m^3 of volume and under the hypothesis that no air is trapped in the mixture (the material contains more water than at saturation state):

$$\begin{cases} m_s + m_w = \rho_{\text{tot}} \cdot 1 \text{ m}^3 = \rho_{\text{tot}} \\ \frac{m_s}{\rho_s} + \frac{m_w}{\rho_w} = 1 \end{cases} \quad (\text{A.32})$$

From these two equations, the solid fraction volume will be isolated. First, isolating the mass of water contained in $1m^3$:

$$m_w = \rho_w \left(1 - \frac{m_s}{\rho_s} \right) \quad (\text{A.33})$$

Then, with $1m^3$, the sum of masses is the total bulk density:

$$m_s + \rho_w \left(1 - \frac{m_s}{\rho_s} \right) = \rho_{\text{tot}} \quad (\text{A.34})$$

$$m_s \left(1 - \frac{\rho_w}{\rho_s} \right) = \rho_{\text{tot}} - \rho_w \quad (\text{A.35})$$

$$m_s = \frac{\rho_{\text{tot}} - \rho_w}{\rho_s - \rho_w} \cdot \rho_s \quad (\text{A.36})$$

$$V_s = \frac{m_s}{\rho_s} = \frac{\rho_{\text{tot}} - \rho_w}{\rho_s - \rho_w} \quad (\text{A.37})$$

Finally, $V_s = C_v$ for $1m^3$ of total material as V_s would represent the volume of solid contained in $1m^3$ of mixture.

$$\boxed{C_v = \frac{\rho_{\text{tot}} - \rho_w}{\rho_s - \rho_w}} \quad (\text{A.38})$$

A.6 Limitations of exponential yield stress formulations at high solid volume fraction and justification of a power-law approach

Here, this short additional note presents the detailed development of the different yield stress formulations discussed in the main text. It aims to illustrate why exponential and compounded exponential relationships lead to unrealistic values of τ_y for the solid volume fractions encountered in PoliMi's experiments. It justifies the adoption of a power-law formulation for high solid concentration mudflows.

The first formulation tested is the compounded exponential law proposed by (Schipa 2021). It takes the simple exponential law and develop the coefficients α_2 and β_2 with exponential forms, combining two exponential and exploding the values of τ_y for high concentrations of solids:

$$\tau_y = \alpha_2 e^{\beta_2 \Phi_T} \quad (\text{A.39})$$

With $k_1 = 3.5$, $k_2 = -5$ and $k_3 = 2.9$ (as a reminder, the coefficients were fitted to experimental results by (Schipa 2021)).

$$\alpha_2 = \beta_2 = \psi_1 \frac{\phi_T}{\phi_M - \phi_T} = \left(k_1 e^{-k_2 \frac{\phi_g}{\phi_f}} + k_3 \right) \frac{\phi_T}{\phi_M - \phi_T} = 74.43$$

$$\tau_y = 5.8 \cdot 10^{26} [Pa] \quad (\text{A.40})$$

Yield stress is too large to be physically meaningful, the formulation of (Schipa 2021) fails to estimate it, the high fraction volume is in fact outside of its range of application, even if not directly mentioned by the authors.

However, as the yield stress is too large to be calculated with Equation A.40, (U.S. Army Corps of Engineers 2025b) shows a better approximation by giving constant values of α_2 and β_2 for a given range of C_v :

$$\tau_y = 0.005 e^{17.2 C_v} = 2823 Pa \quad (\text{A.41})$$

The value is more correct but is still very high. In reality, values like this dictates that the material will not be able to move when the gate opens in the experiments of PoliMi. The resistance τ_y is too high to allow any movement.

The final expression for hyper-concentrated mudflows as the one in the experiments should satisfy the expression in Equation 4.11. The parameter a is not known and can simply being retrieved by the values obtained with an exponential law. Using the middle value of the spectrum for the exponential law of (U.S. Army Corps of Engineers 2025b), so with $C_v = 0.7$, $\tau_y = 847Pa$, it is possible to fit the coefficient a to this value as it should be correct for the exponential law. The hypothesis is therefore that the factor a does not change much with the solid fraction volume. In this case, $a = 847/0.7^3 = 2470$. Finally:

$$\tau_y = a \cdot C_v^3 = 2470 \cdot 0.771^3 = 1133Pa \quad (\text{A.42})$$

The final value is close to the one expected for the experiment and it was kept for a first estimation. In a second time, iterations with the model to better fit the results will clarify the rheological parameters.

A.7 Verification of high solid fraction volume in PoliMi's experiments

This appendix provides a simplified volumetric analysis aimed at verifying the plausibility of the high solid volume fraction values obtained in the experimental campaign notably for the development of the T06 experiment. The proposed reasoning does not introduce additional assumptions but simply relies on measured bulk and dry densities.

In order to verify the calculated $C_v = 0.77$ of experiment T06, a simple comparison can be developed in a hypothetical 1 m^3 box of material. Quick measurement of the dry density gave (the measurement was carried out before adding any water in one the experiments):

$$\rho_{\text{dry}} = 1633 \text{ kg/m}^3 \quad (\text{A.43})$$

This value corresponds to solid material with voids filled only by air. The volume occupied by the solids in the 1 m^3 box of dry material is therefore:

$$V_s = \frac{\rho_{\text{dry}}}{\rho_s} = 0.62 \text{ m}^3 \quad (\text{A.44})$$

For this specific case, the solid volume fraction is:

$$C_v = \frac{0.62}{1} = 0.62 \quad (\text{A.45})$$

The volume of voids is simply:

$$V_v = 1 - 0.62 = 0.38 \text{ m}^3 \quad (\text{A.46})$$

By filling all these voids with water, the total mass and therefore the bulk density of this 1 m^3 box becomes:

$$\rho_{\text{tot}} = 1633 + \rho_w \cdot 0.38 = 2017 \text{ kg/m}^3 \quad (\text{A.47})$$

The volumetric mass corresponding to a solid fraction $C_v = 0.62$ is therefore more than 250 kg/m^3 lower than the value measured in the experiments ($\rho_{\text{tot}} = 2273 \text{ kg/m}^3$). This indicates that when water is added to the dry material, it lubricates the grains and induces a significant rearrangement. The process of thoughts qualitatively validates the high value of $C_v = 0.77$ obtained in the experimental analysis, which is therefore accepted in the present work.

A.8 Failure of a tailings dam retaining gypsum tailings, Florida



Figure A.2: Aerial view of such tailings dam failures. The picture comes from (Geotechnical Photo Album, University of California Davis 2025)

A.9 Herschel-Bulkley plug-height curves for different values of the power index n

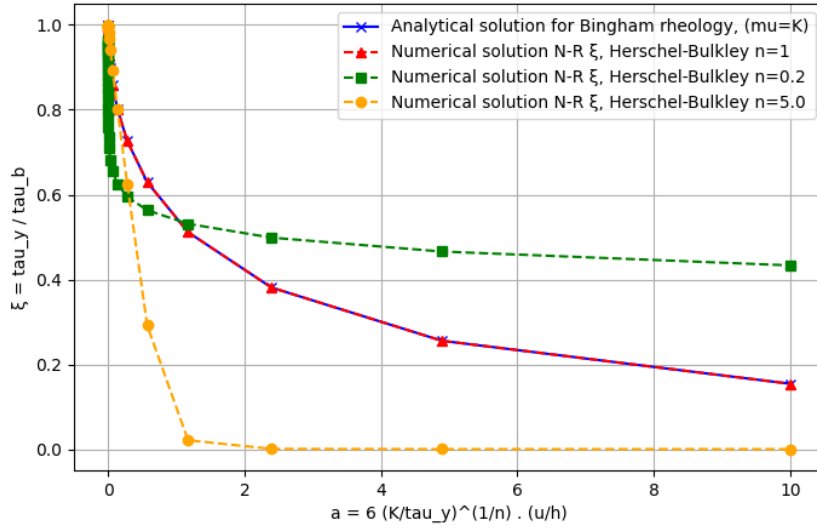


Figure A.3: Relative plug height ξ and parameter a : the logarithmic-like axis has been removed so the visualization of steep and milder approach of $\xi = 1$ is easier.

A.10 Why the formulation of (Pastor et al. 2004) gives faster velocities than the exact solution where it shouldn't be?

To answer this question, here is a copy of the graph in Figure A.4 comparing the relation of (Pastor et al. 2004) to the ones found in this thesis. The graph is at the heart of the explanation.

To put this into context, (Pastor et al. 2004)'s formulation seems to give higher speeds, whereas it overestimates friction and should slow down the speed compared to the exact formulation. This was found while comparing the simulation of the article to both $P2$ and the exact analytical solution. I tried both formulations and $P2$ gave faster velocities than the exact expression.

The formulation of (Pastor et al. 2004) gives lower value of ξ for low velocities (the red curve is under the blue and green one in A.4), so τ_b gets higher for the same low speed compared to the exact solution. Normally, the friction should then be higher for the formulation proposed by (Pastor et al. 2004) than the exact one developed in this document. If the friction is higher, it will be more likely to retain the speeds and slow down the flow. At the opposite, the results showed that (Pastor et al. 2004) was faster than the exact expression.

Actually the reason comes from the implementation of the resolution in Watlab. In order to avoid any numerical noise and the "un-freezing" character of a cell that should have been kept at rest, a limit was set at $\xi = 0.9$: the orange line. Any values above this threshold will be absorbed by the code and set to be $\xi = 1$: all points are sent in the orange large dot. The speed will be nul and the cell will be considered as "frozen". The objective is to avoid noise oscillation between active and stopped cells. Indeed, a cell can be remobilized if its neighbors push hard enough on it: this is the remobilization process. So if every values of ξ are contained below $\xi = 0.9$ or sent to $\xi = 1$, the sort of artificial plateau that is represented in Figure A.4 is not at $\xi = 1$ anymore but at $\xi = 0.9$ (now the plateau is the orange line, where before it was the upper left part of the green curve, where it became constant). If we observe carefully the graphs at $\xi = 0.9$ (around $a = 10^{-2}$) (the orange circle), the red curve passes above the green and blue curve. It means that the formulation of (Pastor et al. 2004) will give lower values of S_f just before the artificial

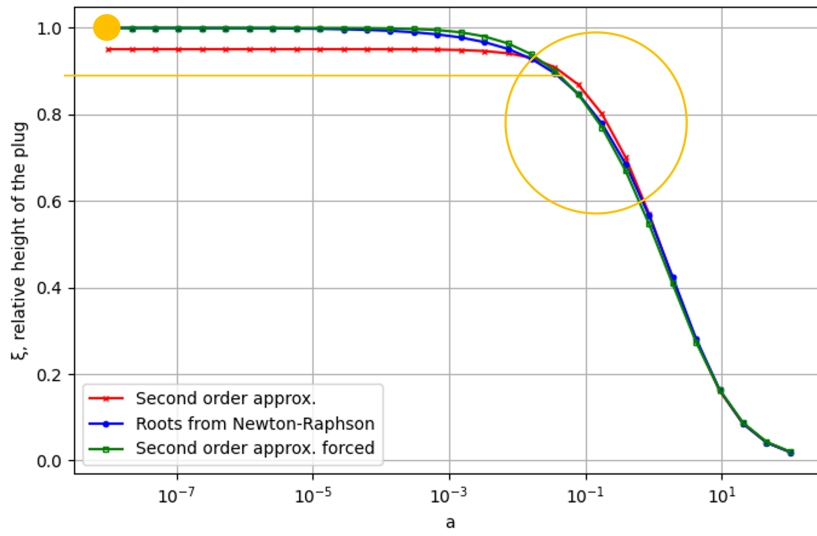


Figure A.4: Comparison of the exact solution of P_3 (with N-R), the approximate solution of P_2 from (Pastor et al. 2004) and the new forced approximate solution of $P_{2,forced}$ developed in this document to calculate ξ as a function of a .

stop $\xi = 0.9$. It results in a faster simulation than the exact expression... Because each time the cell is near to be stopped, the exact formulation will give a lower friction, therefore as a result (Pastor et al. 2004) will be faster in a global view of the simulation.

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